

Infinite dimensional Analysis

2014/4/11

§ 1 Hilbert spaces

Def 1.1 $(\cdot, \cdot) : H \times H \rightarrow K (= \mathbb{R} \text{ or } \mathbb{C})$ is innerproduct if and only if

$$\textcircled{1} (f, \alpha g + \beta h) = \alpha (f, g) + \beta (f, h)$$

$$\textcircled{2} \overline{(f, g)} = (g, f)$$

$$\textcircled{3} (f, f) \geq 0$$

$$\textcircled{4} (f, f) = 0 \Leftrightarrow f = 0$$

$\|f\| = \sqrt{(f, f)}$ is called the ~~scalar~~ norm on H .

Ex $\mathbb{R}^n, \mathbb{C}^n$ Ex $C([a, b])$ $(f, g) = \int_a^b \bar{f}(x)g(x)dx$

Ex $\ell^2 = \{ \{a_n\} ; \sum_n |a_n|^2 < \infty \}$
 $(a, b) = \sum_j \bar{a}_j b_j$

§ 1.2 Geometry of Hilbert spaces

$$\|f + g\|^2 = \|f\|^2 + 2\operatorname{Re}(f, g) + \|g\|^2$$

$$f \perp g \Leftrightarrow (f, g) = 0$$

$$\therefore f \perp g \Rightarrow \|f + g\|^2 = \|f\|^2 + \|g\|^2$$

- $D \subset H$ $\forall f, g \in D, f \neq g \Rightarrow (f, g) = 0$

Then D is called an orthogonal system.

- Let D be orthogonal system and $\forall f \in D$ is $\|f\|=1$. Then D is called an ONS.

- Ex ℓ^2 , $D = \{e_n\}$ ONS

Ex $(([0, 2\pi]), \{e^{inx}/\sqrt{2\pi}\}$ ONS

Theorem 1.2 let $\{e_n\}_{n=1}^{\infty}$ be ONS. Then for $\forall f \in H$ it follows that $\sum_{n=1}^{\infty} |(e_n, f)|^2 < \|f\|^2$

Let $\{e_n\}_{n=1}^{\infty}$ be ONS. Then for $\forall f \in H$ it follows that $\sum_{n=1}^{\infty} |(e_n, f)|^2 < \infty$ and $\sum_{n=1}^{\infty} |(e_n, f)|^2 \leq \|f\|^2$

$d_H(f, g) = \|f - g\|$ defines the metric on H .

(H, d_H) is the topological space

Def 1.84 $H > D$ is an open set if and only if

$\forall f \in D, \exists \varepsilon > 0$ st $B_{\varepsilon}(f) \subset D$, where

$B_{\varepsilon}(f) = \{g \in H ; \|f - g\| < \varepsilon\}$ open ball with center f and radius ε .

Def 1.46 $\{f_n\} \in H$

① f_n strongly converges to f if and only if
 $\|f - f_n\| \rightarrow 0$ ($n \rightarrow \infty$)

② f_n weakly converges to f if and only if
 $(g, f - f_n) \rightarrow 0$ $\forall g \in H$ as $n \rightarrow \infty$.

Def 1.47 $(H, (\cdot, \cdot))$ ^{complete} Inner product space is called
a Hilbert space ~~if and only if~~
I.e. ~~if~~ Cauchy sequence ~~is~~ implies convergent sequence.

Ex ℓ^2 is a Hilbert space

Ex $L^2(\mathbb{R}) / \sim = L^2(\mathbb{R})$ is a Hilbert space

Def 1.3 Let $\{e_n\}_n$ be ONS on H .

Then $\{e_n\}$ is CONS if and only if $\forall f \in H$

$$f = \sum (e_n, f) e_n$$

Theorem 1.4 $\{e_n\}$ ONS (1)~(4) are equivalent

(1) $\{e_n\}$ is CONS

$$(2) (f, g) = \sum (f, e_n)(e_n, g)$$

$$(3) \|f\|^2 = \sum |(f, e_n)|^2$$

$$(4) (f, e_n) = 0 \forall n \Rightarrow f = 0.$$

§ 2 Closed subspace

Suppose that H is a Hilbert space.
 $D \subset H$ is a subspace.

$D^\perp = \{f : (g, f) = 0 \forall g \in D\}$ is called
 the orthogonal complement of D .

Lem 2.1 D^\perp is closed and $(\bar{D})^\perp = D^\perp$

$\because D^\perp \ni f_n, f_n \rightarrow f$.

Then $0 = \lim (g, f_n) = (g, f) \because f \in D^\perp$

$$D \subset \bar{D} \Rightarrow D^\perp \supset (\bar{D})^\perp$$

Let $f \in D^\perp$. Then $(g, f) = 0 \forall g \in D$

Let $g \in \bar{D} \ni g_n \in D$ s.t. $g_n \rightarrow g$

Hence $(g, f) = \lim_n (g_n, f) = 0$,

~~Lemma~~ ~~Theorem~~ 2.2

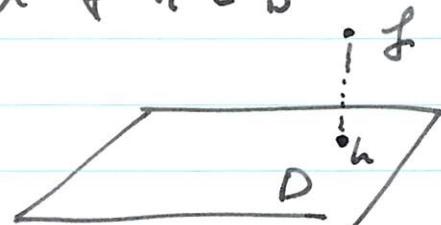
Let D be closed subsp.

Then $\exists' h \in D$ st

$$d(f, D) = \|f - h\|, \text{ and } f - h \in D^\perp.$$

$$d(f, D) = \inf_{h \in D} \|f - h\|$$

is defines the distance between f and D



$\therefore d = d(f, D)$, Then exists $\{h_n\}$ st $\lim_{n \rightarrow \infty} \|f - h_n\| = d$

It can be shown that $\{h_n\}$ is a Cauchy seq.
 Then $\exists h \in H$ s.t. $h_n \rightarrow h$. Since D is closed
 $h \in D$.

$$\because \|f - h_n - dg\|^2 = \|f - h_n\|^2 - 2\Re \alpha (f - h_n, g) + \|g\|^2$$

$$d \leq \|f - h_n\|^2 - 2|\langle f - h_n, g \rangle|/\|g\|^2 \quad (g \in D)$$

$$\text{Take } \alpha = \langle g, f - h_n \rangle / \|g\|^2$$

$$d \leq \|f - h_n\|^2 - 2|\langle f - h_n, g \rangle|/\|g\|^2 + |\langle f - h_n, g \rangle|/\|g\|^2$$

$$\therefore |\langle f - h_n, g \rangle| \leq \sqrt{d_n^2 - d^2} \|g\| \quad \text{--- } \oplus$$

$$\begin{aligned} \therefore |\langle h_n - h_m, g \rangle| &\leq |\langle h_n - f, g \rangle| + |\langle f - h_m, g \rangle| \\ &= (\sqrt{d_n^2 - d^2} + \sqrt{d_m^2 - d^2}) \|g\| \end{aligned}$$

$$\text{Since } h_n - h_m \in \cap_{n=1}^{\infty} D^n \quad //$$

By \oplus we have $|\langle f - h, g \rangle| = 0 \therefore f - h \in D^\perp$

~~closedness is hereditary~~

Suppose that $f - h' \in D^\perp$, and $d = \|f - h'\|$

$$f = f - h + h = f - h' + h'$$

$$\text{Hence } f - h' - (f - h) = h - h' \in D^\perp$$

$$\therefore h - h' = 0 \quad //$$

Theorem 2.3 $D \subset H$ closed subspace,

Then \nexists every f can be uniquely written

$$f = gh, \text{ where } h \in D, g \in D^+.$$

//

Def 2.4 ~~Let D be a closed subspace~~

$D \subset H$ is dense subset if and only if $\overline{D} = H$.

$$\mathcal{L}(D) = \left\{ \sum_{n=1}^N a_n f_n ; f_n \in D \right\}$$

$\mathcal{L}(D)$ is a linear hull of D .

Theorem 2.5 $\{e_n\}$ is CONS. if and only if $\overline{\mathcal{L}(\{e_n\})}$ is dense.

$$\textcircled{1} (\Rightarrow) (e_n, f) = 0 \forall n \rightarrow f = 0$$

$$\therefore \overline{\mathcal{L}(\{e_n\})}^\perp = \{0\} = \overline{(\mathcal{L}(\{e_n\}))^+}$$

$$\therefore \overline{\mathcal{L}(\{e_n\})} = H.$$

$$\therefore \forall f \in H$$

$$\Rightarrow f = g + h = g \in \overline{\mathcal{L}}.$$

$$(\Leftarrow) \quad \overline{\mathcal{L}(\{e_n\})} = H$$

$$\therefore \overline{\mathcal{L}(\{e_n\})}^\perp = \{0\}$$

$$\text{In particular } (e_n, f) = 0 \forall n.$$

$\therefore \{e_n\}$ is CONS.

Def 2.6 H is separable if and only if
 $\exists D \subset H$ st $\#D$ is countable and $\overline{L(D)} = H$.

① Theorem 2.7 \exists CONS in a separable Hilbert sp.

Let $D = \{e_n\}_{n=1}^{\infty}$. By Gram-Schmidt ~~(orthogonal)~~
 orthonormalization

we can construct $\{g_n\}$ ONS st
 $L(\{g_n\}) = L(e_n)$. $\overline{L(\{g_n\})} = H$

Then $\{g_n\}$ is CONS. //

Ex $L^2(\mathbb{R})$ $\mathbb{D}_{[-n, n]}$ x^m

$D = \{ \mathbb{1}_{[-n, n]}(x) x^m; n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\} \}$

is dense, $\#D$ countable,
 Hence $L^2(\mathbb{R})$ is separable. \exists CONS

§ 3 Bounded Operators

$$T: D(T) \rightarrow \mathcal{H}, \quad \begin{aligned} \textcircled{1} \quad T(f+g) &= Tf + Tg \\ \textcircled{2} \quad T\alpha f &= \alpha Tf \end{aligned}$$

T : linear op. $D(T)$ domain of T

In general $D(T) \subsetneq \mathcal{H}$.

Ex $\mathcal{H} = L^2(\mathbb{R})$, $T = -i \frac{d}{dx}$, $D(T) = C_0^\infty$

$\mathcal{H} = L^2(\mathbb{R})$, $T = M_g$, $D(M_g) = \{g; fg \in L^2\}$

differential op multiplication op

- Sum $D(T+S) = D(T) \cap D(S)$
- product $D(TS) = \{f; Sf \in D(T)\}$

Def 3.1 $A: \mathcal{H} \rightarrow K$, linear op. (\mathcal{H}, K Hilbert space)

A is a b'dd operator if and only if $\exists c \geq 0$ st

$$\|Af\|_K \leq c\|f\|_{\mathcal{H}} \quad \forall f \in D(A)$$

$\textcircled{1}$ $\sup_{\substack{f \in D(A) \\ f \neq 0}} \frac{\|Af\|}{\|f\|} = \|A\| \therefore \|Af\| \leq \|A\| \cdot \|f\|$

$\|A\|$ operator norm of A

$\textcircled{2}$ $f_n \rightarrow f \Rightarrow Af_n \rightarrow Af \quad (f_n, f \in D(A))$

A b'dd $\rightarrow A$ continuous.

Theorem 3.2 T is bdd $\Leftrightarrow T$ is continuous.

Theorem 3.3 T is densely defined and bdd.

Then $\exists^* \bar{T}$ st $D(\bar{T}) = \mathcal{H}$, $\bar{T} = T$ on $D(T)$, $\|\bar{T}\| = \|T\|$.

$\because \forall f \in \mathcal{H} \exists^* f_n \in D(T) \text{ st } f_n \rightarrow f (n \rightarrow \infty)$

$\{Tf_n\}$ is a Cauchy sequence.

$\exists \lim_n Tf_n = g$ Define $\bar{T}f = g$ (well-defined)

① $\bar{T}f = Tf \quad f \in D(T)$

$$\|\bar{T}f\| = \|Tf\| \leq \lim_n \|Tf_n\| = c \|f\|$$

② \bar{T} is bdd

$$\begin{aligned} \|T\| &= \sup_{f \in D(T)} \|Tf\| / \|f\| = \sup_{f \in D(T)} \|\bar{T}f\| / \|f\| \\ &\leq \sup_{f \in \mathcal{H}} \|\bar{T}f\| / \|f\| = \|\bar{T}\|. \end{aligned}$$

$$\therefore \|T\| \leq \|\bar{T}\|$$

$$\|\bar{T}f\| = \lim \|Tf_n\| \leq \|T\| \cdot \|f_n\| = \|T\| \|f\|$$

$$\therefore \|\bar{T}\| \leq \|T\|$$

③ $\|\bar{T}\| = \|T\|$

In what follows we assume that the bdd operators are defined on \mathcal{H} .

§4 Unbdd operators

(Quantum Mechanics) $[P, Q] = -i$

Ex $P = -\frac{d}{dx}, Q = Mx$

~~HYBRID~~

Lemma. Either P or Q is unbdd.

Suppose that P and Q are bdd

$$P^n Q = Q P^n - i n P^{n-1}$$

$$2 \|D_m^m Q\| \geq n \|P^{n-1}\| \therefore 2 \|P\| \cdot \|Q\| \geq n,$$

Def 4.1 $A : \mathcal{H} \rightarrow K$ is unbdd if and only if
 A is not bdd.

Def 4.2 $A : \mathcal{H} \rightarrow K$ is closed if and only if
 $f_n \in D(A)$, $f_n \rightarrow f$, $Tf_n \rightarrow g \Rightarrow f \in D(T)$
 $\& Tf = g$

Ex B'dd op is closed.

Def 4.3 $A : \mathcal{H} \rightarrow K$ densely defined

$$\begin{cases} D(A^*) = \{f \in K ; (f, Ag) = (\overset{\exists}{f}^*, g) \quad \forall g \in D(A)\} \\ A^* f = f^* \end{cases}$$

A^* is called the adjoint of A .

- $A \subset B \Rightarrow A^* \supset B^*$, $(A + B)^* \supset A^* + B^*$
- $(BA)^* \supset A^* B^*$

- f^* can be uniquely determined
 $\because (f^*, g) = (h, g) \Rightarrow f^* = h$

Lemma

Theorem 4.4 $A : \mathcal{H} \rightarrow K$ densely defined.

A^* is a closed operator.

$$\because f_n \rightarrow f, A^* f_n \rightarrow \cancel{h}$$

$$(f_n, Ag) = (A^* f_n, g) \rightarrow (h, g)$$

$$(f, Ag)$$

$D(A^*)$
is not necessarily
dense

"

Ex $M_f^* = M_{\bar{f}}$, $M_f^{**} = M_{\bar{f}}^*$ $\therefore M_f$ is a closed op.

$\Gamma(T) = \{(f, Tf); f \in D(T)\} \subset \boxed{\mathcal{H} \times K}$
is called the graph of T .

- $\Gamma(T)$ is a closed subspace $\Leftrightarrow T$ is closed
- Def 4.5 $T : \mathcal{H} \rightarrow K$ closable if and only if
 $\exists S$ closed st $T \subset \boxed{S}$
- In general there are infinite number of closed extensions ~~for~~ when T is closable.

$$T \subset \boxed{S} \Rightarrow \Gamma(T) \subset \Gamma(\boxed{S}) \Rightarrow \overline{\Gamma(T)} \subset \Gamma(\boxed{S})$$

$$\overline{\Gamma(T)} = \Gamma(\overline{T}) ?$$

Lemma 4.6 Let T be closed, i.e. $T \subset S$ closed
 Then $\exists \bar{T}$ closed s.t. $\overline{P(T)} = P(\bar{T})$

$\because D(\bar{T}) = \{ f \in \mathcal{F} ; \exists g \text{ s.t. } (f, g) \in \overline{P(T)} \}$
 $\bar{T} f = g$

\bar{T} is well-defined \therefore Suppose that $(f, g) \in \overline{P(T)}$
 $(f, g') \in \overline{P(T)}$

Hence $(0, g-g') \in \overline{P(T)} \subset P(S)$
 $\therefore g-g' = 0 = 0 \therefore g=g'$.

Then $\overline{P(T)} = P(\bar{T})$ //

\bar{T} is the smallest extension of T .

$\therefore T \subset S$ closed $P(T) \subset P(S)$
 $\therefore \overline{P(T)} \subset P(S) \therefore P(\bar{T}) \subset P(S)$.

Def 4.7 \bar{T} is called the closure of T .

Lemma 4.8 Let T be densely defined.

~~⊕ Suppose that T is closable then $T^* = (\bar{T})^*$~~

① T is closed $\Rightarrow D(T^*)$ is "dense" and $T = T^{**}$

② T is closable $\Rightarrow (\bar{T})^* = T^*$

③ T is closable $\Leftrightarrow D(T^*)$ is dense and
 Furthermore $\bar{T} = T^{**}$

∴ Let us see relationship between $G(\tau^*)$ and $G(\tau)^+$.

$$\mathcal{H} \oplus \mathcal{H} \Rightarrow (x_1, y_1, z_1) = (x_1, y_1)_{\mathcal{H}} + (x_2, y_2)_{\mathcal{H}}$$

$$G(\tau)^\perp \Rightarrow (y, z) \Leftrightarrow ((x, \tau x), (y, z)) = 0$$

$$(x, y) + (\tau x, z) = 0 \quad \therefore (\tau x, z) = (x - y)$$

$$\therefore z \in D(\tau^*) \text{ & } \tau^* z = -y$$

$$\therefore G(\tau)^+ \rightarrow (-\tau^* z, z) \quad U: \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}$$

$$\therefore U[G(\tau)^+] = G(\tau^*) \quad U^* U = U U^* = I, \quad U^2 = -I$$

$$\therefore U[G(\tau^*)^+] = [U G(\tau^*)]^+ = [G(\tau)^+]^\perp = \overline{G(\tau)}$$

$$\textcircled{1} \quad D(\tau^*)^+ \rightarrow v \Leftrightarrow ((0, v), (\tau^* z, -z)) = (0, \tau^* z) - (v, z) = 0$$

$$\therefore (0, v) \in [U G(\tau^*)]^\perp = \overline{G(\tau)} = G(\tau) \quad \therefore v = 0$$

$$\textcircled{2} \quad \text{Also } U[G(\tau^*)^+] = G(\tau^{**}) \quad \therefore \tau^{**} = \overline{\tau}$$

$$\textcircled{3} \quad G(\tau^*) = U[G(\tau)^+] = U[\overline{G(\tau)}]^\perp = G((\bar{\tau})^*)$$

$$\textcircled{3} \quad (\Rightarrow) \quad \tau \subset S \Rightarrow S^* \subset \tau^* \quad D(S^*) \text{ is dense}$$

Then $D(\tau^*)$ is dense

T^* densely defined \therefore we can consider T^{**}

$$(\Leftarrow) G(T^{**}) = \overline{G(T^*)^\perp} = \overline{G(T)} \supseteq G(T)$$

closed op

$$\therefore T^{**} = \overline{T}.$$

Example $\mathcal{H} = L^2(\mathbb{R})$, $T = \partial_x$, $D(T) = C_0^\infty(\mathbb{R})$

$$\text{Then } (\partial f, g) = \int \bar{f} \partial g = - \int \bar{f} \partial g \quad \because g \in D(\partial^*)$$

$g \in C_0^\infty$

$\therefore \partial$ is closable op. $\exists \overline{\partial}$.

Theorem 4.9 closed graph theorem

Let T be closed and $D(T) = \mathcal{H}$

Then T is bdd.

§ 5 Spectrum of closable operators

$$Av = \lambda v, \lambda \in \mathbb{C}, v \in \mathbb{R}^n$$

λ eigenvalue, v eigenvector.

⇒ We extend A to closable operators.

Def 5.1 $A: \mathcal{H} \rightarrow \mathcal{H}, \lambda \in \mathbb{C}$.

$\lambda \in \mathfrak{F}(A)$ if and only if

① $(A - \lambda)$ is injective i.e., $(A - \lambda)^{-1}$ is well-defined
and $(A - \lambda)^{-1}$ is bounded on $D(A - \lambda)^{-1} = \text{Ran}(A - \lambda)$

② $\text{Ran}(A - \lambda)$ is dense

$\mathfrak{F}(A)$ is called the residual set of A .

Prop 5.2 Let A be closable.

$$\text{① } \mathfrak{F}(A) = \mathfrak{F}(\bar{A})$$

$$\text{② } \forall \lambda \in \mathfrak{F}(A), \frac{\text{Ran}(\bar{A} - \lambda)}{(A - \lambda)^{-1}} = (\bar{A} - \lambda)^{-1}.$$

Def 5.1' Let $A: \mathcal{H} \rightarrow \mathcal{H}$ be closed $\lambda \in \mathbb{C}$

$\lambda \in \mathfrak{F}(A)$ if and only if $(A - \lambda)$ is bijective

bdd/ $(A - \lambda)^{-1}$ is bdd.

In particular $(A - \lambda)^{-1}$ is bdd
closed graph thm

~~Let A be closed~~

$\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the spectrum of A .

① $\sigma_p(A)$ denotes the set of eigenvalues of A
 $\sigma_p(A) \ni \lambda \Leftrightarrow \begin{cases} f \text{ not } (A-\lambda)f \\ (\neq 0) \end{cases}$

$A - \lambda$ is not injective $\therefore \lambda \notin \sigma_p(A) \subset \sigma(A)$

② $\sigma_c(A) = \{ \lambda \in \mathbb{C} ; A - \lambda \text{ injective}$
 $\text{Ran}(A - \lambda) \text{ dense}$
 $(A - \lambda)^{-1} \text{ is unb'dd} \}$

③ $\sigma_r(A) = \{ \lambda \in \mathbb{C} ; \begin{cases} A - \lambda \text{ injective} \\ \text{Ran}(A - \lambda) \text{ is not dense} \end{cases} \}$

$$A - \lambda \Rightarrow \begin{cases} (1) \text{ not injective} & \lambda \in \sigma_p(A) \\ (2) \text{ injective} & \begin{cases} \text{Ran}(A - \lambda) \text{ not dense} & \lambda \in \sigma_r(A) \\ \text{Ran}(A - \lambda) \text{ dense} & \begin{cases} (A - \lambda)^{-1} \text{ unb'dd} & \lambda \in \sigma_c(A) \\ (A - \lambda)^{-1} \text{ b'dd} & \lambda \in \rho(A) \end{cases} \end{cases} \end{cases} \end{cases}$$

Example M_f is bdd $\Leftrightarrow f \in L^\infty$

, $M_x - \lambda \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$

Then $(M_x - \lambda) f = (\lambda - \lambda) f$ bijective

$(M_x - \lambda)^{-1} f = \frac{1}{\lambda - \lambda} f$ bdd

$\therefore \lambda \in \sigma(M_x)$

• $\lambda \in \mathbb{R}$

$M_x - \lambda$ injection $(M_x - \lambda) f = 0 \Rightarrow f = 0$

$\text{Ran}(M_x - \lambda) = D((M_x - \lambda)^{-1})$ dense

$\underline{\lambda = 0} \quad \sum_n |f_n|^2 < \infty \quad \forall f \in L^2 \quad \times$

$(M_x - \lambda)^{-1}$ is unbdd $\therefore \mathbb{R} \subset \sigma_c(M_x)$

• Example $\mathcal{H} = \ell^2, S : \ell^2 \rightarrow \ell^2$ shift by one

$$\begin{cases} (S_a)_n = a_{n-1} & n \geq 2 \\ (S_a)_1 = 0 \end{cases}$$

① $S_a = 0 \Leftrightarrow a = 0$ injection

② $(1, 0, 0, \dots) \notin \text{Ran } S \quad \therefore 0 \in \sigma_r(S)$

③ $\sigma_p(S) = \emptyset$

$$\sigma_r(S) = \{ \lambda \in \mathbb{C}; |\lambda| < 1 \}$$

$$\sigma_c(S) = \{ \lambda \in \mathbb{C}; |\lambda| \neq 1 \}$$

- Generalization -

(X, μ) σ -finite measure space

$F : X \rightarrow \mathbb{A}$ meas function
 μ -a.e. finite.

$\lambda \in \text{ess ran}(F) \Leftrightarrow \mu\{x; |F(x) - \lambda| < \varepsilon\} > 0$

$(\mathbb{R}^d, \lambda) \leftarrow$ Lebesgue meas.

$F : \mathbb{R}^d \rightarrow \mathbb{A}$ cont
Then ess. ran $\bar{F} = \overline{F(\mathbb{R}^d)}$.

Thm M_F : multiplication op in $L^2(X, d\mu)$.

- $\sigma(M_F) = \text{ess. ran}(F)$
- $\lambda \in \sigma_p(M_F) \Leftrightarrow \mu(F^{-1}(\{\lambda\})) > 0$
- $\mu(F^{-1}(\{\lambda\})) = 0 \forall \lambda \Rightarrow \sigma_p(M_F) = \emptyset$

Neumann's theorem

Prop Let A be bdd st $\|A\| < 1$

Then $(I - A)$ is bijective and

$(I - A)^{-1}$ is bdd and $(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$

$\therefore (B(\mathbb{Z}), \|-\|)$ is complete.

$$\sum \|A\|^n < \frac{1}{1 - \|A\|} < \infty$$

$$S = \sum_{n=0}^{\infty} A^n \in B(\mathbb{Z}).$$

$$S_N = \sum_{n=0}^N A^n.$$

$$(I - A)S_N \rightarrow (I - A)S$$

$$S_N(I - A) \rightarrow S(I - A)$$

$$(I - A)S_N = \sum_{n=0}^N A^n - \sum_{n=1}^{N+1} A^n$$

$$= I - A^{N+1} \rightarrow I.$$

Similarly $S_N(I - A) \rightarrow I$.

$$\text{Hence we have } S(I - A) = (I - A)S = I$$

$\therefore I - A$ has bdd inverse

$\therefore I - A$ bijective

~~YEAH BABY~~

A closed. Then
 Lemma 5.3 $S(A)$ is open. hence $G(A)$ is closed

$\because \lambda_0 \in S(A)$

$$\begin{aligned} A - \lambda &= A - \lambda_0 + (\lambda_0 - \lambda) \\ &= [I - \underline{(\lambda - \lambda_0)(A - \lambda_0)^{-1}}] (A - \lambda_0) \end{aligned}$$

$$\|K\| = |\lambda_0 - \lambda| \cdot \|(\lambda - \lambda_0)^{-1}\|$$

Let $\|(\lambda - \lambda_0)^{-1}\| = \frac{1}{\varepsilon}$. Then for $|\lambda_0 - \lambda| < \varepsilon$

$$\|(\lambda - \lambda_0)(A - \lambda_0)^{-1}\| < 1 \quad \& \text{ it follows that}$$

$I - K$ is bijection (and vector see $(I - K)^{-1} = \sum_{n=0}^{\infty} K^n$) $\therefore (I - K)f = 0 \therefore f = Kf.$ ✓

$\therefore A - \lambda = (I - K)(A - \lambda_0)$ bijection

& $(A - \lambda)^{-1} = (A - \lambda_0)^{-1}(I - K)^{-1}$ bdd

$$\therefore \lambda \in G(A)$$

Cor 5.4 A is bdd ~~not~~ Then

$$G(A) \subset \{ \lambda \in \mathbb{C}; |\lambda| \leq \|A\| \}$$

It is enough to show that

$$\therefore \sqrt{|\lambda|} \geq \|A\| \quad \lambda \in S(A).$$

$$\|A/\lambda\| < 1 \quad \therefore \left(1 - \frac{A}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{A}{\lambda}\right)^n$$

$\therefore A - \lambda = \lambda \left(\frac{A}{\lambda} - 1\right)$ is bijective and
 $(A - \lambda)^{-1}$ is bdd.

§6 Symmetric Op & Self-adjoint Op.

Def 6.1 $A: \mathcal{H} \rightarrow \mathcal{H}$ is called symmetric

$D(A)$ is dense and $A \subset A^*$

Example $\mathcal{H} = L^2(\mathbb{R})$, $+i\frac{d}{dx} = T$ $D(T) = C_0^\infty$

$$\begin{aligned} (Tf, g) &= \int \overline{Tf} \cdot g dx = i \int \frac{d}{dx} f \cdot g \\ g &\in C_0^\infty \\ &= -i \int g \cdot \frac{d}{dx} f \\ &= \int \bar{f} \cdot Tg \end{aligned}$$

$\therefore D(T^*) \supset C_0^\infty$ dense

$T \subset T^*$ T is symmetric op

T : closable \bar{T} , $(\bar{T})^* = T^*$

Lemma 6.2 Let A be sym. Then it is closable and \bar{A} is also sym.

$\therefore A \subset A^*$ $D(A)$ is dense. $D(A^*)$ is dense
Hence A is closable

~~$A \subset A^* \Rightarrow A \subset A^* \Rightarrow A \subset (\bar{A})^*$~~

$A \subset A^* \Rightarrow A^* \supset A^{**} = \bar{A} \Rightarrow (\bar{A})^* \supset \bar{A}$,

\therefore It is enough to show $\mathbb{C} \setminus \mathbb{R} \subset S(A)$.

Let $z \in \mathbb{C} \setminus \mathbb{R}$.

$$\|(A - z)\varphi\|^2 = \cancel{\|(\lambda - z)\varphi\|^2} = 2\operatorname{Re}(\langle A\varphi, z\varphi \rangle) + \|z\|\|\varphi\|^2.$$

$$z = x + iy$$

$$\begin{aligned} \|(A - z)\varphi\|^2 &= 2\operatorname{Re}(\langle (A - x)\varphi, iy\varphi \rangle) + y^2\|\varphi\|^2 \geq 0 \\ &\geq y^2\|\varphi\|^2 \quad (y \neq 0) \end{aligned}$$

$\therefore (A - z)\varphi = 0 \Rightarrow \varphi = 0 \therefore A - z$ is injective.

Moreover $\|(A - z)^{-1}\varphi\| \leq \frac{1}{y^2}\|\varphi\| \quad \forall \varphi \in D(A)$.
 \therefore b'dd.

$$\textcircled{*} - \mathcal{H} = \overline{\operatorname{Ker} T} \oplus \overline{\operatorname{Ran}(T^*)} \quad \text{if } T \text{ : closed op.}$$

In particular

$$\mathcal{H} = \overline{\operatorname{Ran}((A - z^*)^*)} = \overline{\operatorname{Ran}(A - z)}.$$

$\therefore \operatorname{Ran}(A - z)$ is dense. //

Lemma Let T be closed. Then $\operatorname{Ker} T$ is closed and $\textcircled{*}$ holds.

$\textcircled{1}$ let $f \in \operatorname{Ker} T$, $g \in D(T^*)$

$$0 = (Tf, g) = (f, T^*g) \quad \therefore \operatorname{Ker} T \perp \operatorname{Ran}(T^*)$$

$\textcircled{2} = \{ f + g ; f \in \operatorname{Ker} T, g \in \operatorname{Ran}(T^*) \}$ is dense

$\therefore h \in D^+ \Leftrightarrow (f + g, h) \quad \text{① } h \in \operatorname{Ker} T^+$

$$\text{② } (T^*g, h) = 0 \quad \bar{T} \cdot T$$

$$\overline{\operatorname{Ker} T \oplus \operatorname{Ran}(T^*)} = \operatorname{Ker} T \oplus \overline{\operatorname{Ran}(T^*)} = \mathcal{H} \quad \therefore (g, (T^*f)h) = 0 \quad \therefore Th = 0$$

$\therefore h \in \operatorname{Ker} T \quad \therefore h = 0$

$$G_r(A) = \emptyset \quad \text{or} \quad \Leftrightarrow \lambda \in \mathbb{R}$$

$$\mathcal{H} = \overline{\text{Ker}(A - \lambda)} \oplus \overline{\text{Ran}(A - \lambda)}$$

$\text{Ker}(A - \lambda) = \{0\} \Rightarrow \text{Ran}(A - \lambda)$ is dense
Hence $\lambda \notin G_r(A)$.

A self-adjoint

A symmetric op When is it self-adjoint?

~~Thm 6.6~~ let A be sym.

$$\begin{array}{c} \textcircled{1} \ A \text{ is self-adjoint} \Leftrightarrow \textcircled{2} \ A \text{ is closed and} \\ \textcircled{1} \Leftrightarrow \text{Ran}(A \pm i) = \mathcal{H} \end{array}$$

Thm 6.6 let A be a sym, closed op.

A is s.a. $\Leftrightarrow G(A) \subset \mathbb{R}$

(\Leftarrow) \Leftrightarrow OK

$\Leftarrow i \in S(A) \Leftrightarrow A \pm i$ is bijective

$$\therefore \text{Ran}(A \pm i) = \mathcal{H}$$

$$\overline{D(A)} \subset \overline{D(A^*)}$$

Since $A \subset A^*$, it is enough to show $D(A) \supset D(A^*)$

$$\text{let } f \in D(A^*) \quad (A^* - i)f = (A - i)g, \ g \in D(A)$$

$$\therefore (A^* - i)(f - g) = 0 \quad \text{But} \quad \overline{\text{Ker}(A^* - i)} \oplus \overline{\text{Ran}(A^* - i)} = \mathcal{H}$$

$$\therefore f \in D(A),$$

$$\text{Then } \text{Ker}(A^* - i) = 0 \quad \therefore f = g$$

A symmetric op.

Def 6.7 let A be symmetric.

A is essentially self-adjoint if and only if \bar{A} is self-adjoint.

Lemma 6.8 let A be ess. s.a. Then

\sim A has only one self-adjoint extension \bar{A} .

$\therefore A \subset B$ self-adjoint

$$\therefore \bar{A} \subset B \Rightarrow B^* \subset (\bar{A})^* = \bar{A} \subset B \therefore \bar{A} = B,$$

\Downarrow
 B

~~(A is a D.A.)~~ ~~is sym.~~
 ~~A is also sym.~~ ~~\bar{A} is ess. s.a.~~
~~and D is a core of A .~~

\sim From the proof of Thm 6.6, we can see that for symmetric A ,

$$\textcircled{Q} \quad \text{Ran}(A \pm i) = \mathcal{H} \Leftrightarrow A \text{ is self-adjoint}$$

Lemma 6.9 Let A be sym.

$$\overline{\text{Ran}(A \pm i)} = \mathcal{H} \Leftrightarrow A \text{ is ess. s. a.}$$

$$\textcircled{P} \quad (\Leftarrow) \quad \overline{\text{Ran}(A \pm i)} = \overline{\text{Ran}(\bar{A} \pm i)} = \mathcal{H}$$

Then \bar{A} is s.a.

$$(\Leftarrow) \quad \overline{\text{Ran}(\bar{A} \pm i)} = \overline{\overline{\text{Ran}(A \pm i)}} = \mathcal{H} \quad //$$

Ex. $F : \mathbb{R}^d \rightarrow \mathbb{R}$ $F \in L_{loc}^2(\mathbb{R}^d)$
 i.e. $\int_{\mathbb{R}^d} |F|^2 dx < \infty$

$D(M_F) \subset C^\infty(\mathbb{R}^d)$

let $g \in \text{Ran}(M_F|_{C_0} + i)^{\perp}$

Hence $(f, (M+i)g f) = 0$

$$\int \overline{(F+i)g} \cdot f = 0 \quad \forall f \in C_0$$

$$\therefore (F+i)g = 0 \quad \therefore g = 0$$

$$\therefore \text{Ran}(M+i) = 0 \quad \therefore M \text{ is ess. s.a.}$$

Ex. $AC^2[0,1] = \{ f : [0,1] \text{ absolutely cont} ; \int_{[0,1]} |f|^2 < \infty \}$

$$A_\alpha f = -i \frac{d}{dx} f \quad f \in AC^2$$

$$D(A_\alpha) = \{ f \in AC^2 ; f(0) = \alpha f(1) \} \quad \alpha \in \mathbb{C} \quad |\alpha| = 1$$

A_α self-adjoint

$$G(A_\alpha) = \{ 2\pi n - \theta_\alpha \}_{n \in \mathbb{Z}}$$

$$\alpha = e^{i\theta_\alpha}$$

Lemma 6.10 let A be symmetric $\textcircled{1}$ - $\textcircled{3}$ are equivalent.

$\textcircled{1}$ A is ess. s.g.

$\textcircled{2}$ $\ker(A^* \pm i) = \{0\}$

$\textcircled{3}$ $\overline{\text{Ran}(A \pm i)}$ is dense

$\therefore \textcircled{1} \rightarrow \textcircled{2}$ \bar{A} s.g. $\therefore \ker((\bar{A})^* \pm i) = \{0\}$
 $\ker(A^* \pm i)$

$\textcircled{2} \rightarrow \textcircled{3}$ $\mathcal{H} = \overline{\ker(A^* \pm i)} \oplus \overline{\text{Ran}(A^* \mp i)}$
 $\therefore \overline{\text{Ran}(\bar{A} \mp i)} = \mathcal{H}$
 $\therefore \overline{\text{Ran}(A \mp i)} = \mathcal{H}$

~~$\text{Ran}(\bar{A} \mp i) \supset \text{Ran}(A \mp i)$~~
 ~~$\therefore \text{Ran}(\bar{A} \mp i) = \text{Ran}(A \mp i)$~~
 ~~$\therefore \text{Ran}(A \mp i) = \overline{\text{Ran}(A \mp i)}$~~

General theory let T be closable.

The $\overline{\text{Ran}(T)} = \overline{\text{Ran}(\bar{T})}$

$\overline{\text{Ran}(T)} \subset \overline{\text{Ran}(\bar{T})}$ ok $\exists g_n \in D(T)$ s.t. $g_n \rightarrow f$

let $f \in \text{Ran}(\bar{T})$. $\exists g_n \in D(T)$ s.t. $g_n \rightarrow f$
 $Tg_n \rightarrow \bar{T}g_n = f$
 $\therefore f \in \overline{\text{Ran}(T)}$ (by definition)
 $\therefore \overline{\text{Ran}(\bar{T})} \subset \overline{\text{Ran}(T)}$,

$\textcircled{3} \rightarrow \textcircled{1}$ $\overline{\text{Ran}(A \pm i)} = \mathcal{H}$
 $\overline{\text{Ran}(\bar{A} \pm i)} \therefore \bar{A}$ is s.g.

• Let A be closed

$$\overline{A|_D} = A \Leftrightarrow D \text{ is a core of } A.$$

• let A be symmetric. $A|_D$ is also symmetric
(D ~~is dense~~ is dense)

$$\overline{A|_D} \text{ is s.a.} \Leftrightarrow A \text{ is ess. s.a. on } D.$$

Ex. $F: \mathbb{R}^d \rightarrow \mathbb{R}$ ~~such that~~ $F \in L^2_{loc}(\mathbb{R}^d)$

$$\text{i.e. } \int_{\mathbb{R}^d} |F|^2 < \infty$$

$$D(M_F) \supset C_0^\infty(\mathbb{R}^d)$$

$$M_F|_{C_0^\infty(\mathbb{R}^d)} = M.$$

$$\frac{1}{\text{Ran}(M+i)} = \mathcal{Z} \quad \therefore \text{let } g \in \text{Ran}(M+i)^\perp$$
$$(g, (M+i)f) = 0 \quad \forall f \in D(M)$$

$$\int \overline{(F+i)g} \cdot f = 0 \quad \forall f \in C_0^\infty$$

$$\therefore (F+i)g = 0 \text{ a.e.}$$

$$\therefore g = 0$$

$\therefore M$ is ess. on $C_0^\infty(\mathbb{R}^d)$. $(\text{de Bois - Raymond})$
 Lemma

Cor: $M_F|_S$ is also ess. s.a.

Cor: $M_\alpha|_{C_0^\infty}, M_\alpha|_S$ is ess. s.a.

$$\text{Ex 2. } Uf(k) = (2\pi)^{-\frac{1}{2}} \int f(n) e^{-ik \cdot n} dn$$

$$U: \mathcal{F} \rightarrow \mathcal{F} = \{ f \in C^\infty; \sup_{|x|} |f^{(\beta)}_{(x)}| < \infty \}$$

$$(Uf, Ug) = (f, g), \quad \forall f, g \in \mathcal{F}$$

$$\text{In particular } \|Uf\|^2 = \|f\|_h^2 \quad \text{b'dd}$$

$$\exists \bar{U}: L^2 \rightarrow L^2 \text{ st } \|\bar{U}f\| = \|f\| \quad \forall f \in L^2$$

- \bar{U} is called the Fourier transf. on L^2 -

$$U(-i \frac{d}{dx} f) = k Uf. \quad \cancel{\text{if } f \in D(U)}$$

$$U \bar{U} = I \quad \text{or} \quad P = \bar{U} U \quad \text{on } \mathcal{F}$$

- $\bar{U} M_k U = P$
- $D(P) = \{ Uf; Uf \in D(M_k) \}$

Lemma P is self-adjoint, and ess. s.a on \mathcal{F}

$$\text{Let } (\text{let}) \quad (P^* \pm i)g = q$$

$$\therefore (f \cdot (\bar{U} M_k U)^* g) = \pm i (f \cdot g)$$

$$\therefore (\bar{U} M_k U f, g) = \pm i (f, g)$$

$$\therefore (M_k U f, g) = \pm i (U f, g)$$

$$\therefore U g \in \text{Ker}(M_k^* \pm i) \therefore g = 0$$

Let $Pf_n \rightarrow g$, $f_n \rightarrow f$

Then $\bar{U}' M_h U f_n \rightarrow g$ $U f_n \rightarrow U f$

$\therefore M_h U f_n \rightarrow U g$ $\therefore U f \in D(M_h)$

$$\Rightarrow M_h U f = U g$$

$$\therefore \bar{U}' M_h U f = g$$

$$\therefore Pf = g$$

Hence P is closed

$\therefore P$ is self-adjoint.

$P|_{\mathcal{S}} = \tilde{P}$ we can ~~only~~ also show that

$$\ker(\tilde{P}^* + i) = 0$$

$\therefore \tilde{P}$ is ess. s.a.

$$i) (f \cdot (\tilde{P} \pm i)^* g) = 0 \quad \forall f \in \mathcal{S}$$

$$\therefore (\tilde{P}f, g) = \pm i(f, g) \Rightarrow$$

$$(M_h U f, U g) = \pm i(U f, U g) \quad U f \in \mathcal{J}$$

$$M_h \mathcal{J} = M$$

$$\therefore (M F, U g) = \pm i(F, U g) \quad \forall F \in \mathcal{S}$$

$$\therefore U g = 0 \Rightarrow g = 0 \quad \text{since } \ker(M^* + i) = 0$$

1

i) P is called the generalized differential op.

$$= \frac{d}{dx}$$

$\exists x \quad \mathcal{H} = L^2([0,1])$. $P = -i\frac{d}{dx}$, $D(P) = \bigcup_{n=1}^{\infty} C_c^{\infty}([0,1])$

$$e^x \in L^2$$

$$(e^x, Pf) = (-ie^x, f)$$

$$\therefore e^x \in D(P^*) = D((\bar{P})^*) \quad \& \quad (\bar{P})^* e^x = -ie^x$$

let \bar{p} be s.a. Hence $(\bar{P})^* = \bar{p}$ and

$$\bar{p} e^x = -ie^x \quad -i \in \sigma(\bar{p}) \quad \text{contradiction.} //$$

$\exists x \quad \mathcal{H} = L^2([0,1]), \quad P = i\frac{d}{dx}, \quad D(P) = AC^2[0,1]$.

Ex 3. $AC^2[0,1] = \{ f : [0,1] \rightarrow \mathbb{C} ; \int_{[0,1]} |f|^2 dx < \infty \}$
 abs. cont.

f abs. cont. on $[0,1]$

$$\Leftrightarrow f(x) = \int_0^x u(u) du + C$$

f abs. cont. \Rightarrow ① cont

② f is differentiable a.e. $x \in [0,1]$

$$f'(x) = u(x) \text{ a.e. } x \in [0,1]$$

• $A_\alpha = -i \frac{d}{dx}, \quad \alpha \in \mathbb{C}$

• $D(A_\alpha) = \{ f \in AC^2[0,1] ; f(0) = \underline{\alpha} f(1) \}$
 $T = T \subset \mathbb{C} \quad \alpha \in \mathbb{C} \rightarrow |\alpha| = 1.$

Lemma A_α is closed and symmetric.

(Symmetric)

$$(f, A_\alpha g) = (-i) \int \bar{f} \frac{d}{dx} g = -i \left[\bar{f} g \right]_0^1 + i \int \overline{\frac{df}{dx}} f g$$

$$(f, g \in D(A_\alpha)) = -i \left[\bar{f} g \Big|_0^1 - \bar{f}(1)g(1) \right] + i \int \bar{f}' g$$

$$= \int \overline{-i \frac{df}{dx}} g.$$

(closedness) Suppose that $A_\alpha f_n \rightarrow g$, $f_n \rightarrow f$.
 we shall prove that $f \in D(A_\alpha)$ and $g = A_\alpha f$.

$$f_n(x) = \int_0^x f_n'(m) dm = i \langle x, A_\alpha f_n \rangle + f_n(0)$$

$$\therefore \lim_n f_n(x) - f_n(0) = i \langle x, g \rangle. \quad \text{--- (4)}$$

\therefore (4) is true in the sense of L^2 .

$$\begin{aligned} \therefore f_n(0) &= f_n(x) - f_n(x) + f_n(0) \\ &\rightarrow i \langle x, g \rangle + f \quad \text{in } L^2 \end{aligned}$$

$$\therefore \exists_{n_j} f_{n_j}(0) \rightarrow i \langle x, g \rangle + f \quad \text{a.e.} \\ = c \quad \text{constant.}$$

$$\therefore f(x) = -i \int_0^x g dm + c \quad \therefore f \in AC^2[0,1] \\ (c = f(0)) \quad \text{and } g = A_\alpha f.$$

Similarly we can see that $f_n(1) - f_n(0) \rightarrow i \langle x_1, g \rangle$ in L^2

$$\therefore f_{n_j}(1) \rightarrow c + i \langle x_1, g \rangle$$

$$\therefore \alpha(c + i \langle x_1, g \rangle) = c \quad \therefore \alpha f(1) = f(0) \\ \therefore f \in D(A_\alpha).$$

$$-iy' = \lambda y \quad \therefore y' = i\lambda y \quad \therefore y = e^{i\lambda x} c$$

$$\text{boundary condition: } \alpha e^{i\lambda c} = c \quad \therefore \alpha e^{i\lambda} = 1$$

$$\therefore \alpha = e^{i\theta_\alpha} \quad \therefore \quad \lambda + \theta_\alpha = 2\pi n \quad \therefore \lambda = 2\pi n - \theta_\alpha$$

$$\mathcal{G}_p(A_\alpha) \subset \{2\pi n - \theta_\alpha\}_{n \in \mathbb{Z}}$$

$\varphi_n(x) = e^{2\pi i n x} e^{i\theta_\alpha x}$ is e.v.

$$\therefore \mathcal{G}_p(A_\alpha) = \{2\pi n - \theta_\alpha\}_{n \in \mathbb{Z}}$$

$$\left\{ \varphi_n \right\} \text{ CONS} \quad \therefore \text{Ran}(A_\alpha + i)^+ = \text{hol.} \quad //$$

$\therefore)$

A symmetric $\Rightarrow A$ is closable $\Rightarrow \overline{A}$ sym. closed.

~~A has sym~~ Every sym op can be extended to the sym. closed op.

Def 6.3 A densely defined op A in \mathcal{H}

is self-adjoint $\Leftrightarrow A^* = A$ i.e. $D(A^*) = D(A)$ and $A^* f = Af$

Ex. $f: \mathbb{R}$ -valued meas. function a.e.-finite

M_f is self-adjoint. $\therefore M_f^* = M_{\bar{f}} = M_f$.

Prop 6.4 Let A be s.a. ^{and B be s.a.} ~~and~~ $A \subset B$. ~~not closed~~.
Then $A = B$.

$\because A \subset B \Rightarrow B \supset A^* \supset B^* \supset B \supset A$
 $\therefore A = B = B^*$

The spectrum of self-adjoint operator

Theorem 6.5 Let A be self-adjoint.

① $\sigma(A) \subset \mathbb{R}$

② $\sigma_r(A) = \emptyset$

§7 Spectral Theory (I) Stieltjes integral

- (A): $\mathbb{C}^n \rightarrow \mathbb{C}^n$ linear map s.t. $\overline{\lambda}^* = \lambda^*$
- $\exists U : \mathbb{C}^n \rightarrow \mathbb{C}^n$ unitary s.t. $U^* (\lambda) U = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$
- $\therefore A = U \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^*$ $\uparrow j \in \mathbb{N}$
- $= \sum_{j=1}^n \lambda_j P_j$ $P_j^2 = P_j$, $P_j^* = P_j$ projection
- $A : \mathcal{H} \rightarrow \mathcal{H}$ s.a. $A = \int_{\sigma(A)} \lambda dE(\lambda)$

Review of Stieltjes integral

- $f : [a, b] \rightarrow \mathbb{C}$
- $\pi : a = \lambda_0 < \lambda_1 < \dots < \lambda_n = b$: division of $[a, b]$
- $S_\pi = \sum_{k=1}^n |f(\lambda_k) - f(\lambda_{k-1})|$
- $\sup_{\pi} S_\pi \leq M \Leftrightarrow f$: b'dd variation.
- Infimum of M s.t. $\sup_{\pi} S_\pi \leq M$ is called the total variation of f , which is denoted by $V[f]$.

Suppose that $f : [a, b] \rightarrow \mathbb{C}$ is b'dd variation and right-cont i.e. $f(\lambda+0) = f(\lambda) \quad \forall \lambda \in [a, b]$. Then

$$\lim_{|\pi| \rightarrow 0} \sum_{k=1}^n u(\lambda_k) (f(\lambda_k) - f(\lambda_{k-1})) = \int_a^b u(\lambda) df(\lambda)$$

for a cont function u on $[a, b]$ and

$$\left| \int_a^b u df \right| \leq \sup_{\lambda} |u(\lambda)| V[f] \quad \text{Stieltjes integral}$$

Let u, v cont, f b'dd variation and right cont.

$$g(\mu) = \int_a^\mu u df \text{ is also b'dd variate and right-cont}$$

$$\int \nu dg = \int \nu u df.$$

Def 7.1 Let \mathcal{H} be a Hilbert space.

$\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ is called spectral family or resolution of identity if and only if it satisfies

- ① $E(\lambda)$ is projection $E(\lambda)^* = E(\lambda)$, $E(\lambda)^{\dagger} = E(\lambda)$
- ② $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\})$
- ③ $\lim_{\lambda \rightarrow +\infty} E(\lambda) = I$, $\lim_{\lambda \rightarrow -\infty} E(\lambda) = 0$
- ④ $E(\lambda+0) = E(\lambda)$ strongly.

Ex $\mathcal{H} = L^2(\mathbb{R})$, $E(\lambda) := M_{X_{(-\infty, \lambda]}}$

Ex $\mathcal{H} = L^2(\mathbb{R}^d)$ $F: \mathbb{R}^d \rightarrow \mathbb{R}$ cont

$$E(\lambda) = M_{X_{F^{-1}((-\infty, \lambda])}}$$

Lemma 7.2 Let $\varphi, \psi \in \mathcal{H}$.

$f: [a, b] \ni \lambda \mapsto (\varphi, E(\lambda)\psi)$ is b'd d variation and right-cont, and $V[f] \leq \|\varphi\| \cdot \|\psi\|$.

(Remark) $E(\lambda) \leq E(\mu)$ if $\lambda \leq \mu$

$E(\lambda) - E(\mu) = \Delta$ is projection $\Delta^* = \Delta$

$$(E(\lambda) - E(\mu))(E(\lambda) - E(\mu))$$

$$\Delta^2 = \Delta$$

$$= E(\lambda) + E(\mu) - E(\mu) - E(\mu) = E(\lambda) - E(\mu)$$

—

Prop (Riesz's representation theorem)

Let $F \in \mathcal{L}^*$. Then $\exists \bar{\Phi}_F \in \mathcal{L}$ s.t $F(\bar{\Phi}) = (\bar{\Phi}_F, \bar{\Phi})$ and $\|F\| = \|\bar{\Phi}_F\|$.

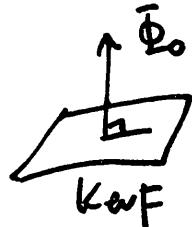
\because ① $\ker F = \mathcal{L}$ or $\{0\}$

$$F(\bar{\Phi}) = 0 = (0, \bar{\Phi}) \quad \therefore \bar{\Phi}_F = 0$$

~ ② $\ker \bar{F} \neq \mathcal{L}$ ($\ker F$ closed)

$$(\ker F)^+ \xrightarrow{\exists} \bar{\Phi}_0 (\neq 0)$$

$$\text{Let } \bar{\Phi}_F = \overline{F(\bar{\Phi}_0)} \bar{\Phi}_0 / \|\bar{\Phi}_0\|^2$$



~~PROOF~~

(i) $\bar{\Phi} \in \ker F$

$$F(\bar{\Phi}) = 0 = (\bar{\Phi}_F, \bar{\Phi}) \quad \text{ok}$$

(ii) $\bar{\Phi} \in \perp \bar{\Phi}_0$

$$F(\perp \bar{\Phi}_0) = \alpha F(\bar{\Phi}_0) = (\bar{\Phi}_F, \perp \bar{\Phi}_0)$$

$$(iii) \bar{\Phi} \in \mathcal{L} \Rightarrow \bar{\Phi} = \underbrace{\bar{\Phi}_1}_{\in \ker F} + \underbrace{\bar{\Phi}_2}_{\perp \bar{\Phi}_0} \quad \bar{\Phi} - \underbrace{\frac{F(\bar{\Phi})}{F(\bar{\Phi}_0)} \bar{\Phi}_0}_{\in \ker F} = \underbrace{\frac{F(\bar{\Phi})}{F(\bar{\Phi}_0)} \bar{\Phi}_0}_{\perp \bar{\Phi}_0}$$

we can identify

Hence $F \cong \bar{\Phi}_F$. Then $\mathcal{L}^* \cong \mathcal{L}$.

Lemma 7.3 Let $f \in C(1/2)$

Then \exists densely defined linear map $A(f)$ s.t.

$$D(A(f)) = \{ \varphi + f\zeta; \int |f|^2 d(\varphi, E(\lambda)\varphi) < \infty \}$$

$$(\varphi, A(f)\varphi) = \int f d(\varphi, E(\lambda)\varphi).$$

Furthermore it holds that

$$\textcircled{1} \quad \|A(f)\varphi\|^2 = \int |f|^2 d(\varphi, E(\lambda)\varphi)$$

$$\textcircled{2} \quad E(\lambda)A(f) \subset A(f)E(\lambda)$$

$$\textcircled{3} \quad A(f)^* = A(f^*) \text{ and } A(f) \text{ is closed}\\ \text{in particular } f \text{ is real } \Rightarrow A(f) \text{ is s.o.}$$

$$\textcircled{4} \quad f \text{ is bdd} \Rightarrow A(f) \text{ is bdd and } \|A(f)\| \leq \|f\|_{L^\infty}$$

~~$\textcircled{5} \quad \text{if } f \in L^2 \Rightarrow A(f) \text{ is closable.}$~~

$\therefore \textcircled{6} \quad D(A(f)) \text{ is dense. } \therefore \varphi \in \mathcal{H} \quad E_n \varphi = \varphi_n \rightarrow \varphi.$

$$\int |f|^2 d(\varphi_n, E(\lambda)\varphi_n) = \int_{-n}^n |f|^2 d(\varphi, E(\lambda)\varphi) \leq \sup_{-n \leq \lambda \leq n} \|f\|^2_{L^2} \|\varphi\|^2 < \infty.$$

$$\therefore (\varphi_n, E(\lambda)\varphi_n) = \begin{cases} 0 & \lambda < -n \\ (\varphi, \underbrace{E(n) - E(-n)}_{\|\varphi_n\|^2} \varphi) & -n \leq \lambda \leq n \\ \|\varphi_n\|^2 & \lambda > n \end{cases} \quad E_j = E(\lambda_j) - E(\lambda_{j-1})$$

Existence:

$$\textcircled{6}' \quad \varphi \in D(A(f)) \Rightarrow |(\varphi, A(f)\varphi)| < \infty$$

$$|\int f d(\varphi, E(\lambda)\varphi)| \stackrel{\text{1.1.1}}{\leq} \sum_{j=1}^n |f(\lambda_j)| |\varphi, E_j \varphi|$$

$$\leq \left(\sum_j |f(\lambda_j)|^2 \|E_j \varphi\|^2 \right)^{1/2} \left(\sum_j \|E_j \varphi\|^2 \right)^{1/2}$$

$$= \left[\int |f|^2 d(\varphi, E(\lambda)\varphi) \right]^{1/2} \|\varphi\| < \infty$$

$$\mathcal{H} \ni \varphi \mapsto \int f d(\varphi, E(\lambda)\varphi) \in \mathbb{C} \quad \text{cont. func.} \quad \exists K_f \text{ s.t.} \\ A(f): \varphi \mapsto \varphi_f \quad \text{def. by } (\varphi, \varphi_f)_E$$

(6)" $D(A(f))$ is linear subspace

$$\therefore \varphi_1, \varphi_2 \in D(A(\ell)) \quad \alpha \varphi_1 + \beta \varphi_2 = \varphi$$

$$(\varphi, E(\lambda) \varphi) \leq 2|\alpha|^2 \|E(\lambda)\varphi_1\|^2 + 2|\beta|^2 \|E(\lambda)\varphi_2\|^2$$

$$\therefore \int |f|^2 d(\varphi, E(\lambda)\varphi) \leq 2|\alpha|^2 \int |f|^2 d(\varphi_1, E(\lambda)\varphi_1) \\ + 2|\beta|^2 \int |f|^2 d(\varphi_2, E(\lambda)\varphi_2) < \infty.$$

- (1) $\|A(f)\varphi\|^2 = (\underline{A(f)}\varphi, A(f)\varphi) = \int f d(A(f)\varphi, E(\lambda)\varphi)$

$$(A(f)\varphi, E(\lambda)\varphi) = (E(\lambda)\varphi, A(f)\varphi)^* = \left[\int f d(E(\lambda)\varphi, E(\mu)\varphi) \right]^* \\ = \int_{-\infty}^{\lambda} f^* d(\varphi, E(\mu)\varphi)$$

$$\therefore \|A(f)\varphi\|^2 = \int |f|^2 d(\varphi, E(\lambda)\varphi).$$

(2) $\int |f|^2 d(E(\lambda)\varphi, E(\mu)E(\lambda)\varphi) = \int_{-\infty}^{\lambda} |f|^2 d(\varphi, E(\mu)\varphi) < \infty$

$$\therefore E(\lambda)\varphi \in D(A(f)) \quad \text{sic}$$

$$(\phi, E(\lambda)A(f)\varphi) = \int f d(E(\lambda)\phi, E(\mu)\varphi) \\ = (\phi, A(f)E(\lambda)\varphi).$$

$$\therefore E(\lambda)A(f) = A(f)E(\lambda) \subset D(A(f)).$$

$$\textcircled{3} \quad (A(f)\varphi, \varphi) = (\varphi, A(f)\varphi)^* = \int f^* d\langle \varphi, E(\lambda)\varphi \rangle \stackrel{(\varphi, E(\lambda)\varphi)}{=} (\varphi, A(f^*)\varphi)$$

$$\therefore A(f^*) \subset A(f)^*$$

We shall show that $A(f^*) \supset A(f)^*$ i.e. $D(A(f))^* \subset D(A(f^*))$
 $\varphi \in D(A(f)^*)$

$$\Leftrightarrow \|A(f)^*\varphi\| = \lim_n \|E_n A(f)^* \varphi\|$$

$$= \lim \sup |(\varphi, E_n A(f)^* \varphi)|$$

$$= \lim \sup |(A(f) E_n \varphi, \varphi)|$$

$$= \lim \sup |(E_n A(f) E_n \varphi, \varphi)|$$

$$= \lim \sup |(\varphi, A(f^*) E_n \varphi)|$$

$$= \lim \|A(f^*) E_n \varphi\|$$

$$= \lim \left[\int_{-n}^n |f|^2 d(\varphi, E(\lambda)\varphi) \right]^{1/2} < \infty$$

$$\begin{aligned} E_n \varphi &\in D(A(\lambda)) \\ E_n A(f) &= A(f) E_n \\ &\sim D(A(\lambda)) \end{aligned}$$

$$\exists \frac{1}{k} \in C_0 \cap D(A(\lambda))$$

$$A(f) = A(f^*)^* \leftarrow \text{closed}$$

$$A(f)^* = A(f^*) = A(f) \leftarrow \text{s.g.}$$

$$\textcircled{4} \quad \|A(f)\varphi\|^2 = \int |f|^2 d(\varphi, E(\lambda)\varphi) \leq \|f\|_\omega^2 \|\varphi\|^2.$$

$$A(f) \text{ is denoted by } A(f) = \int f dE(\lambda)$$

In particular $A = \int \lambda dE(\lambda)$ is called a s.a. op associated with $\{E(\lambda)\}$.

Example $F \in C(\mathbb{R})$ 1:1.

$$E(\lambda) = M \chi_{F^{-1}(-\infty, \lambda]} \quad \text{spectral family.}$$

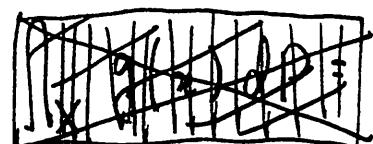
The self-adjoint operator associated with $E(\lambda)$ is M_F .

Let A be a.s.a. QSS. with $\{E(\lambda)\}$. We shall show that $A = M_F$.
 $\therefore \underline{\text{STEP 1}} \quad D(M_F) = D(A)$

$$\begin{aligned} (\varphi, E(\lambda)\varphi) &= \int |\varphi(x)|^2 \chi_{(-\infty, \lambda]}(F(x)) dx \\ &= \int |\varphi(F^{-1}F(x))|^2 \chi_{(-\infty, \lambda]}(F(x)) dm \quad \downarrow \text{Lebesgue meas.} \\ &= \int |\varphi(F^{-1}(y))|^2 \chi_{(-\infty, \lambda]}(y) dP(y), \text{ where } P(B) = |\{x; F(x) \in B\}| \end{aligned}$$

In general $(X, \mathcal{B}_X, P) \xrightarrow{f} (Y, \mathcal{B}_Y)$

$(P \circ f^{-1})(A) = P(f^{-1}(A)) = P_f(A)$ is meas. on (Y, \mathcal{B}_Y, P_f)



$$\int_Y 1_A(y) dP_f = P_f(A) = \int_X 1_A(f(x)) dP$$

$$\Rightarrow \int_Y g(y) dP_f = \int_X g \circ f^{-1}(x) dP$$

$$= \int_{-\infty}^{\lambda} |\varphi \circ F^{-1}(y)|^2 dP(y) \quad \text{is abs cont} \quad \because | \cdot |^2 \in L^1(dP)$$

$$\begin{aligned} \therefore \int |\lambda|^2 d(\varphi, E(\lambda)\varphi) &= \int |\lambda|^2 |\varphi \circ F^{-1}(\lambda)|^2 dP(\lambda) \\ &= \int |F(\lambda)|^2 |\varphi(\lambda)|^2 d\lambda < \infty, \end{aligned}$$

STEP 2 $A = MF$

$$(\varphi, A\varphi) = \int \lambda d(\varphi, E(\lambda)\varphi)$$

$$\begin{aligned} (\varphi, E(\lambda)\varphi) &= \int_{-\infty}^{\infty} \lambda \overline{\varphi(F(y))} \varphi(F(y)) dP(y) \\ &= \int F(\lambda) \overline{\varphi(\lambda)} \varphi(\lambda) d\lambda \quad // \end{aligned}$$

Theorem 7.4 (von Neumann) Spectral theorem-
Suppose that A is a s.a. on \mathcal{H} . Then $\{E(\lambda)\}$
spectral family such that

$$A = \int \lambda dE(\lambda) \quad \text{i.e.,}$$

$$(\varphi, A\varphi) = \int \lambda d(\varphi, E(\lambda)\varphi),$$

$$D(A) = \{ \varphi \in \mathcal{H} \mid \int |\lambda|^2 d(\varphi, E(\lambda)\varphi) < \infty \}$$

§8 Spectral theory II (spectral meas.)

We shall show only outline without proofs.

$\mathcal{B}(\mathbb{R}^d)$: Borel σ -field on \mathbb{R}^d

$\mathcal{P}(\mathbb{R})$: The family of projections on \mathbb{R} .

Def 8.1 $E(\cdot) : \mathcal{B}(\mathbb{R}^d) \rightarrow \mathcal{P}(\mathbb{R})$ is called spectral meas. or projection-valued meas. if and only if

$$\textcircled{1} E(\emptyset) = 0, \quad E(\mathbb{R}^d) = \mathbb{1}$$

$$\textcircled{2} E\left(\bigcup_{j=1}^{\infty} B_j\right) = \text{s-lim } \sum_j E(B_j) \quad \text{when } B_i \cap B_j = \emptyset$$

$$\textcircled{3} E(A \cap B) = E(A) \cdot E(B) \quad (i \neq j)$$

Ex. $\mathcal{H} = L^2(\mathbb{R}^d)$, $E(B) f = \chi_B(x) f \quad B \in \mathcal{B}(\mathbb{R}^d)$

Rem $(\varphi, E(\cdot)\varphi) = P(\cdot)$ $\stackrel{\mathbb{C}}{\in}$ is signed meas..

$$\therefore B_i \cap B_j \Rightarrow P\left(\bigcup_{j=1}^{\infty} B_j\right) = \lim_{\downarrow} \sum_j P(B_j)$$

Lemma 8.2 Let $\{E(B); B \in \mathcal{B}(\mathbb{R}^d)\}$ be spectral meas.

Then $E(\lambda) = E((-\infty, \lambda])$ is the spectral family.

and

$$\int f(\lambda) d(\varphi, E(\lambda)\varphi) = \int f(\lambda) dP_{\varphi\varphi}(\lambda).$$

for contf.

$\exists \delta$ $\text{Supp } E := \mathbb{R}$

$E(a+\delta) = E(a-\delta) \Leftrightarrow a$ is called ^{as} point of constancy

$A = \{a \in \mathbb{R} : a \text{ is point of constancy}\}$ open set

$A^c \subset \mathbb{R} \setminus A^c = \text{Supp } E$ closed

- Carathéodory's extension theorem -

(X, \mathcal{F}, μ) \mathcal{F} : finitely additive set family.
 μ : Jordan meas.

Suppose that ① σ -finite
 ② $\bigcup_{j=1}^{\infty} B_j \in \mathcal{F} \rightarrow \sum_{j=1}^{\infty} \mu(B_j)$

Then $(X, \mathcal{G}(\mathcal{F}), \bar{\mu})$ st. $\bar{\mu} = \mu$ on \mathcal{F} .

Prop. 8.3 \mathcal{F} : The family of disjoint unions of rectangles in \mathbb{R}^d . i.e., $\mathcal{F} \ni A = \bigcup_{j=1}^n I_j$
 $I_j = [a_j^1, b_j^1] \times \dots \times [a_j^d, b_j^d]$

$E_0: \mathcal{F} \rightarrow P(\mathbb{R})$ st. ①-③

$\Rightarrow \exists^1$ Spectral meas. E st. $E = E_0$ on \mathcal{F}
 in $G(\mathcal{F}) = B(\mathbb{R}^d)$

Ex. Let $\{E(\omega)\}$ be spectral family.

$(a, b] \rightarrow E(b) - E(a) \in P(\mathbb{R})$

$= E_0((a, b])$, which satisfies ①-③.

$\exists^1 E: B(\mathbb{R}^d) \rightarrow P(\mathbb{R})$ st. $E = E_0$ on \mathcal{F} .

In particular $E(\lambda) = E((-\infty, \lambda])$

② $\{E(\omega)\}_{\lambda}$ spectral family $\Leftrightarrow \{E(B): B \in B(\mathbb{R}^d)\}$

Thm 8.4 $E : \mathcal{B}(\mathbb{R}^1) \rightarrow \mathcal{P}(\mathbb{R})$

f : Borel meas on \mathbb{R}^1 . $\exists T_f$ s.t. $\xrightarrow{\text{meas}}$

$$D(T_f) = \left\{ \varphi \in \mathcal{H}; \int |f|^2 d(E(\omega)\varphi) \right\}$$

$$(T_f \varphi) = \int f(x) d(E(x)\varphi)$$

① $E(B) T_f \subset T_f E(B)$

② $f \in L^\infty \Rightarrow T_f \text{ bdd}, \|T_f\| \leq \|f\|_\infty$

③ $T_f^* \subset (T_f)^*$

④ $E(\{\lambda; |f(\omega)| = \infty\}) = 0 \Rightarrow T_f^* = (T_f)^*$

In particular if f is real, $E(\{\lambda; |f(\omega)| = \infty\}) = 0$

$\Rightarrow T_f$ is self-adjoint.

⑤ $T_g T_f \subset T_{g+f}$

⑥ $T_g T_f \subset T_{fg}$

⑦ $\sigma(T_f) = \overline{f(\text{supp } E)}$

A : self-adjoint $\exists E(\lambda)$ spectral family

s.t. $\int \lambda dE(\lambda) = A$ $\int f(\lambda) dE(\lambda) = f(A)$

$E(\lambda) \subset \tilde{E}(\lambda)$ \leftarrow spectral meas. f cont

$$\int f(\lambda) d\tilde{E}(\lambda) = f(A) \quad f: \mathcal{V}_{\text{meas}}^{\text{Borel}}$$

$$F - \Delta f = |k|^2 F f$$

§ 8 Schrödinger operators.

$$-\frac{1}{2}\Delta + V = H \quad \text{on } L^2(\mathbb{R}^d)$$

- $\tilde{F}^{-1} M_{|k|^2} F = -\Delta \quad \text{on } \mathcal{S}(\mathbb{R}^d) \quad \text{ess. s.a.}$

$$\overline{-\Delta|_{\mathcal{S}(\mathbb{R}^d)}} = -\Delta \quad \text{s.a.}$$

- V : multiplication

$$-\frac{1}{2}\Delta + V = H$$

$$D(H) = D(-\Delta) \cap D(V) \quad \leadsto \text{s.a.?}$$

Perturbation of S.A. op.

- Def 8.1 let A, B be densely defined linear op.

s.t. $D(B) \supset D(A)$

$$\|B\varphi\| \leq \exists a \|A\varphi\| + \exists b \|\varphi\| \quad \forall \varphi \in D(A)$$

Then B is said to be A -bounded

- The infimum of such a is called the relative bound of B wrt A .

- If the relative bound is zero, we say that B is infinitesimally small w.r.t A . and we write $B \ll A$.

- A sym $\inf_{\varphi \in \Omega(A)} (\varphi, H\varphi) = M > -\infty$
 $\|\varphi\|_1 = 1$ or φ b'dd below.
 - A s.a. $\Leftrightarrow \inf \sigma(H) = M$
 - A sym / A is s.a. $\stackrel{\textcircled{A}}{\Leftrightarrow} \text{Ran}(A^{\frac{1}{2}}i) = \mathcal{H}$
 $\Leftrightarrow \text{Ran}(A^{\frac{1}{2}}i^{\frac{1}{2}}\lambda) = \mathcal{H}$
Proof is the same as \textcircled{A} .
-

Thm 8.2 (The Kato-Rellich theorem)

Suppose that A is s.a. B is sym. A -bdd
with $a \neq 1$. Then $A+B$ is s.a. on $D(A)$
and ess.s.a. on any core of A .

Furthermore if A is bdd below by M , then

$A+B$ is also bdd below by $-\max\left\{\frac{b}{1-a}, a(M+b)\right\}$

∴ It is enough to show that $\text{Ran}(A+iB \pm i\mu) = \mathcal{H}$

$$A+iB+i\mu = B(A+i\mu)^{-1}(A+i\mu) + A+i\mu$$

$$= [B(A+i\mu)^{-1} + I](A+i\mu)$$

Hence where $\|B(A+i\mu)^{-1}\varphi\| \leq a \|A(A+i\mu)^{-1}\varphi\| + b \|(A+i\mu)^{-1}\varphi\|$

$$\leq \left(a + \frac{b}{\mu}\right) \|\varphi\| \quad \mu \gg 1 \Rightarrow B(A+i\mu)^{-1} \text{ is b'dd and } \|\cdot\| \leq$$

$[+ B(A + i\mu)^{-1}]$ is onto. Hence $(A + B + i\mu)$ is onto.

Suppose that D is a core of A i.e., $A|_D$ ess.s.s.g.

$$D(\overline{A|_D}) = D(A) \subset D(\overline{A + B|_D})$$

$$\therefore \forall \varphi \in D(A) \exists \varphi_n \in D \text{ s.t. } \begin{array}{l} \varphi_n \rightarrow \varphi \\ A\varphi_n \rightarrow A\varphi \end{array}$$

$\therefore (A + B)\varphi_n = A\varphi_n + B\varphi_n$ is also convergent
 $\downarrow \quad \downarrow$ sequence.
 $A\varphi \quad B\varphi$

$$\therefore \varphi \in D(\overline{A + B|_D}) \quad \because D(\overline{A|_D}) \subset D(\overline{A + B|_D})$$

$\overset{\text{"}}{D(A)}$

$$\therefore D(A + B) \subset D(\overline{A + B|_D})$$

$\overset{\uparrow}{\text{s.g.}} \quad \overset{\uparrow}{\text{sym}}$

There is no nontrivial s.a. ext.

$$\text{Hence } A + B = \overline{A + B|_D}.$$

Let $t \in \mathbb{R}$, s.t. $-t < M$

$$\mathbb{A}^{x^t}$$

$$\therefore \overline{B(A+t)^{-1}\varphi}$$

$$\begin{aligned} \|B(A+t)^{-1}\varphi\| &\leq a \|A(1+t)^{-1}\varphi\| + b \|(1+t)^{-1}\varphi\| \\ &\leq \left(a + \frac{b}{t}\right) < 1 \end{aligned}$$

$$(A + B\tau t)^{-1} = \cancel{(A + B\tau t)}^{-1} = \left\{ (B(A\tau t)^{-1} + I)(A\tau t)^{-1} \right\}^{-1}$$

$$= (A\tau t)^{-1} (B(A\tau t)^{-1} + I)^{-1}$$

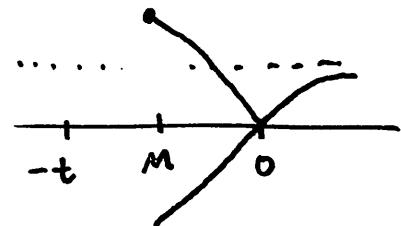
$-t < M$ & $\|B(A\tau t)^{-1}\| < 1 \Rightarrow (A\tau B\tau t)^{-1}$ bdd.

$$\|B(A\tau t)^{-1}\varphi\| \leq a \|A(A\tau t)^{-1}\varphi\| + b \|(A\tau t)^{-1}\varphi\|$$

\leq

$$\int_M^\infty \left| \frac{\lambda}{(A\tau t)} \right|^2 d(\varphi, E(\lambda)\varphi) \leq \sup_{\lambda > M} \frac{|\lambda|^2}{|\lambda + t|^2}$$

$$\begin{aligned} \therefore \|A(A\tau t)^{-1}\| &\leq \sup_{\lambda > M} \frac{|\lambda|}{\lambda + t} \\ &= \max \left\{ \frac{|M|}{M+t}, 1 \right\} \end{aligned}$$



$$\| (A\tau t)^{-1}\varphi\|^2 = \int_M^\infty \frac{1}{(\lambda + t)^2} d(\varphi, E(\lambda)\varphi) \leq \frac{1}{|M+t|^2}$$

$$\therefore \|(A\tau t)^{-1}\| \leq \frac{1}{M+t}$$

$$\therefore \|B(A\tau t)^{-1}\| \leq a \max \left\{ \frac{|M|}{M+t}, 1 \right\} + \frac{b}{M+t}$$

$$\textcircled{1} \quad \frac{|M|}{M+t} \approx \frac{a|M|}{M+t} + \frac{b}{M+t} < 1 \quad \therefore -M + (a|M| + b) < t$$

$$\textcircled{2} \quad 1 \text{ or } 2 \quad a + \frac{b}{M+t} < 1 \quad \therefore -M + \frac{b}{1-a} < t$$

$$a(M+t) + b < M+t$$

$$aM + b - M < t - at = (1-a)t$$

$$-(1-a)M + b < (1-a)t$$

Lemma (the uncertainty principle (lemma))

$$\varphi \in C_0^\infty(\mathbb{R}^3)$$

$$\int_{\mathbb{R}^3} \frac{1}{4r^2} |\varphi(r)|^2 dr \leq \int_{\mathbb{R}^3} |D\varphi|^2 dr$$

This is the example of Hardy's inequality:

$$\text{let } d \geq 3, \quad f \in \overline{C_0^\infty(\mathbb{R}^d)}, \quad S(\mathbb{R}^d)$$

$$\int \frac{1}{|x|^2} |f(x)|^2 dx \leq \frac{4}{(d-2)^2} \int |Df|^2$$

$$\textcircled{1} \quad \int \left| \partial_i f - \alpha \frac{x_i}{|x|^2} f \right|^2 dx$$

$$= \int (\partial_i f)^2 - 2\alpha \frac{x_i}{|x|^2} f + \alpha^2 \frac{x_i^2}{|x|^4} |f|^2$$

$\underbrace{\partial_i f}_{\text{d.f.}}$

$$\begin{aligned} -2 \int \alpha \frac{x_i}{|x|^2} \partial_i f \cdot f &= \int -\alpha \frac{x_i}{|x|^2} \partial_i f f^2 \\ &= \alpha \int \partial_i \frac{x_i}{|x|^2} \cdot |f|^2 \\ &= \alpha \int \left(\frac{1}{|x|^2} - 2 \frac{x_i^2}{|x|^4} \right) |f|^2 \end{aligned}$$

$$i=1, \dots, d \quad \alpha \neq 0 \in \mathbb{R}^d$$

$$\therefore 0 \leq \int |Df|^2 + \alpha \left(\frac{d}{|x|^2} - \frac{2}{|x_1|^2} \right) |f|^2 + \alpha^2 \frac{|f|^2}{|x_1|^2}$$

$$= \int |D\varphi|^2 + \alpha(d-2) + \alpha^2 \frac{|f|^2}{|x_1|^2}$$

$$\therefore -[\alpha(d-2) + \alpha^2] \int \frac{|f|^2}{|x_1|^2} \leq \int |Df|^2$$

$$- \left(\alpha + \frac{d-2}{2} \right)^2 + \frac{(d-2)^2}{4} \quad \alpha = -\frac{d-2}{2} \text{ by } \star.$$

$$\therefore \int \frac{|f|^2}{|x_1|^2} \leq \frac{4}{(d-2)^2} \int |D\varphi|^2. \quad \square$$

Ex. $d=3$. $V(r) = -\frac{q}{r}$ Coulomb potential

$$\|V\varphi\|^2 = \int \frac{q^2}{|x_1|^2} |\varphi|^2 \leq q^2 4 \int |\Delta \varphi|^2$$

$$= 4q \| \Delta \varphi \| \cdot \| \Delta \varphi \| \leq 4q \left(\varepsilon \| \Delta \varphi \| + \frac{1}{4\varepsilon_0} \| \varphi \|^2 \right)$$

$$\therefore \|V\varphi\| \leq \varepsilon \| \Delta \varphi \| + b_\varepsilon \| \varphi \|^2 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^3)$$

$\exists \varphi_n \in \mathcal{S}$ s.t. $\Delta \varphi_n \rightarrow \Delta \varphi$

$\forall \varphi \in D(\mathbb{R}) \quad \varphi_n \rightarrow \varphi$

V : closed op $\therefore \|V\varphi\| \leq \varepsilon \| \Delta \varphi \| + b_\varepsilon \| \varphi \|^2$.

$\therefore -\Delta + V$ is s.a. on $D(1-\delta)$.

支國復習: $-\frac{1}{2}\Delta + V$

- $\|V\varphi\| \leq a \| -\frac{1}{2}\Delta \varphi \| + b \|\varphi\|$ $\forall \varphi \in D(-\Delta) \subset D(V)$
 \Leftrightarrow and $a < 1 \Rightarrow -\frac{1}{2}\Delta + V$ is s.a. on $D(-\Delta)$
 and ess. s.a. on any cone of $-\frac{1}{2}\Delta$.

But $V(m = x^2)$ does not satisfy \oplus .

Theorem. Suppose that $V \in L^2_{loc}(\mathbb{R}^d)$ and $V \geq 0$.
 Then $-\frac{1}{2}\Delta + V$ is ess. s.a. on $C_0^\infty(\mathbb{R}^d)$
 $\therefore (-\frac{1}{2}\Delta + V + 1)^* u = 0$ is proven.

More precisely

$$\begin{aligned} \langle (H+1)u, f \rangle &= 0 \quad \forall u \in C_0^\infty \Leftrightarrow f = 0 \\ \Leftrightarrow \ker \left[(H+1) \Big|_{C_0^\infty} \right]^* &= 0 \quad \therefore (H+1) \Big|_{C_0^\infty} \text{ is ess.s.a.} \end{aligned}$$

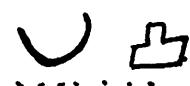
$$\langle -\frac{1}{2}\Delta u, f \rangle = \langle \alpha.(V+1)u, f \rangle$$

$$\int \# \Delta u \cdot f = - \int \bar{u} (V+1) f \quad \left(\frac{1}{2} \text{ ESS} \right)$$

$$\# \Delta f = (V+1)f \text{ is in } L^1_{loc}$$

distributional derivative \swarrow Kato's inequality

$$\begin{aligned} |\Delta f| &\geq R(\operatorname{sgn} \# \Delta f) = R \operatorname{sgn} f (V+1)f \\ &= (V+1)|f| \quad \therefore |\Delta f| \geq 0 \Rightarrow |f| = 0. \end{aligned}$$



$$w_\varepsilon = |f| * g_\varepsilon \quad g_\varepsilon(x) = \frac{1}{\varepsilon^\alpha} g(x/\varepsilon)$$
$$\int g_\varepsilon dx = 1$$
$$g > 0, \quad g \in C^\infty.$$

जैसे ($w_\varepsilon, \Delta w_\varepsilon$)

$$\Delta w_\varepsilon = \Delta(|f| * g_\varepsilon) = \Delta|f| * g_\varepsilon \geq 0$$

$$\therefore w_\varepsilon = 0 \Rightarrow w_\varepsilon \rightarrow |w| \leq 0 \quad \therefore w = 0.$$

§9 Canonical Commutation Relations

$$d=1 \quad P = -i \frac{d}{dx}, \quad Q = M_x = x, \quad \mathcal{H} = L^2$$

$$\begin{cases} D(P) = \left\{ \psi \in \mathcal{D} : F^{-1} D(M_{\frac{\psi}{h}}) \in F\psi \right\} \\ P\psi = F^{-1} M_h F\psi \quad \text{for } P \cong Q \end{cases}$$

\mathcal{H} Hilbert space, P, Q ~~abstr.~~ op.

$$[P, Q] = PQ - QP \quad \text{Commutator}$$

This immediately see that $[P, Q] = -i$

$$\therefore P Q f - Q P f = -i((x_f)' - x f') = -i f.$$

• von Neumann:

$$H_{\text{Schr}} = \frac{1}{2} P^2 + V(x) \quad \text{on } L^2(\mathbb{R})$$

$$H_{\text{abs}} = \frac{1}{2} P^2 + V(Q) \quad \text{on abstract H space}$$

$$\text{where } [P, Q] = -i$$

$$H_{\text{abs}} \cong H_{\text{Schr}} ?$$

• (\mathcal{H}, D, P, Q) CCR representation.

\mathcal{H}, K Hilbert space A, B s.a. in \mathcal{H} and K
 $U: \mathcal{H} \rightarrow K$ unitary operator respectively

$$UAU^{-1} = B \Leftrightarrow U: D(A) \rightarrow D(B) \text{ onto, 1:1}$$

i.e. $U^*f \in D(A) \Leftrightarrow f \in D(B)$
 and $UAU^{-1}f = Bf \Leftrightarrow f \in D(B)$
 or $Af = U^*Bu \Leftrightarrow u \in D(B)$

Lemma 9.1 \mathcal{H}, K Hilbert space A, B s.a. in \mathcal{H} ,
 respectively. $\exists U: \mathcal{H} \rightarrow K$ unitary such that U^*f .

$UAU^{-1} = B$. Then spectral meas

$$(1) \quad U E_A(\lambda) U^{-1} = E_B(\lambda) \quad \lambda \in \text{B}(m)$$

(2) $U f(A) U^{-1} = f(B)$

$$\text{But } U E_A(\cdot) U^{-1} = F(\cdot) \quad \cdot \in \text{B}(m)$$

$$C = \int \lambda \varphi dF(\lambda). \quad \text{Hence}$$

$$\varphi \in D(C) \Leftrightarrow \int \lambda^2 d(\varphi, F(\lambda)\varphi) < \infty$$

$$\Leftrightarrow \int \lambda^2 d(U\varphi, E_A U^{-1}\varphi) < \infty$$

$$\Leftrightarrow U\varphi \in D(A)$$

$$\therefore U^*D(C) = D(A) \quad \therefore D(C) = UD(A) = D(B)$$

$\forall \varphi \in K, \forall \varphi \in D(B)$

$$(\varphi, B\varphi) = \int \lambda d\mathbb{P} \varphi, E_B(\lambda) \varphi$$

$$(\tilde{u}'\varphi, A(\tilde{u}'\varphi)) = \int \lambda d(\tilde{u}'\varphi, E_A(\lambda) \tilde{u}'\varphi)$$

$$= \int \lambda d(\varphi, F(\lambda)\varphi) = (\varphi, C\varphi)$$

$\therefore B = C$ Since the spectral measure is unique
 $F(\lambda)$ is "the" spectral measure associated with B i.e., $F(\lambda) = E(\lambda)$,

Let f be a Borel mes function.

$$\varphi \in D(f(B))$$

$$\Leftrightarrow \int |f(\lambda)|^2 d(\varphi, E_B(\lambda)\varphi) < \infty$$

$$\Leftrightarrow \int |f(\lambda)|^2 d(\varphi, \cup E_A(\lambda) \cup' \varphi) < \infty$$

$$\Leftrightarrow \int |f(\lambda)|^2 d(\tilde{u}'\varphi, E_A(\lambda) \tilde{u}'\varphi) < \infty$$

$$\Leftrightarrow \tilde{u}'\varphi \in D(f(A))$$

$$\therefore D(f(A)) = D(f(B))$$

$$\text{i.e., } D(f(B)) = D(f(A)\tilde{u}') \subset D(f(A)) \\ = D(u f(A) u')$$

$$f(B)u\varphi$$

$\psi \in K, \psi \in D(B)$

$$\begin{aligned}
 (\Phi \circ f(A) \bar{u}^* \psi) &= \int f(\lambda) d(\bar{\psi}, E_A(\lambda) \bar{u}^* \psi) \\
 &= \int f(\lambda) d(\psi, E_B(\lambda) \psi) \\
 &= \cancel{\int f(\lambda) d(\psi, E_B(\lambda) \psi)} \quad \int (\psi, f(B) \psi). //
 \end{aligned}$$

In particular $P \hat{=} Q$

$$e^{itP} \hat{=} e^{itk}$$

"

$$F e^{itP} F^{-1}$$

-1:

Lemma 9.2 Let $f \in L^2$. Then

$$e^{itP} f = f(\cdot + t) \quad \text{i.e. shift by } t.$$

$$\begin{aligned}
 \therefore e^{itP} f &= F^{-1} e^{itk} F f && \text{Let } f \in \mathcal{S} \\
 &= F^{-1} e^{itk} (2\pi)^{-\frac{1}{2}} \int \widehat{f}(k) e^{ikx} dx \\
 &= F^{-1} (2\pi)^{\frac{1}{2}} \int \widehat{f}(k) e^{ik(x+t)} dx = f(\cdot + t)
 \end{aligned}$$

$$\begin{aligned}
 \therefore \| e^{itP} f - f(\cdot + t) \| &\\
 \leq \| e^{itP} f - g \| + \| g(\cdot + t) - f(\cdot + t) \| & < \varepsilon.
 \end{aligned}$$

$$\begin{aligned}
 F^{-1} k F & \int k e^{-ikx} = \int i \frac{d}{dx} e^{-ikx} \\
 F \left(-i \frac{d}{dx} \right) F^{-1} & \xrightarrow{e^{-ikx}} i(-i)k = k
 \end{aligned}$$

$$\begin{aligned}
 F P f &= \int -if'(x) e^{-ikx} dx \\
 &= \int -if(k) (-ik)(-1) dx \\
 &= \int k f(k) e^{-ikx} = k \hat{f} \quad F P F^T F \hat{f} = k F \hat{f} \\
 &\quad F P P^T = k
 \end{aligned}$$

Prop 9.3 Let H be a Hilbert space and A be a self-adjoint op. Then $U_t = e^{itA}$ satisfies unitary and

- (1) $U_0 = 1$,
- (2) $U_t U_s = U_{t+s}$
- (3) $\lim_{t \rightarrow 0} U_t = 1$.

Strongly continuous unitary group.

$$\text{Def: } (U_t \varphi, \psi) = \int e^{it\lambda} d(\varphi, E_\lambda(\psi)).$$

Let $t=0 \therefore$ then

$$(U_0 \varphi, \psi) = (\varphi, U_0 \psi) \quad \forall \varphi, \psi.$$

unitary 1:1, and, leave the scalar product invariant

$$\therefore \|U_t \varphi\|^2 = \int |1| d(\varphi, E(\lambda)\varphi) = \|\varphi\|^2 \text{ isometry.}$$

$$U_t U_{-t} \subset 1 \text{ as bounded.}$$

$$\therefore U_t U_{-t} = 1 \text{ onto.}$$

$$(U_t \varphi, U_t \psi) = \frac{1}{4} \sum i^n (U_t \varphi + i \tilde{U}_t \varphi, U_t \psi + i \tilde{U}_t \psi)$$

polarization identity

$$(2) U_t U_s \subset U_{t+s} \text{ and bdd.}$$

$$(3) \lim_{t \rightarrow 0} (U_t \varphi, \psi) = \int e^{it\lambda} d(\varphi, E(\lambda)\psi)$$

$$= (\varphi, \psi) \text{ Lebesgue.}$$

$$\therefore \|(\varphi - U_t \varphi)\|^2 = \|\varphi\|^2 - 2 \operatorname{Re} (\varphi, U_t \varphi) \rightarrow 0,$$

Prop 9.4 Let $\{U_t = e^{itA}\}_{t \in \mathbb{R}}$.

$$\textcircled{1} \quad \text{For } \varphi \in D(U), \quad s\text{-}\lim_{t \rightarrow 0} \frac{U_t \varphi - \varphi}{t} = iA\varphi$$

$$\textcircled{2} \quad \exists \quad s\text{-}\lim_{t \rightarrow 0} \frac{U_t \varphi - \varphi}{t} \Leftrightarrow \varphi \in D(A)$$

\textcircled{3} For $\varphi \in n(\mathbb{R}) \Rightarrow \exists U_t \varphi \in D(A)$ and

$$AU_t \varphi = U_t A\varphi.$$

$$\frac{dU_t \varphi}{dt} = AU_t \varphi = iU_t A\varphi.$$

$$\therefore \textcircled{1} \left\| \frac{U_t \varphi - \varphi}{t} - iA\varphi \right\|^2 = \int \left| \frac{e^{it\lambda} - 1}{t} - i\lambda \right|^2 d(\varphi_E \varphi)$$

$$\text{as } \leq 4\lambda^2 \int_0^\infty e^{i\lambda t} dt$$

Here by the Lebesgue dominated conv.,

$$\textcircled{2} \quad \left\| \frac{U_t - 1}{t} \varphi \right\|^2 = \int \left| \frac{e^{it\lambda} - 1}{t} \right|^2 d(\varphi_E \varphi)$$

$$\therefore \int |\lambda|^2 d(\varphi_E \varphi) = \int \lim_{t \rightarrow 0} \left| \frac{e^{it\lambda} - 1}{t} \right|^2 d(\varphi_E \varphi)$$

$$\leq \lim_{t \rightarrow 0} \int |.-|^2 d(\varphi_E \varphi) = \lim_{t \rightarrow 0} \left\| \frac{U_t - 1}{t} \varphi \right\|_\infty^2.$$

Fatou's lemma. $\liminf_{t \rightarrow 0} \left\| \frac{U_t - 1}{t} \varphi \right\|_\infty^2 \leq \infty$

$$\textcircled{3} \quad \text{Hence } \varphi \in D(A) \Leftrightarrow \int |\lambda|^2 d(\varphi_E \varphi) < \infty$$

$$\therefore \varphi \in D(A_U) \therefore U_t \varphi \in D(A)$$

$$= D(U_t A)$$

③ $AU_t \subset F(A)$ where $F(\lambda) = \lambda e^{it\lambda}$
 $U_t A \subset F(A)$

$\varphi \in D(A) \Rightarrow \varphi \in D(F(A)) \therefore \int |\lambda|^2 |e^{it\lambda}|^2 < \infty$
 $\therefore U_t \varphi \in D(A) //$

Prop 9.4 (Stone's theorem) Let $\{U_t\}_{t \in \mathbb{R}}$ be a strongly cont one-parameter unitary group. Then \exists_1 A s.a. s.t $U_t = e^{itA} //$

Ex. $\mathcal{H} = L^2$, $U_t : f \mapsto f(\cdot + t)$

$\{U_t\}_{t \in \mathbb{R}}$ is ~~not~~ strongly cont one-parameter unitary group.

Then $U_t = e^{it \frac{d}{dx}}$ $A = -i \frac{d}{dx}$.

$$\text{s-lim}_{t \rightarrow 0} \left(\frac{U_t - 1}{t} \right) \varphi = i A \varphi = \frac{d}{dx} \varphi$$

$$\frac{\varphi(x+t) - \varphi(x)}{t} \rightarrow \frac{d}{dx} \varphi.$$

Lemma 9.5

$$e^{itQ} e^{isP} = e^{-its} e^{isP} e^{itQ}$$

∴ Let $f \in S$

$$e^{itQ} e^{isP} f = e^{itx} f(x+s)$$

$$e^{isP} e^{itQ} f = \underline{e^{itx} f(x)} \rightarrow e^{it(x+s)} f(x+t) //$$

Def 9.6 If Hilbert space

with $\{A, B\}$ satisfies

$$e^{itA} e^{isB} = e^{-its} e^{isB} e^{itA}.$$

$\{A, B\}$ satisfies Wylie relation.

Ex $\{Q, P\}$ satisfies Wylie relation

Suppose that

Lemma 9.10 Set $\{A, B\}$ satisfies Wylie relation.

for $\varphi \in D(PQ) \cap D(P)$, $\varphi \in D(QP)$ and

$$[Q, P]\varphi = i\varphi.$$

∴ $e^{itQ} \underline{e^{isP} f} = e^{-its} e^{isP} e^{itQ} f$

The ^{rhs} \underline{f} is differentiable with respect to t .
Hence $e^{isP} f$ is in the domain of Q .

$$iQ e^{isP} f = e^{isP} (it + iQ) f. \text{ follows.}$$

$$\therefore Q e^{isP} f = e^{isP} (it + Q) f$$

Lhs is differentiable in s ($\because f \in D(PQ) \cap D(P)$)

$$\therefore Q \left(\frac{e^{isP} - 1}{s} \right) f \rightarrow \text{conv.}$$

$$\left(\frac{e^{isP} - 1}{s} \right) f \rightarrow i Pf$$

Ω is closed $\therefore i Pf \in D(Q)$ and we have

$$\Omega P f = if + PQ f$$

$$\therefore P\Omega f - QPf = -if. \quad //$$

Wylie Relation \Rightarrow cor.

thanks

§ 10 von Neumann's uniqueness thm.

\mathcal{H} : Separable Hilbert space

P, Q satisfies Wylie relation

$$\underline{e^{itQ} e^{isP} = e^{-its} e^{isP} e^{itQ}}$$

Schrödinger representations

$$\text{let } \Omega = \pi^{-1/4} e^{-1/2} \in L^2.$$

$$D = \text{Span}\{ e^{isQ} e^{itP} \Omega; s, t \in \mathbb{R} \}$$

$$\mathcal{M} = \{ e^{isQ}, e^{itP}; s, t \in \mathbb{R} \} \\ \subset B(L^2)$$

Lemma 10.1

① $\mathcal{M}D \subset D$ i.e. D is an invariant subspace w.r.t. \mathcal{M} .

② D is dense

③ ① follows from Wylie relations.

② It is enough to show that if $(f, g) = 0$ for $\forall g \in D \Rightarrow f = 0$.

$$\begin{aligned} \text{we have } (f, e^{isQ} \Omega) &= 0 \quad \forall s \in \mathbb{R} \\ &= \int \bar{f}(x) e^{isx} \Omega(x) dm = 0 \end{aligned}$$

$$\therefore \widehat{f\Omega}(s) = 0 \quad \therefore f\Omega = 0 \quad \Omega > 0 \quad \therefore f = 0,$$

let us introduce notation $W(s, t)$ by

$$W(s, t) = e^{ist/2} e^{isQ} e^{itP}$$

$$a(s, t) = \bar{e}^{(s^2 + t^2)/4}$$

$$A = \int a(s, t) W(s, t) ds dt$$

Lemma 10.2 let $P = -i \frac{d}{dx}$, $Q = Mx$.

$$\text{Then } \mathcal{O}_P(A) \setminus \{0\} = \{2\pi k\}$$

$$\text{Ker}(k + 2\pi) = \{c \in \mathbb{C} : c \in G\}$$

$$\therefore \cancel{(Af)}(x) = \int \bar{e}^{-(s^2 + t^2)/4} e^{isx} f(x+t) ds dt$$

$$(n \pi + y) = \int \bar{e}^{s^2/4} \bar{e}^{-(y-x)^2/4} e^{isx} f(y) dy ds$$

$$\int \bar{e}^{-\frac{s^2}{4}} \bar{e}^{is(y+x)/2} ds = \int \bar{e}^{-\frac{u^2}{2}} e^{iu(y+x)/\sqrt{2}} du$$

$$s/\sqrt{2} = u \quad ds = \sqrt{2} du \quad = \sqrt{2\pi} \cdot \sqrt{2} e^{-(y+x)^2/4}$$

$$= 2\sqrt{\pi} \int \bar{e}^{(y+x)^2/4} \bar{e}^{-(y-x)^2/4} f(y) dy = 2\sqrt{\pi} \int \bar{e}^{(\frac{y^2+x^2}{2})/2} f(y) dy$$

$$= C \cdot \Omega$$

$$\left(\int \bar{e}^{-\frac{s^2}{4}} e^{is(y-x)/2} ds \right) = \int \bar{e}^{-\frac{u^2}{2}} e^{i\sqrt{2}ux} \sqrt{2} du = \sqrt{2} \int \bar{e}^{-(\sqrt{2}u)^2/2} du = 2\sqrt{\pi} e^{-2x^2/2}$$

$$\begin{aligned}
 A_f &= 2\sqrt{\pi} \int f(y) e^{-\frac{x^2}{2} - \frac{y^2}{2}} dy \\
 &= \left(2\pi^{\frac{3}{4}} \int f(y) e^{-\frac{y^2}{2}} dy \right) \Omega
 \end{aligned}$$

$\frac{\pi}{\sqrt{2}} = u$
 $dx = \sqrt{2} du$
 $\int e^{-\frac{u^2}{2}} \sqrt{2} du = \sqrt{2}\sqrt{\pi}$
 $\int e^{-x^2} = \sqrt{\pi}$

$$\begin{aligned}
 A_\Delta \Omega &= \lambda \Omega, \quad \lambda = 2\pi^{\frac{3}{4}} \int \pi^{\frac{1}{4}} e^{-\frac{y^2}{2}} dy \\
 &= 2\pi \Omega
 \end{aligned}$$

$$A_f = 2\pi(\Omega, f)\Omega$$

one rank op

$$A\Omega = 2\pi\Omega$$

$$A_f f = 2\pi A_f$$

$$\therefore Af = \lambda f \quad \lambda \neq 0 \Rightarrow$$

$$A\Omega = 2\pi^{\frac{3}{4}} \bar{4}^{\frac{-1}{4}}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy$$

$$= 2\pi^{\frac{3}{4}} \cdot 2\pi^{\frac{1}{4}} = 2\pi$$

$$Af = 2\sqrt{\pi} e^{-x^2/2} \int f(y) e^{-\frac{y^2}{2}} dy = \lambda f = \left(2\pi^{\frac{3}{4}} \right) f e^{-\frac{x^2}{2}}$$

$$A\Omega = 2\sqrt{\pi} e^{-x^2/2} \int e^{-\frac{y^2}{2}} dy \pi^{1/4} \times \Omega$$

$$= \lambda \Omega$$

$$2\sqrt{\pi} \sqrt{\pi} \pi^{1/4} e^{-x^2/2} = \lambda \pi^{1/4} e^{-x^2/2}$$

$$\therefore \lambda = 2\pi.$$

$$\begin{aligned} & \therefore AAf \\ & = A(C\Omega) \\ & = \underline{CA\Omega} \\ & = \underline{CC'\Omega} \end{aligned}$$

$$G_p(A) \setminus \{0\} = \{2\pi\}. \text{ & } \ker(A - 2\pi) = \text{Im } \Omega.$$

Lemma 10.3 $\mathcal{H}, \mathcal{P}, \Omega$ Wyle relations

$$M = \{ \varphi \in \mathcal{H}; A\varphi = 2\pi\varphi \}$$

$$(i) M = \underline{\text{Ran}(A)} \neq \{0\}$$

$$(ii) \bar{\varphi}_1 \perp \bar{\varphi}_2 \Rightarrow W(n,t)\bar{\varphi}_1 + W(u,v)\bar{\varphi}_2$$

$$\because (\varphi, A W(u,v) A \varphi)$$

$$= \int a(st) a(s't') (\varphi, W(n,t) W(u,v) W(s't') \varphi)$$

$$= \int a(st) a(s't') \left(\varphi, e^{isQ} e^{itP} e^{iuQ} e^{ivP} e^{is'Q} e^{it'P} \varphi \right)$$

$$\times e^{ist/2} e^{iuv/2} e^{is't'/2} \dots$$

$$= 2\pi (\varphi, a(u,v) A \varphi)$$

$$A W(u, v) A = 2\pi a(u, v) A$$

In particular putting $u=v=0$

$$AA = 2\pi A \quad \text{Ie} \quad \mu(A\varphi) = 2\pi A\varphi$$

It implies that $\text{Ran } A \subset M$

$\text{Ran } A \cap M$ is trivial then

$\text{Ran } A = M$ follows. $A \neq 0$: $M \neq \{0\}$

Let $\varphi, \psi \in M$. Then $A\varphi = 2\pi\varphi \therefore \varphi = A\varphi/2\pi$

$$\text{Then } (W(n+t)\varphi, W(u, v)\psi)$$

$$= (\varphi, W(-n-t) W(u, v)\psi)$$

$$= e^{i(-tu+sv)/2} (\varphi, W(u-s, v-t)\psi)$$

$$= \frac{1}{(2\pi)^2} (A\varphi, W(u-s, v-t) A\psi)$$

$$= \frac{1}{(2\pi)^2} (\varphi, 2\pi a(u-s, v-t)\psi)$$

$$= \frac{1}{(2\pi)^2} 2\pi a(u-s, v-t) (\varphi, \psi)$$

$M = \text{cons} \{ \varphi_m \}$.

$H_m = \overline{\text{Span} \{ W(s+t) \varphi_m : s, t \in \mathbb{R} \}}$

$H_m + H_n \oplus H_m \subset \mathcal{H}$

Theorem (von Neumann)

① $e^{itQ} H_m \subset H_m, e^{isP} H_m \subset H_m$

② $\exists U_m : H_m \rightarrow L^2(\mathbb{R}^2)$ unitary st.

$$U_m e^{itQ} U_m^{-1} = e^{it\hat{x}} \quad \hat{x} = Mx$$

$$U_m e^{isP} U_m^{-1} = e^{is\hat{p}} \quad \hat{p} = -i \frac{d}{dx}$$

③ $H = \bigoplus_m H_m$

∴ ① follows from Wylie relation.

② $\text{Span} \{ w(s+t) \varphi_m : s+t \in \mathbb{R} \}$ dense in H_m

$$\mathcal{L} \ni \sum_{ij} \alpha_{ij} W(s_i, t_j) \varphi_m \xrightarrow{U_m} \sum_{ij} \alpha_{ij} W_S(s_i, t_j) \varphi_m$$

$$\text{where } W_S(s+t) = e^{is\hat{x}/2} e^{is\hat{x}} e^{it\hat{p}}.$$

$$\left\| \sum_{ij} \alpha_{ij} W(s_i, t_j) \varphi_m \right\|^2 = \sum_{ij} \alpha_{ij} \bar{\alpha}_{ij} \left(W(s'_i, t'_j) \varphi_m, W(s_i, t_j) \varphi_m \right)$$

$$= \sum_{ij} \alpha_{ij} \bar{\alpha}_{ij} \delta(s_i - s'_i, t_j - t') (\varphi_m, \varphi_m) = \|\sum_{ij} \alpha_{ij} W_S(s_i, t_j) \varphi_m\|^2$$

By the extension to the unitary op.

$$\textcircled{3} \quad K = \bigoplus H_m \quad \begin{matrix} \text{we shall show that} \\ K^\perp = \{0\} \end{matrix}$$

$$W(u, v) H_m \subset H_m$$

$$\text{Hence } W(u, v) K^\perp \subset K^\perp$$

$$\therefore (W(u, v) \varphi, \underset{K}{\underset{\uparrow}{\varphi}}) = (\varphi, W(-u, -v) \varphi) = 0$$

$$\text{Ran}(A) = M \subset K \quad \therefore K^\perp \subset \ker A \quad \text{OK}$$

$$\text{let } \varphi \in K^\perp \quad \xrightarrow{\quad W(u, v) \varphi \in K^\perp \quad} \quad W(u, v) \varphi \in K^\perp$$

$$\therefore A W(u, -v) \varphi = 0$$

$$\therefore (W(-u, v)^* \underset{A}{\downarrow} \varphi, A W(u, -v) \varphi) = 0$$

$$\therefore \int a(n+1) \left(\varphi, \underbrace{W(-u, v) W(n+1) W(u-v) \varphi}_{=0} \right) dt = 0$$

$$= \int e^{i(sv+tu)} a(n+1) (\varphi, W(n+1) \varphi) dt = 0$$

$$\therefore \mathbb{R}^2 \ni (s+t) \mapsto a(n+1) (\varphi, W(n+1) \varphi) = 0$$

$$\therefore W(n+1) \varphi = 0 \Leftrightarrow \varphi = 0,$$

Cor

$$U_m e^{itQ} U_m^{-1} = e^{it\hat{n}}$$

$$U_m e^{itQ} = e^{it\hat{n}} U_m \quad \text{on } D(Q)$$

$$\therefore U_m : D(Q) \rightarrow D(\hat{n}) \quad \text{and}$$

$$U_m Q \not\equiv \hat{n} \quad U_m \therefore U_m Q U_m^{-1} = \hat{n}.$$

- 一般の自由度の場合, degree of
The case of general freedom

$\{Q_j, P_j\}$ satisfies Weyl relation

$$e^{itQ_j} e^{isP_i} = e^{i\delta_{ij}} s.t. e^{isP_i} e^{itQ_j}$$

Hence we can also construct a unitary operator

$$U : \mathcal{H}_m \rightarrow L^2(\mathbb{R}^N) \quad \text{s.t. } Q_j \cong x_j$$

$$P_j \cong -i \frac{d}{dx_j}$$

Non Schrödinger CCR rep.

Let us try to construct CCR-representation (Q_j, P_j) , $j=1, 2$ which however does not satisfy Weyl representation

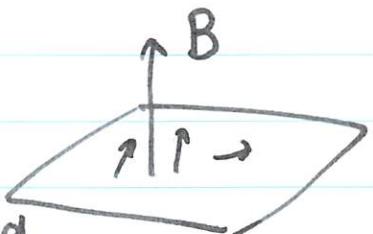
$$\text{i.e. } (Q_j, P_j) \notin (\hat{q}_j, \hat{p}_j)$$

$$\begin{cases} A_1(x) = -\frac{1}{2\pi} \frac{x_2}{|x|^2} \cdot b \\ A_2(x) = \frac{1}{2\pi} \frac{x_1}{|x|^2} \cdot b \end{cases}$$

Then we have $\partial_1 A_2 - \partial_2 A_1 = b \delta(x)$
in the sense of distribution i.e.,

$$Q_j = \hat{q}_j, \quad P_j = \hat{p}_j - \frac{q}{\hbar} A_j(x) \quad (\text{covariant derivative})$$

$$\nabla \times \begin{pmatrix} A_1 \\ A_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\partial_3 A_2 \\ \partial_3 A_1 \\ \partial_1 A_2 - \partial_2 A_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \delta(x) \end{pmatrix}$$



span

$$C_0^\infty(\mathbb{R}) \hat{\otimes} C_0^\infty(\mathbb{R}) = \left\{ f \cdot g : f \in C_0^\infty(\mathbb{R}_{x_1}), g \in C_0^\infty(\mathbb{R}_{x_2}) \right\}$$

is dense in $L^2(\mathbb{R}^2)$.

Lemma 11.1

$$D_1 = C_0(\mathbb{R} \setminus \{0\}) \hat{\oplus} C_0^\infty(\mathbb{R} \setminus \{0\})$$

\hat{P}_1 is ess.sa. on D_1 .

It is enough to show that

$\therefore (\hat{P}_1 + i) D_1$ is dense.

Suppose that

$$\langle \varphi, (\hat{P}_1 + i) f \cdot g \rangle = 0$$

$$\int \overline{\varphi(x_1, x_2)} \cdot (\hat{P}_1 + i) f(x_1) g(x_2) dx_1 dx_2$$

$$\int |\varphi|^2 dx_1 dx_2 < \infty \Leftrightarrow \int |\varphi(x_1, x_2)|^2 dx_1 < \infty \text{ a.e. } x_2$$

Fubini's lemma

Hence it follows that

$$\int |\varphi(x_1, x_2)| |(\hat{P}_1 + i) f(x_1)| dx_1 < \infty \text{ a.e. } x_2$$

$$\text{Let } u(x_2) = \int \varphi(x_1, x_2) (\hat{P}_1 + i) f(x_1) dx_1$$

is finite a.e. x_2 and

$$|u(x_2)|^2 \leq \int |\varphi(x_1, x_2)|^2 dx_1 \int |(\hat{P}_1 + i) f(x_1)|^2 dx_1$$

$$\text{means that } \int |u(x_2)|^2 dx_2 < \infty$$

$$\therefore u \in L^2(\mathbb{R}_{x_2})$$

By the def of u we also see that

$$\langle u, g \rangle = 0 \quad \forall g \in C_0^\infty(\mathbb{R})$$

$$\int \varphi(x_1, x_2) (\hat{P}_1 + i) f(x_1) g(x_2) dx_1 dx_2$$

$$\therefore \mathcal{U} = 0$$

$$\int (\varphi(x_1, x_2))^* (P_i + i) f(x_1) dx_1 = 0 \quad \underline{\text{a.e. } x_2}$$

Since $C_0^\infty(\mathbb{R})$ is a core of P_i , $(P_i + i)|_{C_0^\infty(\mathbb{R})}$ is dom. and $\varphi(x_1, x_2) = 0$ a.e. x_2

More precisely $\exists N \subset \mathbb{R} \text{ s.t. } |N| = 0$

$\varphi(0, x_2) = 0$ as L^2 -function for $x_2 \in N$.

Hence $\int |\varphi(x_1, x_2)|^2 dx_1 dx_2$

$$= \int_{\mathbb{R}^2 \setminus N} |f|^2 + \int_{N \subset \mathbb{R}^2} |f|^2$$

$$= \int_{\mathbb{R}^2 \setminus N} |f(x_1, x_2)|^2 dx_1 dx_2 = 0.$$

Thus P_i is ess. s. a. on D_i

Similarly we see that P_2 is ess. s. a. on D_2 .

Weyl relation of

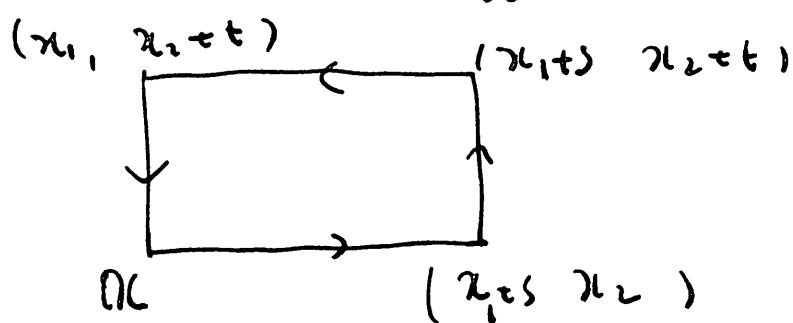
$$(Q_j, P_j) \quad ?$$

Lemma II. 4

$$e^{isQ_j} e^{it\bar{P}_j} = e^{-ist\delta_{ij}} e^{it\bar{P}_i} e^{isQ_j}$$

$$\left\{ \begin{array}{l} e^{is\bar{P}_1} e^{it\bar{P}_2} = e^{-it\bar{\Phi}_{st}} e^{it\bar{P}_2} e^{is\bar{P}_1} \end{array} \right.$$

where $\bar{\Phi}_{st} = \int_{C_{st}} A(x) \cdot dx \leftarrow$ line integral



$$\begin{aligned}
 & \because e^{is\bar{P}_1} e^{it\bar{P}_2} f \\
 &= e^{is\bar{P}_1} \tilde{e}^{-i\oint_0^t A_2(x+se_2) ds} f(x+te_2) \\
 &= \tilde{e}^{-i\oint_0^s A_1(x+s'e_1) ds'} \tilde{e}^{-i\oint_0^t A_2(x+te_2+se_1) ds} \\
 &\quad \uparrow \tilde{e}^* \\
 &\quad f(x+te_2+se_1)
 \end{aligned}$$

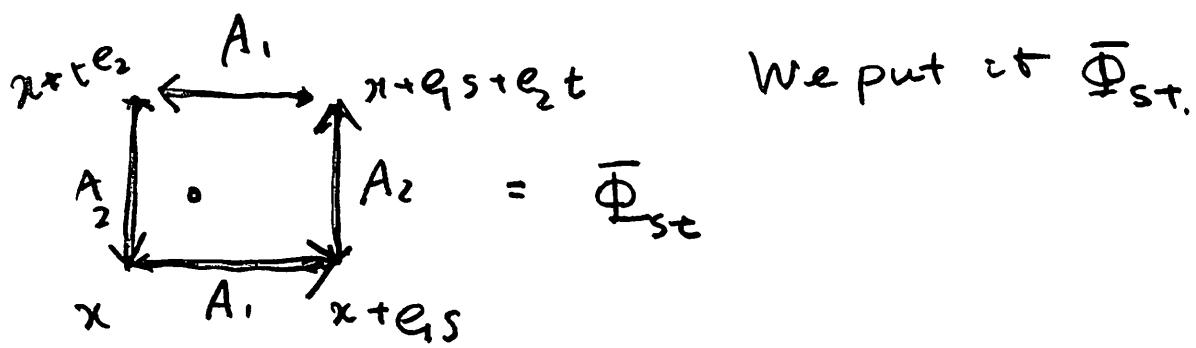
On the other hand we derive that

$$\begin{aligned}
 & e^{it\bar{P}_2} e^{is\bar{P}_1} f \\
 &= \tilde{e}^{-i\oint_0^t A_2(x+s'e_2) ds'} \tilde{e}^{-i\oint_0^s A_1(x+s'e_1+te_2) ds'} \\
 &\quad \uparrow \tilde{e}^* \\
 &\quad f(x+se_1+te_2)
 \end{aligned}$$

Hence we have

$$e^{is\bar{P}_1} e^{it\bar{P}_2} = \frac{-Y+X}{e} \frac{i+t\bar{P}_2}{e} \frac{is\bar{P}_1}{e^Y} e^Y f$$

$$\begin{aligned}
 -Y+X &= -i\oint \left(-\int_0^t A_2(x+s'e_2) ds' + \int_0^s A_1(x+s'e_1+te_2) ds' \right. \\
 &\quad \left. + \int_s^t A_1(x+s'e_1) ds' + \int_0^t A_2(x+te_2+se_1) ds \right)
 \end{aligned}$$



Then the lemma follows
when C includes 0, by Green's formula we have

$$\begin{aligned} \bar{\Phi}_{st} &= \int_C \mathbf{A} \cdot d\mathbf{x} = \int_C A_1 dx_1 + A_2 dy_2 \\ &= \iint_{\square} -\partial x_2 A_1 + \partial x_1 A_2 \, dx_1 dx_2. \end{aligned}$$

When C includes 0, in general

$$\bar{\Phi}_s \neq \iint_{\square} -\partial x_2 A_1 + \partial x_1 A_2 \, dx_1 dx_2$$

- Lemma 11.5

$$[\bar{P}_1, \bar{P}_2] = \iint_{\square} (-\partial_2 A_1 + \partial_1 A_2) \, dx_1 dx_2$$

$\in \mathbb{C}(112/204)$
 $\times \mathbb{C}(112/204)$

If

Def 11.6: For all $s, t \in \mathbb{R}^2$, $\bar{\Phi}_{st} \in \frac{2\pi}{q} \mathbb{Z}$ is satisfies,
then the magnetic field is locally quantized".
we call ~~as~~ as

Especially $q \in \frac{2\pi}{g} \mathbb{Z}$. the $\bar{\Phi} = \iint_{\square} B \, dx_1 dx_2 = q \in \frac{2\pi}{g} \mathbb{Z}$
if $\square \rightarrow 0$.

In particular when the magnetic field is locally \neq
 we see that

$$e^{itP_1} e^{isP_2} = e^{isP_2} e^{itP_1} \quad \text{for all } s, t \in \mathbb{R}.$$

On the other hand the magnetic field is not "really quantized"

$$e^{itP_1} e^{isP_2} \neq e^{isP_2} e^{itP_1}$$

plane $e^{i\frac{\pi}{2}\Phi_{st}}$ appears.

Thm 11.6

$\{P_j, Q_j\}$ is unitarily
~~equivalent to~~ with $\{\hat{P}_j, \hat{Q}_j\}$

① locally quantized \Rightarrow

~~P_1, P_2, Q_1, Q_2~~

② not locally quantized \Rightarrow

$\{P_j, Q_j\}$ is not unitarily
~~equivalent to~~ $\{\hat{P}_j, \hat{Q}_j\}$

§12 Timeop. and Weak Weyl rep.

Def 12.1 \mathcal{H} Hilbert space

Q sym (\neq s.a.), P s.a.

$\{H, \{Q, P\}\}$ is WWR of CCR

$\Leftrightarrow e^{itP} D(Q) \subset D(Q)$ and

$$e^{itP} Q \varphi = (Q + t) e^{itP} \varphi.$$

$$\text{i.e. } e^{itP} Q \subset (Q + t) e^{itP}$$

Remark $\{Q, P\}$ Weyl rep \Rightarrow WWR.

$$\therefore e^{isQ} e^{itP} = e^{-ist} e^{itP} e^{isQ},$$

$\{Q, P\}$

Lemma $\frac{\text{WWIR}}{12.2} \Rightarrow \{\bar{Q}, P\}$ WWIR

$$\because \varphi \in D(\bar{Q}) \quad \exists \varphi_n \in D(Q) \text{ st} \quad \varphi_n \rightarrow \varphi \\ Q\varphi_n \rightarrow Q\varphi$$

$$e^{itP} \bar{Q} \varphi_n = (\bar{Q} + t) e^{itP} \varphi_n$$

$$e^{itP} \bar{Q} \varphi \quad (\bar{Q} + t) e^{itP} \varphi \quad (\because \bar{Q} \text{ is closed.})$$

Cor 12.3 { H, iQ P } wwr and Q s.a

$\Rightarrow \{Q P\}$ is Weyl rep.



$$e^{itP} D(Q) \subset D(Q) \quad \forall t$$

$$\therefore D(Q) \subset e^{-itP} D(Q) \subset D(Q)$$

$$\therefore \underline{e^{itP} D(Q) = D(Q)} \quad \forall t$$

$$\therefore Q \Psi = \underbrace{e^{-itP}}_{\text{equal.}} (Q+t) e^{itP} \Psi$$

$$\therefore Q = e^{itP} (Q-t) e^{-itP}$$

$$\downarrow$$

$$e^{isQ} = e^{itP} e^{is(Q-t)} e^{-itP}$$

$$\therefore e^{isQ} e^{itP} = e^{-ist} e^{itP} e^{isQ}$$

Re/Defn

$$Q = U (Q-t) \bar{U}$$

$$e^{isQ}, \quad U e^{is(Q-t)} \bar{U}^{-1} = U_s$$

$\{U_s\}$ strongly cont one-parameter unitary group

$$U_s = e^{isX} \quad \text{self-adjoint sp.}$$

$$\frac{d}{ds} U_s \Big|_{s=0} = iU(Q-t)U^{-1} = iX$$

$$\therefore X = U(Q-t)U^{-1} = Q \quad //$$

- By Cor 12.3, WWR has a meaning only when
Q is not self-adjoint,

Lemma 12.4 { H, { Q P } } WWR

Then $G_P(Q) = \emptyset$.

∴ Suppose that $P\varphi = \lambda\varphi \quad (\varphi \neq 0)$

$$(\varphi, e^{itP}Q\varphi) = e^{it\lambda}(\varphi, Q\varphi)$$

$$(\varphi, (Q+t)e^{itP}\varphi) = e^{it\lambda}(\varphi, (Q+t)\varphi)$$

$$\therefore t(\varphi, \varphi) = 0 \quad \forall t \quad \therefore (\varphi, \varphi) = 0. //$$

$$i\partial_t \langle e^{itP} \psi | = -i\partial_t \langle e^{itP} \psi | = i\partial_t \langle e^{itP} \psi |$$

$$Q e^{itP} = e^{itP} (-t + 0)$$

Theorem 12.5 $\{H, (Q, P)\}$ WWR \Leftrightarrow

Assume that $P > -\infty$. Then \bar{Q} is not s.a.

(\because) Suppose that \bar{Q} is self-adjoint.
 Then we ~~must~~ see that (Q, P) satisfies
 WR. Hence $\bar{Q} \cong \hat{x}$, $P \cong \hat{P}$.

$S(\hat{P}) = \mathbb{R}$ ~~It is contradiction~~ This contradicts
 $P > -\infty$

Def 12.6

Let H be self-adjoint op. The symmetric
 operator T such that

$\{H, \{T, H\}\}$ is WWR is called
 the time operator associated with H .
Strong

Let $H = -\frac{1}{2} \Delta$ free Hamiltonian. $H > -\infty$.
 Then \bar{T} is not self-adjoint.

$$e^{itH} T = (T + t) e^{itH}$$

Defn 12.7 Let H be s.a.

The symmetric op T st $D(H) \cap D(T) \neq \emptyset$

$\forall \varphi, \psi \in D(T) \cap D(H)$

$$(T\varphi, H\psi) - (H\varphi, T\psi) = i(\varphi, \psi).$$

is called the weak time op.

$$\frac{m}{2} (Q\bar{P}^{\dagger} + \bar{P}^{\dagger}Q) = T_{AB} \quad H = \frac{1}{2m} P^2$$

$$e^{itH} T_{AB} = \frac{m}{2} e^{itH} (Q\bar{P}^{\dagger} + \bar{P}^{\dagger}Q)$$

=

Lemma 12.8 T_{AB} is the weak time op.
of $\frac{1}{2m} P^2$ on $F = \mathcal{F}^1 C_0^\infty(\mathbb{R} \setminus \{0\})$

$\therefore \frac{m}{2} \cdot \frac{1}{2m} ((Q\bar{P}^{\dagger} + \bar{P}^{\dagger}Q) \varphi, P^2 \psi)$
 $- (P^2 \varphi, (Q\bar{P}^{\dagger} + \bar{P}^{\dagger}Q) \psi) = i(\varphi, \psi).$

Formally we have

$$\begin{aligned} P^2 (Q\bar{P}^{\dagger} + \bar{P}^{\dagger}Q) &= \cancel{1/2} P \bar{P}^{\dagger} \cancel{P} + \cancel{P} \bar{P}^{\dagger} \cancel{Q} = \cancel{-1/2} P \bar{P}^{\dagger} \\ 2iP\bar{P}^{\dagger} + QP^2\bar{P}^{\dagger} + \bar{P}^{\dagger}2iP + \bar{P}^{\dagger}Q P^2 \\ &= 4i + (Q\bar{P}^{\dagger} + \bar{P}^{\dagger}Q) P^2 \end{aligned}$$

前半部分是 domain & check 33

$$QF \subset F, \quad PF \subset F, \quad P^{-1}F \subset F$$

$$\therefore F \ni f \therefore f = T^{-1}g$$

$$x T^{-1}g = T^{-1}T x T^{-1}g = T^{-1} - i \frac{d}{dx} g \in L^2(\mathbb{R} \setminus \{0\})$$

$$P T^{-1}g = T^{-1} T P T^{-1}g = T^{-1} g$$

$$P^{-1} T^{-1}g = T^{-1} T P^{-1} T^{-1}g = T^{-1} g$$

$$\text{正确} \Leftrightarrow F^* P F = x \text{ 时}$$

Spectral measure vs

$$\begin{aligned} (\varphi, F^* P F \varphi) &= \int \lambda d(F\varphi, E_P(\lambda) F\varphi) \\ &= \int \lambda d(\varphi, P E_P(\lambda) F\varphi) \\ &= (\varphi, x\varphi) \end{aligned}$$

$$F^* E_P(\lambda) F = E_{x\lambda}(\lambda)$$

Hence we have

$$\begin{aligned} (\varphi, F f(P) F \varphi) &= \int f(\lambda) d(F\varphi, E_P(\lambda) F\varphi) \\ &= (\varphi, f(x)\varphi) \end{aligned}$$

§13 Time op associated with Schrödinger op.

Scattering theory

$$H_0 = -\frac{1}{2}\Delta, \quad H_V = -\frac{1}{2}\Delta + V \quad V: \text{multiplication}$$

Schrödinger op
(potential)

$\sim e^{itH_V} e^{-ith_0} \quad \rightarrow \quad \psi = e^{-ith_V} \psi_0$

$i \frac{\partial}{\partial t} \psi = H_V \psi \quad \text{Schrödinger equation}$

\circ free $e^{-ith_0} \varphi_+$ \circ free $t = +\infty$
 $t = -\infty \rightarrow \varphi_-$ \rightarrow Interaction
 $e^{-ith_V} \varphi$

$\sim \lim_{t \rightarrow -\infty} \| e^{-ith} \varphi_+ - e^{-ith_V} \varphi \| = 0$

$\lim_{t \rightarrow +\infty} \| e^{-ith} \varphi_- - e^{-ith} \varphi \| = 0$

Formally $\varphi_\pm = \lim_{t \rightarrow \mp\infty} e^{ith_V} e^{-ith_0} \varphi_\pm$
 $= \Omega_{\pm} \varphi_\pm$

$\varphi_+ \rightarrow \varphi_\pm$

$\Omega_+ \varphi_+ = \varphi_-$

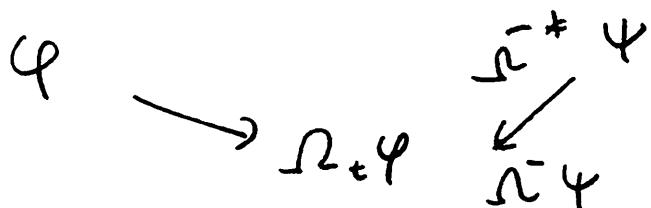
$$(\Omega^+ \Psi, \Omega^- \Psi)$$

Part $e^{-iHt} \Psi$

We want to know how it looks for future
that is, we look at $e^{-iHt} \underline{\Omega^+ \Psi}$

What is ^{the} probability of finding that

the state is the free state $e^{-iHt} \Psi$ asymptotically
in the future.



$$(\Psi, \underbrace{\Omega_-^* \Omega_+ \Psi}_{S})$$

S-operator or S-matrix

Property of wave op.

Intertwining property

Lemma 13.1

$$e^{-isHv} \Omega_{\pm} = \Omega_{\pm} e^{-isH_0}$$

$$\begin{aligned} \therefore \lim_{t \rightarrow \mp\infty} e^{-isHv} e^{itHv} e^{-itH_0} \Psi &= \lim_{t \rightarrow \mp\infty} e^{i(t-s)Hv} e^{-i(t-s)H_0} e^{-isH_0} \Psi \\ &= \Omega_{\pm} e^{-isH_0} \Psi. \end{aligned}$$

In particular $\text{Ran } \Omega^\pm = \mathcal{H}_\pm$

\mathcal{H}_\pm is an invariant subspace for Hv .

$$\bar{e}^{isHv} \Omega_\pm \varphi = \Omega_\pm \bar{e}^{isH_0} \varphi$$

$$\text{thus } e^{-itHv} : \mathcal{H}_\pm \rightarrow \mathcal{H}_\pm \quad "$$

- Cook method -

$$\begin{aligned} S_T \varphi &= \int_0^T \frac{d}{dt} S_t \varphi - \varphi \\ &= \int_0^T e^{itHv} v \bar{e}^{-itH_0} \varphi - \varphi \end{aligned}$$

$$\exists \lim_{T \rightarrow \infty} \int_0^T \| v \bar{e}^{-itH_0} \varphi \| dt < \infty$$

$\Rightarrow \{S_T \varphi\}$ is Cauchy sequence

$$\therefore \lim_{T \rightarrow \infty} S_T \varphi$$

Similarly we can see that

$$\exists \lim_{T \rightarrow \infty} \int_0^T \| v e^{itH_0} \varphi \| dt < \infty$$

$$\exists \lim_{T \rightarrow -\infty} S_T$$

Example Agmon potential

$$V \text{ is Agmon potential} \Leftrightarrow V(x) = \frac{W(x)}{(1+x^2)^{\frac{1}{2}+\epsilon}}$$

s.t. $W(-\Delta + i)^{-1}$ is cpt op.

i.e., T is cpt $\stackrel{\text{def}}{\Leftrightarrow}$ TK is pre compact
for any bdd domain

~~TK is cpt~~

$$\text{eg. } W \in L^\infty \Rightarrow V = \frac{W}{(1+x^2)^{\frac{1}{2}+\epsilon}} = \frac{1}{(1+x^2)^{\frac{1}{2}+\frac{\epsilon}{2}}} \frac{W}{(1+x^4)^{\frac{\epsilon}{2}}}$$

Then Agmon - Kato - Kuroda

Let V be a Agmon potential.

Then $\exists \Omega^+$.

1

$$\text{Ran } \Omega_{\pm} = \boxed{\text{closed}} \text{ Hac} \\ \Leftrightarrow \text{AC.}$$

Proposition

isometry

$$\textcircled{1} \quad \Omega_{\pm} : \mathcal{H} \rightarrow \mathcal{H}_{\pm} = \text{Ran } \Omega_{\pm}$$

$$\textcircled{2} \quad \Omega_{\pm}^* \Omega_{\pm} = \mathbb{1}, \quad \Omega_{\pm} \Omega_{\pm}^* = \text{projections to} \\ \mathcal{H}_{\pm}.$$

$$\textcircled{3} \quad \Omega^* D(H_0) \subset D(H_V) \quad \text{and}$$

$$H_V \Omega^* = \Omega^* H_0 \quad \text{i.e. } H_V \mathcal{H}_{\pm} \subset \mathcal{H}_{\pm}$$

\therefore

(1) trivial

(2) ~~as Ω_{\pm} is isometric it follows that~~

$$\mathbb{1} = \underbrace{e^{isH_0} e^{-isH_V}}_{U_S} \underbrace{e^{isH_V} e^{-isH_0}}_{V_S} f = \Omega_{\pm}^* \Omega_{\pm} \varphi$$

$$V_S \varphi \rightarrow V \varphi$$

$$U_S V_S \varphi - U V \varphi = U_S V_S - U_S V_S \varphi + U_S V \varphi - U V \varphi$$

(3) follows from intertwining properties //

$$H_v \Gamma_{\mathcal{H}_{\pm}} = \bar{H}_v \quad \bar{H}_v \text{ is self-adjoint} \\ D(H_v) \cap \mathcal{H}_{\pm}$$

$$T = \Omega T_{AB} \Omega^*, \quad D(T) = \Omega D(T_{AB})$$

Thm (\bar{H}_v, T) is WWR.

$$\begin{aligned} & \circlearrowleft \bar{e}^{it\bar{H}} \Omega T_{AB} \Omega^* \Omega \varphi \\ &= \bar{e}^{itH} \Omega T_{AB} \Omega^* \Omega \varphi \\ &: \Omega \Omega^* \bar{e}^{itH} \Omega T_{AB} \Omega^* \Omega \varphi \\ &= \Omega \underbrace{\bar{e}^{-itH_0}}_{T_{AB}} \Omega^* \Omega \varphi \\ &= \Omega (T_{AB} - t) \bar{e}^{-itH_0} \Omega^* \Omega \varphi \\ &= \Omega (T_{AB} - t) \Omega^* \Omega \underbrace{\bar{e}^{-itH_0} \Omega^* \Omega \varphi}_{\bar{e}^{itH_v} \Omega \varphi} \\ &= (T - t \Omega \Omega^*) \bar{e}^{-it\bar{H}} \Omega \varphi \\ &= (T - t) \bar{e}^{-it\bar{H}} \Omega \varphi, \quad // \end{aligned}$$