# Enhanced binding of an $N$-particle system interacting with a scalar field II.Relativistic version 

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#### Abstract

An enhanced binding of $N$-relativistic particles coupled to a massless scalar bose field is investigated. It is not assumed that the system has a ground state for the zero-coupling. It is shown, however, that there exists a ground state for sufficiently large coupling. The proof is based on checking the stability condition and showing a uniform exponential decay of infrared regularized ground states.


2010 Mathematics Subject Classification: Primary 81Q10; Secondary 62M15
Keywords: Nelson model, ground state, enhanced biding

## §1. Preliminaries

## §1.1. Introduction

Non-perturbative analysis of eigenvalues embedded in the continuous spectrum has been developed in the last decade and it has been applied to the mathematically rigorous analysis of the spectra of self-adjoint Hamiltonians in quantum field theory. Among other things, stability and instability of a quantum mechanical particle coupled to a quantum field have been investigated.

The Hamiltonian in quantum field theory is realized as a self-adjoint operator of the form

$$
\begin{equation*}
K_{0}+\alpha K_{\mathrm{I}}, \tag{1}
\end{equation*}
$$

[^0]acting in a Hilbert space over $\mathbb{C}$ for each values of coupling constant $\alpha \in \mathbb{R}$. Here $K_{0}$ is the subject term and $K_{\mathrm{I}}$ an interaction term. We are concerned with ground states of $K_{0}+\alpha K_{\mathrm{I}}$ in this paper.

Let $\sigma(T)$ be the spectrum of a self-adjoint operator $T$.
Definition 1.1. (Ground state and ground state energy) Let $T$ be a selfadjoint operator bounded from below. Then the bottom of the spectrum, $E_{0}(T)=$ $\inf \sigma(T)$, is called a ground state energy of $T$. Let $E_{0}(T)$ be an eigenvalue of $T$. Then the eigenvector $f$ associated with $E_{0}(T)$ is called a ground state of $T$, i.e., $T f=E_{0}(T) f$.

Generally the bottom of the spectrum of the zero-coupling Hamiltonian $K_{0}$ is embedded in the continuous spectrum. Hence the spectral analysis of $K_{0}+\alpha K_{\mathrm{I}}$ is regarded as the perturbation problem of embedded eigenvalues. Although an analytic perturbation theory of the discrete spectrum is established for a various type of self-adjoint operators, the perturbation of embedded eigenvalues are crucial and it is not straightforward to apply the perturbation theory of discrete spectra. Then it is subtle to show the existence of a ground state of $K_{0}+\alpha K_{\mathrm{I}}$ even for small values of coupling constant. Moreover it is not necessarily that a ground state exists for $K_{0}+\alpha K_{\mathrm{I}}, \alpha \neq 0$, even when $\inf \sigma\left(K_{0}+\alpha K_{\mathrm{I}}\right)>-\infty$ and $K_{0}$ has ground state.

The existence and the absence of a ground state for physically reasonable Hamiltonians of quantum field theory has been however proven so far under some assumptions. The existence of the ground state of the standard Nelson Hamiltonian [Nel64] was in particular proven in e.g., [BFS98, Spo98, Ger00, Sas05], where the most basic assumptions for proving the existence of a ground state are
(1) infrared regular condition,
(2) the existence of ground state of $K_{0}$.

In particular assumption (2) tells us that Hamiltonians $K_{0}+\alpha K_{\mathrm{I}}$ also has a ground state for arbitrary values of $\alpha$.

It is found however that an interaction with quantum fields enhances the binding energy, which suggests that a Hamiltonian with sufficiently large coupling constants may have a ground state whether $K_{0}$ has a ground state or not. If $K_{0}+\alpha K_{\mathrm{I}}$ with sufficiently large coupling constants has a ground state whether $K_{0}$ has a ground state or not, then it is said that enhanced binding occurs. Enhanced binding is initiated by [HSOT] and in the previous paper [HSO8] enhanced binding is shown for a system of $N$-nonrelativistic particles governed by Schödinger operator and linearly coupled to a massless scalar bose field. In this paper replacing the nonrelativistic particles with relativistic ones, we show the enhanced binding.

Finally we give some comments on related works on enhanced binding. The enhanced binding is studied so far for the various kind of models in quantum field theory. In [HSOT] the enhanced binding of the Pauli-Fierz model with the dipole approximation is studied. In [HSSTI] a complement result of [HSOT] is established, i.e., the absence of ground state for sufficiently small coupling constant is shown. See also [AK033, BLV0.5, BV04, CHH03, CVV03, HVV03] for the related works.

## §1.2. Main results

The total Hamiltonian we consider is of the form

$$
\begin{equation*}
H^{V}=H_{0}+\kappa H_{\mathrm{I}} \tag{2}
\end{equation*}
$$

The operator $H_{0}=H_{0}(\kappa)$ describes the zero coupling Hamiltonian and is given by

$$
\begin{aligned}
& H_{0}=H_{\mathrm{p}}+\kappa^{2} H_{\mathrm{f}} \\
& H_{\mathrm{p}}=\sum_{j=1}^{N}\left(\sqrt{-\Delta_{j}+m_{j}^{2}}-m_{j}+V\left(x_{j}\right)\right)
\end{aligned}
$$

where $m_{j}>0$ is the mass of the $j$-th particle, $V(x)$ an external potential, $H_{\mathrm{f}}$ the free field Hamiltonian, and $\kappa>0$ denotes a scaling parameter. The operator $H_{I}$ describes a particle-boson linear interaction. We notice that there are no pair potentials in $H^{V}$ and $V$ is assumed to be independent of $j$ for simplicity. Introducing a dressing transformation $e^{i T}$ to derive an effective potential $V_{\text {eff }}$, we transform $H^{V}$ as

$$
\begin{equation*}
e^{-i T} H^{V} e^{i T}=h_{\mathrm{eff}}^{V}+\kappa^{2} H_{\mathrm{f}}+H_{\mathrm{R}}(\kappa) \tag{3}
\end{equation*}
$$

where $h_{\text {eff }}^{V}$ is the effective particle Hamiltonian given by

$$
\begin{equation*}
h_{\mathrm{eff}}^{V}=\sum_{j=1}^{N}\left(\sqrt{-\Delta_{j}+m_{j}^{2}}-m_{j}+V\left(x_{j}\right)\right)+V_{\mathrm{eff}}\left(x_{1}, \ldots, x_{N}\right) \tag{4}
\end{equation*}
$$

and $H_{\mathrm{R}}(\kappa)$ a remainder term to be regarded as a perturbation of $h_{\text {eff }}^{V}+\kappa^{2} H_{\mathrm{f}}$. Compensating for deriving $V_{\text {eff }}$ through the dressing transformation, we have the remainder term $H_{\mathrm{R}}(\kappa)$ which is unfortunately no longer linear and is the complicated form:

$$
H_{\mathrm{R}}(\kappa)=\sum_{j=1}^{N}\left(\sqrt{\left(-i \nabla_{j}-\frac{1}{\kappa} A_{j}\left(x_{j}\right)\right)^{2}+m_{j}^{2}}-\sqrt{-\Delta_{j}+m_{j}^{2}}\right)
$$

where $A_{j}$ denotes some quantum vector field. Nevertheless it turns to be a small perturbation for sufficiently large $\kappa$ in some sense.

We are interested in the existence of a ground state of $H^{V}$, equivalently that of $e^{-i T} H^{V} e^{i T}$. We do not however assume the existence of ground states of $H_{0}$. As will be shown below, the enhanced binding is observed by the transformed Hamiltonian (3) rather than $H^{V}$ itself. Since we consider a massless boson, the bottom of the spectrum of $H^{V}$ is the edge of the continuous spectrum and the regular perturbation can not be applied. Then it is not clear whether $e^{-i T} H^{V} e^{i T}$ has a ground state even when $h_{\text {eff }}^{V}$ has a ground state.

The conventional approach is to assume an infrared cutoff in the form factor $\hat{\lambda}$ in $H_{\text {I }}$ by setting $\hat{\lambda}(k) \Gamma_{|k|>\sigma}, H^{V}$ with cutoff $\hat{\lambda} \Gamma_{|k|>\sigma}$ is denoted by $H_{\sigma}^{V}$, and to show the existence of a ground state $\Phi_{\sigma}$ of $H_{\sigma}^{V}$. The vector $\Phi_{\sigma}$ is called an infrared-regularized ground state. Then one is left to show that the sequence of ground states $\Phi_{\sigma}$ has a non-zero weak limit $\Phi$ as $\sigma \rightarrow 0$, which is the desired ground state of $H^{V}$. We show in this paper:
(A) the stability condition for $H^{V}$ is satisfied (Lemma [3.7),
(B) infrared-regularized ground states $\Phi_{\sigma}$ has exponential decay uniformly with respect to the infrared cutoff parameter $\sigma$ (Lemma [3.8),
(C) we prove that (1) stability condition and (2) exponential decay imply the existence of a ground state of $H^{V}$ (Appendix 四),
(D) we show that there exist $\bar{\alpha}>0$ and $\kappa_{0}$ such that for each $\kappa>\kappa_{0}, H^{V}$ has the unique ground state for $|\alpha| \in(\bar{\alpha}, \bar{\alpha}(\kappa))$ with some $\bar{\alpha}(\kappa)$ (Theorem [2.3]).
Statement (D) describes the enhanced binding and this is the main theorem in this paper.

## §1.3. Strategies

We explain more technical improvement of this paper.
(Reduction to the stability condition of $h_{\text {eff }}^{V}$ ) The stability condition is introduced in [GLLOT] to show the existence of ground state of the non-relativistic quantum electrodynamics. The key ingredient in this paper is that we reduce the stability condition of $H^{V}$ to that of $h_{\text {eff }}^{V}$ in Lemma B.Cl. Namely we show that the stability condition of $h_{\text {eff }}^{V}$ implies that of $H^{V}$. These are proven by energy comparison inequality derived by functional integration of the heat semigroup generated by (3) (Lemma [32) and a simple variation principle (Lemma [3.3), hence we focus on showing the stability condition on $h_{\text {eff }}^{V}$ instead of $H^{V}$.
(Uniform exponential localization by functional integrations) Our method is a minor but nontrivial modification of [HS08] and a mixture of [Ger00, GLLOT]. We do not assume the compactness condition on $H_{\mathrm{p}}$, which is entered in
[Ger00]. Instead of this we show an exponential localization of infrared-regularized ground states, $\left\|\Phi_{\sigma}(x)\right\|_{\mathscr{F}} \leq C_{\delta} e^{-\delta|x|}$, which is derived through functional integrations in Lemma [3.8. The crucial point is to show that this localization is uniform on $\sigma>0$, i.e., $C_{\delta}$ and $\delta$ are independent of $\sigma>0$.
(Scaling parameter) The scaling parameter introduced in this paper can be obtained by replacing the annihilation operator $a$ and the creation operator $a^{*}$ with $\kappa a$ and $\kappa a^{*}$, respectively. This scaling is introduced in [Dav77, Dav79, Hir99] and the scaling limit as $\kappa \rightarrow \infty$ is called the weak coupling limit. Roughly speaking at least in the nonrelativistic domain $H_{\mathrm{p}} \cong-\frac{1}{2 m} \Delta+V$, and then

$$
H^{V}=\kappa^{2}\left(\kappa^{-2} H_{\mathrm{p}}+H_{\mathrm{f}}+\kappa^{-1} H_{\mathrm{I}}\right)
$$

with

$$
\kappa^{-2} H_{\mathrm{p}} \cong-\frac{1}{2 m \kappa^{2}} \Delta+\frac{1}{\kappa^{2}} V
$$

Thus we interpret that enhanced binding of $H^{V}$ occurs when sufficiently heavy particle mass and shallow external potential are assumed. Alternate explanation of the scaling parameter is the tool to derive a Markov process from $e^{-t H^{V}}$. Although the scalar product $\left(f \otimes \Omega, e^{-t H^{V}} g \otimes \Omega\right)$ does not define a Markov process, $\left(f, e^{-t\left(h_{\text {eff }}-E_{\text {diag }}\right)} g\right)$ does with generator $h_{\text {eff }}-E_{\text {diag }}$. This can be obtained by the scaling limit:

$$
\left(f \otimes \Omega, e^{-t H^{V}} g \otimes \Omega\right) \rightarrow\left(f, e^{-t\left(h_{\mathrm{eff}}-E_{\mathrm{diag}}\right)} g\right)
$$

as $\kappa \rightarrow \infty$. This can be done in a similar manner to [Hir99]. Precisely speaking, if $h_{\text {eff }}$ has a unique strictly positive ground state $\phi_{\mathrm{p}}$, then there exists a diffusion process $\left(Y_{t}\right)_{t \geq 0}$ such that

$$
\left(f \phi_{\mathrm{p}}, e^{-t\left(h_{\mathrm{eff}}-E_{\text {diag }}\right)} g \phi_{\mathrm{p}}\right)=\mathbb{E}\left[f\left(Y_{0}\right) g\left(Y_{t}\right)\right],
$$

where $\mathbb{E}$ denotes the expectation, and $\left(Y_{t}\right)_{t \geq 0}$ is the so-called $P(\phi)_{1}$ process. See e.g. [GHPS12] for the construction of $P(\phi)_{1}$ process.

The organization of this paper is as follows.
In the remainder of Section 1 we define the Nelson model with a relativistic kinetic term, and introduce a scaling parameter $\kappa>0$. In Section 2 we introduce a dressing transformation, and mention the stability condition and uniform exponential decay of $\Phi_{\sigma}(x)$. In Section 3 we prove the stability condition in Section 3.1 and uniform exponential decay in Section 3.2, and in Section 3.3 we show the enhanced binding.

In Appendix $\mathbb{T}^{\square}$ we show that the relativistic version of the stability condition also implies the existence of the ground state. In Appendix ${ }^{6}$ we review fundamental properties of the bottom of the essential spectrum of relativistic Schrödinger operator. In Appendix we give the functional integral representation of $e^{-t H^{V}}$
and show some inequality used in the proof of exponential decay of infrared regularized ground states. In Appendix 0 we derive some energy comparison inequality of the translation invariant Hamiltonian $\sum_{j=1}^{N}\left(\sqrt{-\Delta_{j}+m_{j}}-m_{j}+V\left(x_{j}\right)\right)$.

## §1.4. Definition

We begin with giving the definition of the Nelson model with $N$-relativistic particles. Throughout we assume $N \geq 2$ and the dimension of state space is $d \geq 3$. The Hamiltonian of the Nelson model can be realized as a self-adjoint operator on the tensor product of $L^{2}\left(\mathbb{R}^{d N}\right)$ and the boson Fock space $\mathscr{F}$ over $L^{2}\left(\mathbb{R}^{d}\right)$,

$$
\begin{equation*}
\mathscr{H}=L^{2}\left(\mathbb{R}^{d N}\right) \otimes \mathscr{F} \tag{5}
\end{equation*}
$$

Here $\mathscr{F}$ is defined by $\mathscr{F}=\oplus_{n=0}^{\infty} L_{\text {sym }}^{2}\left(\mathbb{R}^{d n}\right)$, where $L_{\text {sym }}^{2}\left(\mathbb{R}^{d n}\right)$ is the set of square integrable functions such that $\Psi\left(x_{1}, \cdots, x_{n}\right)=\Psi\left(x_{\sigma(1)}, \cdots, x_{\sigma(n)}\right)$ for any $n$ degree permutation $\sigma$. A vector $\Psi \in \mathscr{F}$ is written as $\Psi=\left\{\Psi^{(n)}\right\}_{n=0}^{\infty}$ with $\Psi^{(n)} \in L_{\text {sym }}^{2}\left(\mathbb{R}^{d n}\right)$, and the Fock vacuum $\Omega \in \mathscr{F}$ is defined by $\Omega=\{1,0,0, \ldots\}$. We denote by $a(f)$ and $a^{*}(f), f \in L^{2}\left(\mathbb{R}^{d}\right)$, the annihilation and creation operator in $\mathscr{F}$, respectively. They satisfy canonical commutation relations:

$$
\begin{equation*}
\left[a(f), a^{*}(g)\right]=(\bar{f}, g) \mathbb{1}, \quad[a(f), a(g)]=0=\left[a^{*}(f), a^{*}(g)\right] \tag{6}
\end{equation*}
$$

and the adjoint relation $a^{*}(f)=(a(\bar{f}))^{*}$ holds. Throughout this paper $(F, G)_{\mathcal{K}}$ denotes the scalar product on Hilbert space $\mathcal{K}$, which is linear in $G$ and antilinear in $F$. We omit $\mathcal{K}$ until confusions arises. We informally write as $a^{\#}(f)=$ $\int a^{\#}(k) f(k) d k, a^{\#}=a, a^{*}$. The second quantization of the closed operator $A$ on $L^{2}\left(\mathbb{R}^{d}\right)$ is denoted by $\mathrm{d} \Gamma(A)$. The free field Hamiltonian $H_{\mathrm{f}}$ is the self-adjoint operator on $\mathscr{F}$, which is given by the second quantization of the multiplication operator $\omega(k)=|k|$ on $L^{2}\left(\mathbb{R}^{d}\right)$ :

$$
\begin{equation*}
H_{\mathrm{f}}=\mathrm{d} \Gamma(\omega) \tag{7}
\end{equation*}
$$

Next we introduce particle Hamiltonian. We suppose that the $N$-relativistic particles are governed by the relativistic Schrödinger operator $H_{\mathrm{p}}$ of the form:

$$
\begin{equation*}
H_{\mathrm{p}}=\sum_{j=1}^{N}\left(\Omega_{j}+V_{j}\right) \tag{8}
\end{equation*}
$$

which is acting on $L^{2}\left(\mathbb{R}^{d N}\right)$, where

$$
\begin{equation*}
\Omega_{j}=\Omega_{j}\left(p_{j}\right)=\sqrt{p_{j}^{2}+m_{j}^{2}}-m_{j} \tag{9}
\end{equation*}
$$

is the $j$-th particle Hamiltonian with momentum $p_{j}=-i \nabla_{x_{j}}$ and mass $m_{j}>0$. $V_{j}=V\left(x_{j}\right)$ denotes an external potential. In this paper, we assume that there is no interparticle potential for simplicity.

The Hamiltonian of the relativistic Nelson model is then defined by

$$
\begin{equation*}
H^{V}=H_{0}+\kappa H_{\mathrm{I}}, \tag{10}
\end{equation*}
$$

where the zero-coupling Hamiltonian $H_{0}$ is given by

$$
\begin{equation*}
H_{0}=H_{\mathrm{p}} \otimes \mathbb{1}+\kappa^{2} \mathbb{1} \otimes H_{\mathrm{f}} \tag{11}
\end{equation*}
$$

and $\kappa>0$ a scaling parameter. $H_{\mathrm{I}}$ denotes the linear interaction given by

$$
\begin{equation*}
H_{\mathrm{I}}=\alpha \sum_{j=1}^{N} \int_{\mathbb{R}^{d N}}^{\oplus} \phi_{j}\left(x_{j}\right) d X \tag{12}
\end{equation*}
$$

under the identification: $\mathscr{H} \cong \int_{\mathbb{R}^{d N}}^{\oplus} \mathscr{F} d X$, where $d X=d x_{1} \cdots d x_{N}$. Here $\alpha \geq 0$ is a coupling constant, and the scalar field $\phi_{j}(x)$ is given by

$$
\begin{equation*}
\phi_{j}(x)=\overline{\frac{1}{\sqrt{2}} \int_{\mathbb{R}^{d}}\left(a^{*}(k) \hat{\lambda}_{j}(-k) e^{-i k x}+a(k) \hat{\lambda}_{j}(k) e^{i k x}\right) d k} \tag{13}
\end{equation*}
$$

for each $x \in \mathbb{R}^{d}$ with ultraviolet cutoff functions $\hat{\lambda}_{j}$. Here $\overline{\{\cdots\}}$ denotes the operator closure. The standard choice of the ultraviolet cutoff is

$$
\hat{\lambda}_{j}(k)=(2 \pi)^{-d / 2} \omega(k)^{-1} \mathbb{1}_{|k| \leq \Lambda},
$$

where $\mathbb{1}_{X}$ denotes the characteristic function of $X$. We do not however fix any special cutoff function.

Throughout this paper we assume the following three conditions:
(V) $V(-\Delta+1)^{-1 / 2}$ is compact.
(UV) $\hat{\lambda}_{j}(-k)=\hat{\lambda}_{j}(k) \geq 0$ and $\hat{\lambda}_{j} \in L^{2}\left(\mathbb{R}^{d}\right)$ for $j=1, \ldots, N$.
(IR) $\hat{\lambda}_{j} / \omega \in L^{2}\left(\mathbb{R}^{d}\right)$ for $j=1, \ldots, N$.
Assumption (V) implies that $V$ is infinitesimally small with respect to self-adjoint operator $\sqrt{-\Delta+m^{2}}-m$ for all $m \geq 0$. Hence, by the Kato-Rellich theorem, $H_{\mathrm{p}}$ is self-adjoint on $D\left(\sum_{j=1}^{N} \Omega_{j}\right)$ and essentially self-adjoint on any core for $\sum_{j=1}^{N} \Omega_{j}$, where $D(A)$ denotes the domain of $A$. (UV) implies that $H_{\mathrm{I}}$ is symmetric. Then (V), (UV) and (IR) also imply that, for arbitrary $\alpha \in \mathbb{R}$ and $\epsilon>0$, it holds that

$$
\left\|H_{\mathrm{I}} \Psi\right\| \leq \epsilon\left\|H_{0} \Psi\right\|+b_{\epsilon}\|\Psi\|, \quad \Psi \in D\left(H_{0}\right)
$$

Therefore, by the Kato-Rellich theorem, $H^{V}$ is self-adjoint on $D\left(H_{0}\right)$ for all $\kappa>0$ and $\alpha \geq 0$. The nonnegativity $\hat{\lambda}_{j}(k) \geq 0$ in (UV) implies that the effective potential is attractive, which is used in Lemma B.ID.

## §2. Existence of a ground state

## §2.1. Dressing transformation

To derive the effective particle Hamiltonian we introduce the so-called dressing transformation $e^{-i T}$, where $T=\frac{\alpha}{\kappa} \sum_{j=1}^{N} \pi_{j}$ and

$$
\pi_{j}=\int_{\mathbb{R}^{d N}}^{\oplus} d X\left(\overline{\frac{i}{\sqrt{2}} \int\left(a^{*}(k) e^{-i k x_{j}} \frac{\hat{\lambda}_{j}(-k)}{\omega(k)}-a(k) e^{i k x_{j}} \frac{\hat{\lambda}_{j}(k)}{\omega(k)}\right) d k}\right)
$$

By (IR), $\pi_{j}$ is self-adjoint on $\mathscr{H}$ and then $e^{i T}$ is unitary.
Lemma 2.1. The unitary operator $e^{i T}$ maps $D\left(H^{V}\right)$ onto itself and

$$
\begin{equation*}
e^{-i T} H^{V} e^{i T}=h_{\mathrm{eff}}^{V} \otimes \mathbb{1}+\kappa^{2} \mathbb{1} \otimes H_{\mathrm{f}}+H_{\mathrm{R}}(\kappa) \tag{14}
\end{equation*}
$$

where the effective Hamiltonian is defined by

$$
\begin{equation*}
h_{\mathrm{eff}}^{V}=\sum_{j=1}^{N}\left(\Omega_{j}+V_{j}\right)+V_{\mathrm{eff}}, \tag{15}
\end{equation*}
$$

with the effective pair potential

$$
\begin{align*}
V_{\mathrm{eff}} & =\alpha^{2} \sum_{1 \leq i<j \leq N} W_{i j}\left(x_{i}-x_{j}\right),  \tag{16}\\
W_{i j}(x) & =-\int_{\mathbb{R}^{d}} \frac{\hat{\lambda}_{i}(-k) \hat{\lambda}_{j}(k)}{\omega(k)} e^{-i k x} d k . \tag{17}
\end{align*}
$$

Here $H_{\mathrm{R}}(\kappa)$ is the remainder term given by

$$
\begin{align*}
H_{\mathrm{R}}(\kappa) & =\sum_{j=1}^{N}\left(\Delta \Omega_{j}-\frac{\alpha^{2}}{2}\left\|\hat{\lambda}_{j} / \sqrt{\omega}\right\|^{2}\right)  \tag{18}\\
\Delta \Omega_{j} & =\Omega_{j}\left(p_{j}+\frac{\alpha}{\kappa} A_{j}\right)-\Omega_{j}\left(p_{j}\right) \tag{19}
\end{align*}
$$

with a vector field

$$
\begin{aligned}
& A_{j}=\left(A_{j 1}, \cdots, A_{j d}\right) \\
& A_{j l}=\int_{\mathbb{R}^{d N}}^{\oplus}\left(\frac{1}{\sqrt{2}} \int_{\mathbb{R}^{d}} k_{l}\left(a^{*}(k) e^{-i k x_{j}} \frac{\hat{\lambda}_{j}(-k)}{\omega(k)}+a(k) e^{i k x_{j}} \frac{\hat{\lambda}_{j}(k)}{\omega(k)}\right) d k\right) d X
\end{aligned}
$$

Proof. We directly see that

$$
\begin{aligned}
& e^{-i T} p_{j} e^{i T}=p_{j}+\frac{\alpha}{\kappa} A_{j}, \\
& e^{-i T} \phi_{j} e^{i T}=\phi_{j}-\frac{\alpha}{\kappa} \sum_{i=1}^{N} \int_{\mathbb{R}^{d}} \frac{\hat{\lambda}_{i}(k) \hat{\lambda}_{j}(-k)}{\omega(k)} e^{-i k\left(x_{j}-x_{i}\right)} d k, \\
& e^{-i T} H_{\mathrm{f}} e^{i T}=H_{\mathrm{f}}-\frac{1}{\kappa} H_{\mathrm{I}}+\frac{\alpha^{2}}{2 \kappa^{2}} \sum_{i, j=1}^{N} \int_{\mathbb{R}^{d}} \frac{\hat{\lambda}_{i}(-k) \hat{\lambda}_{j}(k)}{\omega(k)} e^{-i k\left(x_{i}-x_{j}\right)} d k .
\end{aligned}
$$

Together with them, the lemma follows.
(UV) and (IR) imply that $V_{\text {eff }}$ is bounded. Therefore $H_{\text {eff }}^{V}$ is a self-adjoint operator on $D\left(\sum_{j=1}^{N} \Omega_{j}\right)$.

## §2.2. Main results

Recall that $E_{0}(T)=\inf \sigma(T)$ for a self-adjoint operator $T$.
Theorem 2.2. (Existence of ground state) Assume (V), (UV) and (IR). Suppose that $E_{0}\left(h_{\text {eff }}^{V}\right) \in \sigma_{\text {disc }}\left(h_{\text {eff }}^{V}\right)$. Then there exists $\kappa_{0}>0$ such that $H^{V}$ has the unique ground state for any $\kappa>\kappa_{0}$.

In order to show the enhanced binding, we introduce an assumption on $V$. (EN)
(1) $\inf _{x \in \mathbb{R}^{d}} V(x)>-\infty$ and $\liminf _{|x| \rightarrow \infty} V(x)=0$;
(2) $\sqrt{-\Delta}+N V$ acting in $L^{2}\left(\mathbb{R}^{d}\right)$ has a negative energy ground state;
(3) $V$ is $d$-dimensional relativistic Kato-class, i.e.,

$$
\lim _{t \downarrow 0} \sup _{x \in \mathbb{R}^{d}} \mathbb{E}_{\mathrm{P}}^{x}\left[\int_{0}^{t} V\left(X_{s}\right) d s\right]=0
$$

where $\mathbb{E}_{\mathrm{P}}^{x}$ denotes the expectation on a probability space $\left(\mathcal{D}, \mathcal{B}, \mathrm{P}^{x}\right)$, and $\left(X_{t}\right)_{t \geq 0}$ denotes the $d$-dimensional Lévy process with the characteristic function $\mathbb{E}_{\mathrm{P}}^{x}\left[e^{i u X_{t}}\right]=e^{-t\left(\sqrt{u^{2}+m^{2}}-m\right)} e^{i u x}$.
Assumption (EN)(1) is used only to show spatial exponential decay of the infrared regularized ground state $\Phi_{\sigma}$. The second assumption (EN)(2), which is used in (5.5), is a crucial assumption for showing the enhanced binding. Intuitively a sufficiently strong interaction engages $N$ particles through linear interaction of the quantum field, and consequently the total Hamiltonian can be regarded as $\sqrt{-\Delta}+N V$. This intuitive description is justified in this paper. (EN)(3) is used to show the continuity of ground state energy of a translation invariant Hamiltonian in Lemma 3.

We state the main results in this paper.
Theorem 2.3. (Enhanced binding) Let us suppose (V), (UV) and (IR). Assume (EN) and $N \geq 2$. Then there exist $\bar{\alpha}>0$ and $\kappa_{0}>0$ such that for each $\kappa>\kappa_{0}, H^{V}$ has the unique ground state for $|\alpha| \in(\bar{\alpha}, \bar{\alpha}(\kappa))$ with some constant $\bar{\alpha}(\kappa)$.

Remark 2.4. In Theorem [2.2, $h_{\text {eff }}^{V}$ has a ground state. In Theorem [..3] we do not assume the existence of a ground state of $H_{\mathrm{p}}$, i.e., the zero-coupling Hamiltonian $H_{0}$ does not necessarily have a ground state.

Remark 2.5. In the case of $N=1$, we can not apply our method to show the enhanced binding. Although in this case the enhanced binding may also occur, it is crucial to estimate dressing transformed Hamiltonian ([4]). We do not then discuss this case.

Example 2.6. We give examples of $V$ satisfying (V) and (EN), but $\sqrt{-\Delta+1}-$ $1+V$ has no ground state in the dimension $d \geq 3$. Suppose that $\tilde{V}$ satisfies

$$
|\tilde{V}(x)| \leq c(1+|x|)^{-\epsilon}
$$

with some $c>0$ and $\epsilon>0$. It involves $\tilde{V}=-e^{-x^{2}}$. Then (V) is satisfied with $V=\delta \tilde{V}$ for all constant $\delta>0$. Let $\tilde{V} \not \equiv 0, \tilde{V} \leq 0$ and $\tilde{V} \in L^{d}\left(\mathbb{R}^{d}\right) \cap L^{d / 2}\left(\mathbb{R}^{d}\right)$. Let $\delta>0$ be sufficiently small constants and set

$$
\begin{equation*}
H_{\delta}=\sqrt{-\Delta+1}-1+\delta \tilde{V} \tag{20}
\end{equation*}
$$

Let $E_{\delta}(\cdot)$ be the spectral measure of $H_{\delta}$. Since $\tilde{V}(\sqrt{-\Delta+1})^{-1}$ is compact, the essential spectrum of $H_{\delta}$ is $\sigma_{\text {ess }}\left(H_{\delta}\right)=[0, \infty)$ for all $\delta>0$. By the relativistic version of the Lieb-Thirring bound [Dau8.3], we have

$$
\begin{equation*}
\operatorname{dim} \operatorname{Ran} E_{\delta}((-\infty, 0]) \leq c_{1} \delta^{d} \int_{\mathbb{R}^{d}}|\tilde{V}(x)|^{d} d x+c_{2} \delta^{d / 2} \int_{\mathbb{R}^{d}}|\tilde{V}(x)|^{d / 2} d x \tag{21}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are positive constants independent of $\tilde{V}$. Hence $H_{\delta}$ has no ground state for sufficiently small $\delta$ such that the right-hand side of (Z1) is strictly smaller than one. Similarly $\sigma_{\text {ess }}(\sqrt{-\Delta}+N \delta \tilde{V})=[0, \infty)$ follows. $\sqrt{-\Delta}+N \delta \tilde{V}$ has however a negative eigenvalue for sufficiently large $N$, since $\inf \sigma(\sqrt{-\Delta}+N \delta \tilde{V})<0$ for sufficiently large $N$, which implies that $\sqrt{-\Delta}+N \delta \tilde{V}$ has a ground state for sufficiently large $N$. Therefore for sufficiently small $\delta, V=\delta \tilde{V}$ satisfies (V) and (EN), but $\sqrt{-\Delta+1}-1+\delta \tilde{V}$ has no ground state.

## §2.3. Stability condition and exponential decay

In order to prove Theorems 2.2 and we investigate the stability condition. First of all we introduce cluster Hamiltonians. Let $C_{N}=\{1,2, \cdots, N\}$. For each $\beta \subset C_{N},(\beta \neq \emptyset)$, we define

$$
\begin{align*}
H^{0}(\beta) & =\sum_{j \in \beta}\left(\Omega_{j}+\kappa \alpha \phi_{j}\right)+\kappa^{2} H_{\mathrm{f}}  \tag{22}\\
H^{V}(\beta) & =H^{0}(\beta)+\sum_{j \in \beta} V_{j} \tag{23}
\end{align*}
$$

acting on $L^{2}\left(\mathbb{R}^{d|\beta|}\right) \otimes \mathscr{F}$, where $\phi_{j}=\int_{\mathbb{R}^{d|\beta|}}^{\oplus} \phi_{j}\left(x_{j}\right) d X_{\beta}, X_{\beta}=\left(x_{j}\right)_{j \in \beta}$. Clearly $H^{V}=H^{V}\left(C_{N}\right)$. Let

$$
\begin{equation*}
E^{0}(\beta)=\inf \sigma\left(H^{0}(\beta)\right), \quad E^{V}(\beta)=\inf \sigma\left(H^{V}(\beta)\right) \tag{24}
\end{equation*}
$$

For the case of $\beta=\emptyset$, we set $E^{0}(\emptyset)=E^{V}(\emptyset)=0$. The lowest two cluster threshold is defined as the minimal energy of systems such that only the particles involved in $\beta$ are bound by the origin but others are sufficiently remote from the origin. It is defined by

$$
\begin{equation*}
\Sigma^{V}=\min \left\{E^{V}(\beta)+E^{0}\left(\beta^{c}\right) \mid \beta \varsubsetneqq C_{N}\right\} \tag{25}
\end{equation*}
$$

The gap between the ground state energy $E^{V}$ and the lowest two cluster threshold $\Sigma^{V}$ is related to the existence of ground state by the proposition below. Let $H_{\sigma}^{V}$ be defined by $H^{V}$ with $\hat{\lambda}_{j}$ replaced by $\hat{\lambda}_{j}(k) \mathbb{1}_{|k|>\sigma}$.
Proposition 2.7. (Case $\sigma>0$ ) Suppose that $E^{V}<\Sigma^{V}$. Then $H_{\sigma}^{V}$ has the unique ground state. We denote the ground state by $\Phi_{\sigma}$.
(Case $\sigma=0$ ) Suppose that $E^{V}<\Sigma^{V}$ and there exists $0<\delta$ independent of $\sigma$ such that $\sup _{0<\sigma<\bar{\sigma}}\left\|\left(e^{\delta|X|} \otimes \mathbb{1}\right) \Phi_{\sigma}\right\|_{\mathscr{H}}<\infty$ with some $\bar{\sigma}>0$. Then $H^{V}$ has a ground state.

Proof. The proof is a minor modification of [Ger00, GLLOT], and it is given in Appendix 0.0 for the case $\sigma>0$, and in Appendix 4.2 for the case $\sigma=0$.

The condition $\Sigma^{V}>E^{V}$ is called the stability condition. For our model the uniform exponential decay of $\left\|\Phi_{\sigma}(x)\right\|_{\mathscr{F}}$ may be derived from the stability condition, but we do not check it. So we need not only stability condition but also uniform exponential decay.

## §3. Proof of the main theorem

In order to show Theorems 22 and $[2]$, by Proposition 2.2 it is enough to show both (1) stability condition and (2) the uniform exponential decay of $\left\|\Phi_{\sigma}(x)\right\| \mathscr{F}$.

## §3.1. Stability condition

It is however not straightforward to show the stability condition, so we will make a detour and the discussion will be reduced to that of effective particle Hamiltonian $h_{\text {eff }}^{V}$. Let us define the lowest two cluster threshold of $h_{\text {eff }}^{V}$ in a similar way to $H^{V}$ and we shall compare it with $\Sigma^{V}$. For $\beta \subset C_{N}$, we define effective cluster Hamiltonians by

$$
\begin{align*}
& h_{\mathrm{eff}}^{0}(\beta)=\sum_{j \in \beta} \Omega_{j}-\alpha^{2} \sum_{i, j \in \beta, i<j} W_{i j}\left(x_{i}-x_{j}\right),  \tag{26}\\
& h_{\mathrm{eff}}^{V}(\beta)=h_{\mathrm{eff}}^{0}(\beta)+\sum_{j \in \beta} V_{j} . \tag{27}
\end{align*}
$$

We set

$$
\begin{equation*}
\mathcal{E}^{0}(\beta)=\inf \sigma\left(h_{\mathrm{eff}}^{0}(\beta)\right), \quad \mathcal{E}^{V}(\beta)=\inf \sigma\left(h_{\mathrm{eff}}^{V}(\beta)\right) \tag{28}
\end{equation*}
$$

and $\mathcal{E}^{V}=\mathcal{E}^{V}\left(C_{N}\right)$. Then the lowest two cluster threshold of $h_{\text {eff }}^{V}$ is defined by

$$
\begin{equation*}
\Xi^{V}=\min \left\{\mathcal{E}^{V}(\beta)+\mathcal{E}^{0}\left(\beta^{c}\right) \mid \beta \varsubsetneqq C_{N}\right\} . \tag{29}
\end{equation*}
$$

Constants $c^{V}$ and $d^{V}$ are such that $\left\|\sum_{j=1}^{N} \Omega_{j} \Psi\right\| \leq c^{V}\left\|h_{\text {eff }}^{V} \Psi\right\|+d^{V}\|\Psi\|$ and set

$$
\begin{equation*}
\mathcal{G}(t)=\left(\sum_{j=1}^{N}\left\|\hat{\lambda}_{j} / \omega\right\|\left\|\hat{\lambda}_{j}\right\|\right) t^{2}+\left(\sum_{j=1}^{N} \sqrt{2} m_{j}\left\|\hat{\lambda}_{j} / \omega\right\|\right)|t|+\sqrt{2} N\left(c^{V}\left|\mathcal{E}^{V}\right|+d^{V}\right) . \tag{30}
\end{equation*}
$$

The next lemma is a key ingredient of this paper.
Lemma 3.1. We assume that $\Xi^{V}-\mathcal{E}^{V}>0$, and $\alpha$ and $\kappa$ satisfy $\Xi^{V}-\mathcal{E}^{V}>$ $\mathcal{G}(\alpha / \kappa)$. Then the stability condition $\Sigma^{V}-E^{V}>0$ holds.

In order to prove Lemma B. 3 , we prepare two lemmas. We set

$$
\begin{equation*}
E_{\mathrm{diag}}=\frac{\alpha^{2}}{2} \sum_{j=1}^{N}\left\|\hat{\lambda}_{j} / \sqrt{\omega}\right\|^{2} \tag{31}
\end{equation*}
$$

Lemma 3.2. For all $\beta \subset C_{N}$, it follows that

$$
\begin{equation*}
E^{\#}(\beta) \leq \mathcal{E}^{\#}(\beta)+\frac{\alpha^{2}}{2} \sum_{j \in \beta}\left\|\hat{\lambda}_{j} / \sqrt{\omega}\right\|^{2}, \quad \#=0, V \tag{32}
\end{equation*}
$$

In particular, it holds that $\Xi^{V} \leq \Sigma^{V}+E_{\text {diag }}$.
Proof. See Proposition [6.3 in Appendix [6.

Lemma 3.3. For all $\kappa>0$ it follows that $E^{V} \leq \mathcal{E}^{V}+\mathcal{G}(\alpha / \kappa)-E_{\text {diag }}$,
Proof. For arbitrary $\epsilon>0$, we can choose a normalized vector $v \in C_{0}^{\infty}\left(\mathbb{R}^{d N}\right)$ such that $\left\|\left(h_{\text {eff }}^{V}-\mathcal{E}^{V}\right) v\right\| \leq \epsilon$. Set $\Psi=v \otimes \Omega$. Then, by Lemma ㄹ..T, we have

$$
E^{V} \leq \mathcal{E}^{V}+\epsilon+\left(\Psi,\left(-E_{\text {diag }}+\sum_{j=1}^{N} \Delta \Omega_{j}\right) \Psi\right)
$$

Since $\pi_{j}$ commutes with $p_{i}, i \neq j$, by setting $T_{j}=\alpha \pi_{j} / \kappa$, we can see that $\Delta \Omega_{j}=$ $e^{-i T_{j}} \Omega_{j} e^{i T_{j}}-\Omega_{j}$ and

$$
\left|\left(\Psi, \Delta \Omega_{j} \Psi\right)\right|=\left|\left(\left(e^{i T_{j}}-1\right) \Psi, \Omega_{j} e^{i T_{j}} \Psi\right)+\left(\Psi, \Omega_{j}\left(e^{i T_{j}}-1\right) \Psi\right)\right|
$$

Hence we have

$$
\left|\left(\Psi, \Delta \Omega_{j} \Psi\right)\right| \leq \frac{|\alpha|}{\kappa}\left\|\pi_{j} \Psi\right\| \cdot\left\|\Omega_{j} e^{i T_{j}} \Psi\right\|+\frac{|\alpha|}{\kappa}\left\|\pi_{j} \Psi\right\| \cdot\left\|\Omega_{j} \Psi\right\|
$$

The right-hand side above is identical with

$$
=\frac{|\alpha|}{\sqrt{2} \kappa}\left\|\hat{\lambda}_{j} / \omega\right\|\left(\left(\Psi,\left(p_{j}+\frac{|\alpha|}{\kappa} A_{j}\right)^{2} \Psi\right)^{1 / 2}+\left(\Psi, p_{j}^{2} \Psi\right)^{1 / 2}\right)
$$

Then we have

$$
\left|\left(\Psi, \Delta \Omega_{j} \Psi\right)\right| \leq \frac{|\alpha|}{\sqrt{2} \kappa}\left\|\hat{\lambda}_{j} / \omega\right\|\left(2\left\|\Omega_{j} \Psi\right\|+2 m_{j}+\frac{\sqrt{2}|\alpha|}{\kappa}\left\||k| \hat{\lambda}_{j} / \omega\right\|\right)
$$

and

$$
\begin{aligned}
E^{V} \leq & \mathcal{E}^{V}+\epsilon+\sum_{j=1}^{N} \frac{|\alpha|}{\sqrt{2} \kappa}\left\|\hat{\lambda}_{j} / \omega\right\|\left(2 m_{j}+\frac{\sqrt{2}|\alpha|}{\kappa}\left\||k| \hat{\lambda}_{j} / \omega\right\|\right) \\
& +\sum_{j=1}^{N} \frac{\sqrt{2}|\alpha|}{\kappa}\left\|\hat{\lambda}_{j} / \omega\right\|\left(c^{V}\left(\left|\mathcal{E}^{V}\right|+\epsilon\right)+d^{V}\right)-E_{\text {diag }}
\end{aligned}
$$

Since $\epsilon>0$ is arbitrary, the lemma follows.
Proof of Lemma [3.]: By Lemmas [5.2] and [3.3], we have

$$
\begin{equation*}
\Sigma^{V}-E^{V} \geq \Xi^{V}-\mathcal{E}^{V}-\mathcal{G}(\alpha / \kappa)>0 \tag{33}
\end{equation*}
$$

Then the lemma is proven.

## §3.2. Exponential decays

It is proven that the functional integration is a strong tool to show an exponential localization of bound state in quantum mechanics. That can be also applied in quantum field theory.

Let $\left(X_{t}\right)_{t \geq 0}=\left(X_{t}^{1}, \ldots, X_{t}^{N}\right)_{t \geq 0}$ be the $N$ independent $d$-dimensional Lévy processes on a probability space $\left(\mathcal{D}, \mathcal{B}, \mathrm{P}^{x}\right), x \in \mathbb{R}^{d N}$, with the characteristic function

$$
\begin{equation*}
\mathbb{E}_{\mathrm{P}}^{0}\left[e^{-i u \cdot X_{t}}\right]=e^{-t \sum_{j=1}^{N}\left(\sqrt{u_{j}^{2}+m_{j}^{2}}-m_{j}\right)}, \quad u=\left(u_{1}, \ldots, u_{N}\right) \in \mathbb{R}^{d N} \tag{34}
\end{equation*}
$$

Here and in what follows $\mathbb{E}_{m}^{x}[\cdots]$ denotes the expectation with respect to a path measure $m^{x}$ starting from $x$. Let $W_{\text {eff }}=W_{\text {eff }}\left(x_{1}, . ., x_{N}\right)=\sum_{j=1}^{N} V\left(x_{j}\right)+V_{\text {eff }}(x)$.
Proposition 3.4. There exists $\sigma_{0}>0$ such that for all $\sigma \leq \sigma_{0}$,

$$
\begin{equation*}
\left\|\Phi_{\sigma}(X)\right\|_{\mathscr{F}} \leq e^{t\left(E^{V}+E_{\text {diag }}+\epsilon(\sigma)\right)}\left(\mathbb{E}_{\mathrm{P}}^{X}\left[e^{-2 \int_{0}^{t} W_{\text {eff }}\left(X_{s}\right) d s}\right]\right)^{1 / 2}\left\|\Phi_{\sigma}\right\|_{\mathscr{H}} \tag{35}
\end{equation*}
$$

for each $X \in \mathbb{R}^{d N}$, where $\epsilon(\sigma)>0$ satisfies $\lim _{\sigma \rightarrow 0} \epsilon(\sigma)=0$.
Proof. See Proposition 6.4.
From Proposition 3.4 it suffices to estimate

$$
e^{t\left(E^{V}+E_{\mathrm{diag}}\right)} \mathbb{E}_{\mathrm{P}}^{X}\left[e^{-2 \int_{0}^{t} W_{\mathrm{eff}}\left(X_{s}\right) d s}\right]^{1 / 2}
$$

in order to show the exponential decay of $\left\|\Phi_{\sigma}(X)\right\|_{\mathscr{F}}$. To estimate this we divide $W_{\text {eff }}$ into two parts. Let

$$
B_{R}=\left\{x=\left(x_{1}, \ldots, x_{N}\right) \in \mathbb{R}^{d N}| | x \mid \geq 2 R \text { and } \min \left\{\left|x_{i}-x_{j}\right|, i \neq j\right\} \leq|x| / 2\right\}
$$

Define $V_{\text {eff, } \infty}^{R}=V_{\text {eff }} \mathbb{1}_{B_{R}}$ and $V_{\text {eff }, 0}^{R}=V_{\text {eff }} \mathbb{1}_{B_{R}^{c}}$. Then

$$
\begin{equation*}
W_{\mathrm{eff}}=V+V_{\mathrm{eff}, 0}^{R}+V_{\mathrm{eff}, \infty}^{R} \tag{36}
\end{equation*}
$$

By the Riemann Lebesgue lemma $\lim _{|x| \rightarrow \infty} W_{i j}(x)=0$. Then notice that

$$
\begin{aligned}
& \lim _{|x| \rightarrow \infty}\left(V(x)+V_{\mathrm{eff}, 0}^{R}(x)\right)=0 \\
& \left\|V_{\mathrm{eff}, \infty}^{R}\right\|_{\infty} \leq \frac{\alpha^{2}}{2} \sum_{i \neq j} \int \frac{\hat{\lambda}_{i}(k) \hat{\lambda}_{j}(-k)}{\omega(k)} d k .
\end{aligned}
$$

The Lévy measure $\nu_{j}(d x)=\nu_{j}(x) d x$ associated with the Lévy process $\left(X_{t}^{j}\right)_{t \geq 0}$ is given by

$$
\begin{equation*}
\nu_{j}(x)=2\left(\frac{m_{j}}{2 \pi}\right)^{\frac{d+1}{2}} \frac{1}{|x|^{\frac{d+1}{2}}} \int_{0}^{\infty} \xi^{\frac{d-1}{2}} e^{-\frac{1}{2}\left(\xi+\xi^{-1}\right) m_{j}|x|} d \xi, \quad x \in \mathbb{R}^{d} \tag{37}
\end{equation*}
$$

We note that $\nu_{j}(x) \leq C e^{-c|x|}$ with some constants $C, c \geq 0$.
Proposition 3.5. There exist $\eta>0, C_{1}>0$ and $C_{2}>0$ such that

$$
\begin{equation*}
\mathrm{P}^{0}\left(\sup _{0 \leq s \leq t}\left|X_{s}\right|>a\right) \leq C_{1} e^{-\eta a} e^{C_{2} t} \tag{38}
\end{equation*}
$$

for all $a>0$.
Proof. We see that

$$
\mathrm{P}^{0}\left(\sup _{0 \leq s \leq t}\left|X_{s}\right|>a\right)=\mathbb{E}_{\mathrm{P}}^{0}\left[\mathbb{1}_{\sup _{0 \leq s \leq t}\left|X_{s}\right|-a>0}\right] \leq e^{-\eta a} \mathbb{E}_{\mathrm{P}}^{0}\left[e^{\eta \sup _{0 \leq s \leq t}\left|X_{s}\right|}\right] .
$$

It is known that $\mathbb{E}_{\mathrm{P}}^{0}\left[e^{\eta\left(\sup _{0 \leq s \leq t}\left|X_{s}\right|\right)}\right]<C_{1} e^{C_{2} t}$ for sufficiently small $0<\eta$ [CMS.90]. Hence the proposition follows.

We define $\mathscr{B}=\left\{X_{s} \in B_{R}^{c}\right.$ for all $\left.0 \leq s \leq t\right\}$. Since $V_{\text {eff }, \infty}^{R}\left(X_{s}\right)=0$ for on $\mathscr{B}$, we have

$$
\begin{equation*}
\mathbb{E}_{\mathrm{P}}^{X}\left[e^{-2 \int_{0}^{t} W_{\text {eff }}\left(X_{s}\right) d s}\right]=\mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}} e^{-2 \int_{0}^{t}\left(V+V_{\text {eff }, 0}^{R}\right)\left(X_{s}\right) d s}\right]+\mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}} e^{-2 \int_{0}^{t} W_{\text {eff }}\left(X_{s}\right) d s}\right] \tag{39}
\end{equation*}
$$

By the Schwartz inequality

$$
\begin{align*}
& \mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}} e^{-2 \int_{0}^{t} W_{\text {eff }}\left(X_{s}\right) d s}\right] \\
& \leq \mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}} e^{-4 \int_{0}^{t} V_{\text {eff }, \infty}^{R}\left(X_{s}\right) d s}\right]^{1 / 2} \mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}} e^{-4 \int_{0}^{t}\left(V+V_{\text {eff }, 0}^{R}\right)\left(X_{s}\right) d s}\right]^{1 / 2} \tag{40}
\end{align*}
$$

We will estimate terms in (39) and (401). Set

$$
\begin{aligned}
& W_{a}^{R}(x)=\inf \left\{V(y)+V_{\mathrm{eff}, \infty}^{R}(y) \| x-y \mid<a\right\} \\
& W_{\infty}^{R}=\inf _{x \in \mathbb{R}^{d N}}\left(V(x)+V_{\mathrm{eff}, \infty}^{R}(x)\right)
\end{aligned}
$$

Lemma 3.6. Suppose (1) of (EN). Let $R>0$ and $a>0$. Then for all $X \in \mathbb{R}^{d N}$ and $t>0$ it follows that

$$
\begin{equation*}
\mathbb{E}_{\mathrm{P}}^{X}\left[e^{-2 \int_{0}^{t}\left(V\left(X_{s}\right)+V_{\mathrm{eff}, \infty}^{R}\left(X_{s}\right)\right) d s}\right] \leq e^{-2 t W_{a}^{R}(x)}+C_{1} e^{-2 t W_{\infty}^{R}} e^{C_{2} t} e^{-\eta a} \tag{41}
\end{equation*}
$$

where $C_{1}, C_{2}$ and $\eta$ are given in (58).
Proof. Set $A=\left\{\sup _{0 \leq s \leq t}\left|X_{s}\right|<a\right\} \subset \mathcal{D}$. Since $\left(X_{t}\right)_{t \geq 0}$ under the probability measure $\mathrm{P}^{X}$ and $\left(X_{t}+X\right)_{t \geq 0}$ under $\mathrm{P}^{0}$ are identically distributed, we have the identity:

$$
\mathbb{E}_{\mathrm{P}}^{X}\left[e^{-2 \int_{0}^{t}\left(V\left(X_{s}\right)+V_{\mathrm{eff}, \infty}^{R}\left(X_{s}\right)\right) d s}\right]=\mathbb{E}_{\mathrm{P}}^{0}\left[e^{-2 \int_{0}^{t}\left(V\left(X_{s}+X\right)+V_{\mathrm{eff}, \infty}^{R}\left(X_{s}+X\right)\right) d s}\right]
$$

Then we have

$$
\begin{aligned}
\mathbb{E}_{\mathrm{P}}^{0}\left[\mathbb{1}_{A} e^{-2 \int_{0}^{t}\left(V\left(X_{s}+X\right)+V_{\mathrm{eff}, \infty}^{R}\left(X_{s}+X\right)\right) d s}\right] & \leq e^{-2 t W_{a}^{R}(x)}, \\
\mathbb{E}_{\mathrm{P}}^{0}\left[\mathbb{1}_{A^{c}} e^{-2 \int_{0}^{t}\left(V\left(X_{s}+X\right)+V_{\mathrm{eff}, \infty}^{R}\left(X_{s}+X\right)\right) d s}\right] & \leq e^{-2 t W_{\infty}^{R}} \mathbb{E}_{\mathrm{P}}^{0}\left[\mathbb{1}_{A^{c}}\right] \leq e^{-2 t W_{\infty}^{R}} C_{1} e^{C_{2} t} e^{-\eta a}
\end{aligned}
$$

by Proposition [3.5. Then the lemma follows.
Lemma 3.7. Let $X \in \mathbb{R}^{d N}$ and set $R=|X|$. Then it follows that

$$
\begin{equation*}
\mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}} e^{-4 \int_{0}^{t} V_{\mathrm{eff}, \infty}^{R}\left(X_{s}\right) d s}\right] \leq e^{4\left\|V_{\text {eff }, \infty}\right\| \infty t} C_{1} e^{C_{2} t} e^{-\eta R} \tag{42}
\end{equation*}
$$

where $C_{1}, C_{2}$ and $\eta$ are given in (38).
Proof. Since $\mathbb{E}_{\mathrm{P}}^{X}\left[e^{-4 \int_{0}^{t} V_{\text {eff }, \infty}^{R}\left(X_{s}\right) d s}\right] \leq \mathbb{E}_{\mathrm{P}}^{X}\left[e^{4\left\|V_{\text {eff }, \infty}\right\| \infty \int_{0}^{t} \mathbb{1}_{B_{R}}\left(X_{s}\right) d s}\right]$, we can see that

$$
\begin{aligned}
& \mathbb{E}_{\mathrm{P}}^{X}\left[e^{-4 \int_{0}^{t} V_{\mathrm{eff}, \infty}^{R}\left(X_{s}\right) d s}\right] \\
& \leq \sum_{n=0}^{\infty} \frac{\left(4\left\|V_{\mathrm{eff}, \infty}\right\|_{\infty}\right)^{n}}{n!} \int_{0}^{t} d s_{1} \cdots \int_{0}^{t} d s_{n} \mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}} \prod_{j=1}^{n} \mathbb{1}_{B_{R}}\left(X_{s_{j}}\right)\right] \\
& =\mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}}\right]+\sum_{n=1}^{\infty} \frac{\left(4\left\|V_{\mathrm{eff}, \infty}\right\|_{\infty}\right)^{n}}{n!} \int_{0}^{t} d s_{1} \cdots \int_{0}^{t} d s_{n} \mathbb{E}_{\mathrm{P}}^{0}\left[\mathbb{1}_{\mathscr{B}^{c}} \prod_{j=1}^{n} \mathbb{1}_{B_{R}}\left(X+X_{s_{j}}\right)\right] .
\end{aligned}
$$

We see that

$$
\begin{align*}
\mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}}\right] & \leq \mathrm{P}^{0}\left(\sup _{0 \leq s \leq t}\left|X_{s}+X\right|>2 R\right) \\
& \leq \mathrm{P}^{0}\left(\sup _{0 \leq s \leq t}\left|X_{s}\right|>2 R-|X|\right)=\mathrm{P}^{0}\left(\sup _{0 \leq s \leq t}\left|X_{s}\right|>R\right) \tag{43}
\end{align*}
$$

By the definition of $B_{R}$ in a similar way to above we have

$$
\begin{aligned}
& \mathbb{E}_{\mathrm{P}}^{X}\left[e^{-4 \int_{0}^{t} V_{\mathrm{eff}, \infty}^{R}\left(X_{s}\right) d s}\right] \\
& \begin{aligned}
\leq & \mathrm{P}^{X}\left(\mathscr{B}^{c}\right)+\sum_{n=1}^{\infty} \frac{\left(4\left\|V_{\mathrm{eff}, \infty}\right\|_{\infty}\right)^{n}}{n!} \\
& \quad \times \int_{0}^{t} d s_{1} \cdots \int_{0}^{t} d s_{n} \mathrm{P}^{0}\left(\left|X_{s_{1}}+X\right|>2 R ; \cdot,\left|X_{s_{n}}+X\right|>2 R\right) \\
\leq & \mathrm{P}^{X}\left(\mathscr{B}^{c}\right)+\sum_{n=1}^{\infty} \frac{\left(4\left\|V_{\mathrm{eff}, \infty}\right\|_{\infty}\right)^{n}}{n!} \int_{0}^{t} d s_{1} \cdots \int_{0}^{t} d s_{n} \mathrm{P}^{0}\left(\left|X_{s_{1}}\right|>R, \cdots,\left|X_{s_{n}}\right|>R\right) .
\end{aligned}
\end{aligned}
$$

By $\mathrm{P}^{0}\left(\left|X_{s_{1}}\right|>R, \cdots,\left|X_{s_{n}}\right|>R\right) \leq \mathrm{P}^{0}\left(\sup _{0 \leq s \leq t}\left|X_{s}\right|>R\right)$ and Proposition [3.5, we have

$$
\begin{aligned}
& \mathbb{E}_{\mathrm{P}}^{X}\left[e^{-4 \int_{0}^{t} V_{\mathrm{eff}, \infty}^{R}\left(X_{s}\right) d s}\right] \\
& \leq \mathrm{P}^{0}\left(\sup _{0 \leq s \leq t}\left|X_{s}\right|>R\right) \\
& \quad+\sum_{n=1}^{\infty} \frac{\left(4\left\|\mid V_{\mathrm{eff}, \infty}\right\|_{\infty}\right)^{n}}{n!} \int_{0}^{t} d s_{1} \cdots \int_{0}^{t} d s_{n} \mathrm{P}^{0}\left(\sup _{0 \leq s \leq t}\left|X_{s}\right|>R\right) \\
& \leq \sum_{n=0}^{\infty} \frac{\left(4\left\|\mid V_{\mathrm{eff}, \infty}\right\|_{\infty}\right)^{n}}{n!} t^{n} C_{1} e^{C_{2} t} e^{-\eta R} \\
& =e^{4\left\|| | V_{\text {eff }, \infty}\right\| \infty t} C_{1} e^{C_{2} t} e^{-\eta R} .
\end{aligned}
$$

Hence the lemma follows.
Lemma 3.8. Let $\Phi_{\sigma}$ be the infrared regularized ground state. Suppose (1) of $(\mathrm{EN})$ and $E^{V}+E_{\text {diag }}<0$. Furthermore we assume that $E^{V}+E_{\text {diag }}+\epsilon(\sigma)<-\gamma$ with some $\gamma>0$ for $\sigma<\bar{\sigma}$, where $\epsilon(\sigma)$ is given in Proposition [3.4. Then there exist $\delta>0$ and $C_{\delta}>$ independent of $\sigma$ such that

$$
\begin{equation*}
\sup _{0<\sigma<\bar{\sigma}}\left\|\Phi_{\sigma}(X)\right\|_{\mathscr{F}} \leq C_{\delta} e^{-\delta \min \{\gamma, \eta\}|X|} \tag{44}
\end{equation*}
$$

where $\eta>0$ is given in Proposition 3.5.
Proof. We set $\tilde{E}=E^{V}+E_{\text {diag }}+\epsilon(\sigma)$. It is enough to estimate

$$
e^{2 t \tilde{E}} \mathbb{E}_{\mathrm{P}}^{X}\left[e^{-2 \int_{0}^{t} W_{\text {eff }}\left(X_{s}\right) d s}\right]
$$

by Proposition [3.4. Recall that $W_{a}^{R}(x)=\inf \left\{W^{R}(y)| | x-y \mid \leq a\right\}$. Then

$$
\begin{equation*}
\lim _{|x| \rightarrow \infty} W_{|x| / 2}^{|x|}(x)=0 \tag{45}
\end{equation*}
$$

Hence there exists a positive constant $R^{*}$ such that $\left|W_{|X| / 2}^{|X|}(X)\right| \leq|\tilde{E}| / 2$ for all $X$ such that $|X|>R^{*}$. Suppose that $|X|>R^{*}$ and let $R=|X|$. We divide $W_{\text {eff }}$ as in (361) for $R$. We have

$$
\begin{aligned}
& e^{2 t \tilde{E}_{\mathbb{E}_{\mathrm{P}}}^{X}}\left[e^{-\int_{0}^{t} W_{\text {eff }}\left(X_{s}\right) d s}\right] \\
& \leq e^{2 t \tilde{E}} \mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}} e^{-2 \int_{0}^{t}\left(V+V_{\text {eff }, 0}^{R}\right)\left(X_{s}\right) d s}\right] \\
& +e^{2 t \tilde{E}}\left(\mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}} e^{-4 \int_{0}^{t}\left(V+V_{\text {eff }, 0}^{R}\right)\left(X_{s}\right) d s}\right]\right)^{1 / 2}\left(\mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}} e^{-4 \int_{0}^{t}\left(V+V_{\text {eff }, \infty}^{R}\right)\left(X_{s}\right) d s}\right]\right)^{1 / 2}
\end{aligned}
$$

Two terms $\mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}} e^{-2 \int_{0}^{t}\left(V+V_{\text {eff }, 0}^{R}\right)\left(X_{s}\right) d s}\right]$ and $\mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}^{c}} e^{-4 \int_{0}^{t}\left(V+V_{\text {eff }, 0}^{R}\right)\left(X_{s}\right) d s}\right]$ can be estimated as

$$
\begin{align*}
& \mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}} e^{-2 \int_{0}^{t}\left(V+V_{\mathrm{eff}, \mathrm{O}}^{R}\right)\left(X_{s}\right) d s}\right] \leq e^{-2 t W_{a}^{R}(x)}+C_{1} e^{-2 t W_{\infty}^{R}} e^{C_{2} t} e^{-\eta a}  \tag{46}\\
& \mathbb{E}_{\mathrm{P}}^{x}\left[\mathbb{1}_{\mathscr{B} c} e^{-4 \int_{0}^{t}\left(V+V_{\mathrm{eff}, 0}^{R}\right)\left(X_{s}\right) d s}\right] \leq e^{-4 t W_{a}^{R}(x)}+C_{1} e^{-4 t W_{\infty}^{R}} e^{C_{2} t} e^{-\eta a} \tag{47}
\end{align*}
$$

by Lemma [.6.6. Let us set $t=t(X)=\epsilon|X|$ and $a=|X| / 2$. Then we can see that $W_{|X| / 2}^{|X|}(X)-\tilde{E}>-\tilde{E} / 2>0$, since $\tilde{E}<0$ by assumption. Hence

$$
\begin{aligned}
e^{2 t \tilde{E}} \mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}} e^{-2 \int_{0}^{t}\left(V+V_{\mathrm{eff}, 0}^{R}\right)\left(X_{s}\right) d s}\right] & \leq e^{\epsilon \tilde{E}|X|}+C_{2} e^{\epsilon C_{2}|X|-\eta|X| / 2-2 \epsilon W_{\infty}^{|X|}|X|} \\
& \leq e^{-\epsilon \gamma|X|}+C_{2} e^{-\left(\eta / 2+2 \epsilon W_{\infty}^{|X|}-\epsilon C_{2}\right)|X|}
\end{aligned}
$$

Similarly we have

$$
e^{4 t \tilde{E}} \mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}} e^{-4 \int_{0}^{t}\left(V+V_{\mathrm{eff}, \mathrm{O}}^{R}\right)\left(X_{s}\right) d s}\right] \leq e^{-2 \epsilon \gamma|X|}+C_{2} e^{-\left(\eta / 2+4 \epsilon W_{\infty}^{|X|}-\epsilon C_{2}\right)|X|}
$$

Finally by Lemma $\sqrt{5.7}$ we have

$$
\begin{aligned}
e^{4 t \tilde{E}} \mathbb{E}_{\mathrm{P}}^{X}\left[\mathbb{1}_{\mathscr{B}} e^{-4 \int_{0}^{t} W_{\text {eff }}\left(X_{s}\right) d s}\right] & \leq C_{1} e^{\left.4 \epsilon \tilde{E}+4\left\|V_{\text {eff }, \infty}\right\|_{\infty} \epsilon+C_{2} \epsilon-\eta\right)|X|} \\
& \leq C_{1} e^{-\left(4 \epsilon \gamma-4\left\|V_{\text {eff }, \infty}\right\|_{\infty} \epsilon-C_{2} \epsilon+\eta\right)|X|}
\end{aligned}
$$

Note that $W_{\infty}^{|X|} \rightarrow 0$ as $|X| \rightarrow \infty$. Take sufficiently small $\epsilon>0$ such that $\eta / 2+$ $\left(2 W_{\infty}^{|X|}-C_{2}\right) \epsilon>0, \eta / 2+\left(4 W_{\infty}^{|X|}-C_{2}\right) \epsilon>0$ and $\left(4 \gamma-4\left\|V_{\text {eff }, \infty}\right\|_{\infty}-C_{2}\right) \epsilon+\eta>0$, then $\left\|\Phi_{\sigma}(X)\right\|_{\mathscr{F}} \leq D_{1} e^{-\min \{\eta, \gamma\} D_{2}|X|}$ follows. Then the lemma is proven.

Corollary 3.9. Suppose (1) of (EN). Then (44) holds for sufficiently small $|\alpha / \kappa|$.
Proof. Notice that $E^{V} \leq \mathcal{E}^{V}+\mathcal{G}(\alpha / \kappa)-E_{\text {diag }}$ in Lemma [3.3. Since $\mathcal{E}^{V}<0$ and $\lim _{t \rightarrow 0} \mathcal{G}(t)=0$, the corollary follows.

## §3.3. Proofs of Theorem [2.2 and Theorem [2.3]

### 3.3.1. Proof of Theorem [2.2]. Proof of Theorem [2.3:

Note that $0<\mathcal{E}^{V}-\Xi^{V}$ is equivalent to $\inf \sigma\left(H_{\text {eff }}^{V}\right) \in \sigma_{\text {disc }}\left(H_{\text {eff }}^{V}\right)$. Uniform exponential decay

$$
\left\|\Phi_{\sigma}(x)\right\|_{\mathscr{F}} \leq C_{\delta} e^{-\delta|x|}
$$

is shown for sufficiently small $|\alpha / \kappa|$ in Lemma [3.8. Then by $\lim _{\kappa \rightarrow \infty} \mathcal{G}(\alpha / \kappa)=0$ and $\Sigma^{V}-E^{V} \geq \Xi^{V}-\mathcal{E}^{V}-\mathcal{G}(\alpha / \kappa)$, there exists $\kappa_{0}$ such that for arbitrary $\kappa>\kappa_{0}$ the stability condition $E^{V}<\Sigma^{V}$ holds. Therefore, by Proposition [..7, $H^{V}$ has a ground state.
3.3.2. Proof of Theorem [2.3. Now we show the enhanced binding. It is enough to show $\mathcal{E}^{V}<\Xi^{V}$, since the uniform exponential decay $\left\|\Phi_{\sigma}(x)\right\|_{\mathscr{F}}<C_{\delta} e^{-\delta|x|}$ is established by Proposition [2.7.

Lemma 3.10. Let $\beta \varsubsetneqq C_{N}$ but $\beta \neq \emptyset$. Then there exists $\alpha_{1}>0$ such that, for all $\alpha$ with $|\alpha|>\alpha_{1}, \mathcal{E}^{0}<\mathcal{E}^{V}(\beta)+\mathcal{E}^{0}\left(\beta^{c}\right)$. In particular $\mathcal{E}^{0}<\Xi^{V}$ holds for $|\alpha|>\alpha_{1}$.

Proof. We have

$$
\begin{aligned}
& \mathcal{E}^{0}=\alpha^{2} \sum_{i<j} W_{i j}(0)+o\left(\alpha^{2}\right), \quad \mathcal{E}^{V}(\beta)=\alpha^{2} \sum_{\substack{i, j \\
i, j \in \beta}} W_{i j}(0)+o\left(\alpha^{2}\right), \\
& \mathcal{E}^{0}\left(\beta^{c}\right)=\alpha^{2} \sum_{\substack{i<j \\
i, j \in \beta^{c}}} W_{i j}(0)+o\left(\alpha^{2}\right)
\end{aligned}
$$

Since $\sum_{\substack{i<j \\ i \in \beta, j \in \beta^{c}}} W_{i j}(0)+\sum_{\substack{i<j \\ i \in \beta^{c}, j \in \beta}} W_{i j}(0)<0$, the lemma holds.
To see the enhanced binding we want to investigate the center of motion of $h_{\mathrm{eff}}^{V}$. Notice that $h_{\mathrm{eff}}^{0}$ commutes with the total momentum $P_{\mathrm{tot}}=\sum_{j=1}^{N} p_{j}$. Then it can be decomposable with respect to the spectrum of $P_{\text {tot }}$. Let $\mathscr{U}=e^{i x_{1} \cdot \sum_{j=2}^{N} p_{j}}$, which diagonalize $P_{\text {tot }}$ as $\mathscr{U} P_{\text {tot }} \mathscr{U}^{-1}=p_{1}$. Hence it also diagonalize $h_{\text {eff }}^{0}$, and we obtain that

$$
\begin{gathered}
\mathscr{U} h_{\text {eff }}^{0} \mathscr{U}^{-1}=\Omega_{1}\left(p_{1}-\sum_{j=2}^{N} p_{j}\right)+\sum_{j=2}^{N} \Omega_{j}\left(p_{j}\right)+\sum_{j \geq 2} \alpha^{2} W_{1 j}\left(x_{j}\right) \\
\quad+\sum_{2 \leq i<j \leq N} \alpha^{2} W_{i j}\left(x_{i}-x_{j}\right), \\
\mathscr{U} h_{\text {eff }}^{V} \mathscr{U}^{-1}=h_{\text {eff }}^{0}+V\left(x_{1}\right)+\sum_{j=2}^{N} V\left(x_{1}+x_{j}\right) .
\end{gathered}
$$

Then we have

$$
\begin{aligned}
& \mathscr{U} h_{\mathrm{eff}}^{0} \mathscr{U}^{-1}=\int_{\mathbb{R}^{d}}^{\oplus} k(P) d P, \\
& k(P)=\Omega_{1}\left(P-\sum_{j=2}^{N} p_{j}\right)+\sum_{j=2}^{N} \Omega_{j}\left(p_{j}\right) \\
& \quad+\sum_{j \geq 2} \alpha^{2} W_{1 j}\left(x_{j}\right)+\sum_{2 \leq i<j \leq N} \alpha^{2} W_{i j}\left(x_{i}-x_{j}\right) .
\end{aligned}
$$

Lemma 3.11. It follows that $\mathcal{E}^{0}=\inf \sigma(k(0))$.

Proof. Set $\inf \sigma(k(P))=E(P)$ for notational simplicity. It can be seen in Appendix $\square$ that

$$
\begin{equation*}
E(0) \leq E(P) \tag{48}
\end{equation*}
$$

holds for all $P$, and that $E(P)$ is continuous in $P$. Then it follows that $(\Phi, H \Phi)=$ $\int_{\mathbb{R}^{d}}(\Phi(P), k(P) \Phi(P)) d P \geq E(0)\|\Phi\|^{2}$ for $\Phi \in D(H)$. Then $E(0) \leq \mathcal{E}^{0}$. On the other hand let us set $\Phi_{\epsilon}=\int_{\mathbb{R}^{d}}^{\oplus} \Phi(P) \mathbb{1}_{[0, \epsilon)}(P) d P$. We have

$$
\left\|\Phi_{\epsilon}\right\|^{2} \mathcal{E}^{0} \leq\left(\Phi_{\epsilon}, H \Phi_{\epsilon}\right) \leq \sup _{|P|<\epsilon} E(P)\left\|\Phi_{\epsilon}\right\|^{2}
$$

Take $\epsilon \downarrow 0$ on both sides we have $\mathcal{E}^{0} \leq E(0)+\delta$ for arbitrary $\delta>0$, since $E(P)$ is continuous in $P$. Hence $E(0) \geq \mathcal{E}^{0}$ and then $\mathcal{E}^{0}=E(0)$ follows.

Lemma 3.12. There exists $\alpha_{2}(P)>0$ such that $\inf \sigma(k(P)) \in \sigma_{\text {disc }}(k(P))$ for every $P \in \mathbb{R}^{d}$ for $|\alpha|>\alpha_{2}(P)$. In particular $k(0)$ has a ground state for $|\alpha|>\alpha_{2}$ with some $\alpha_{2}>0$.

Proof. Notice that $W_{i j}(0)<0, W_{i j}(x)>W_{i j}(0)$ for $x \neq 0$, and $\lim _{|x| \rightarrow \infty} W_{i j}(x)=$ 0 . Set $\mathbf{X}=\left(x_{2}, \ldots, x_{N}\right)$. Let $a=\{2, \ldots, N\}$. Let $\left\{\widetilde{j}_{\beta}\right\}_{\beta \subset a}$ be the Ruelle-Simon partition of unity [CFKS87, Definition 3.4], i.e., $\widetilde{j}_{\beta}(\lambda \mathbf{X})=\widetilde{j}_{\beta}(\mathbf{X})$ for all $\lambda>1$, $|\mathbf{X}|=1$, and there exists a constant $C>0$ such that

$$
\operatorname{supp} \tilde{j}_{\beta} \cap\{\mathbf{X}| | \mathbf{X} \mid>1\} \subset\left\{\mathbf{X} \| \mathbf{X}_{i}-\mathbf{X}_{j}|\geq C| \mathbf{X} \mid \text { for all }(i j) \not \subset \beta\right\}
$$

We set $j_{\beta}(\mathbf{X})=\widetilde{j}_{\beta}(\mathbf{X} / R)$. Then

$$
\begin{equation*}
k(P)=j_{a} k(P) j_{a}+\sum_{\beta \subsetneq a} j_{\beta} k(P) j_{\beta}+o(\mathbb{1}), \tag{49}
\end{equation*}
$$

where $o(\mathbb{1})$ denotes a bounded operator such that $\lim _{R \rightarrow \infty}\|o(\mathbb{1})\|=0$. We set

$$
\begin{aligned}
& k_{\beta}=\sum_{j \in \beta}\left(\Omega_{j}\left(p_{j}\right)+\alpha^{2} W_{1 j}\left(x_{j}\right)\right)+\sum_{i, j \in \beta} \alpha^{2} W_{i j}\left(x_{i}-x_{j}\right) \\
& \bar{k}_{\beta^{c}}=\sum_{j \in \beta^{c}} \Omega_{j}\left(p_{j}\right)+\sum_{i, j \in \beta^{c}} \alpha^{2} W_{i j}\left(x_{i}-x_{j}\right)
\end{aligned}
$$

With the identification $L^{2}\left(\mathbb{R}^{d(N-1)}\right) \cong L^{2}\left(\mathbb{R}^{d|\beta|}\right) \otimes L^{2}\left(\mathbb{R}^{d\left|\beta^{c}\right|}\right)$, we can write

$$
\begin{equation*}
j_{\beta} k(P) j_{\beta}=j_{\beta} \Omega_{1}\left(P-\sum_{j=2}^{N} p_{j}\right) j_{\beta}+j_{\beta}\left(k_{\beta} \otimes \mathbb{1}+\mathbb{1} \otimes \bar{k}_{\beta^{c}}\right) j_{\beta}+I_{\beta} j_{\beta}^{2} \tag{50}
\end{equation*}
$$

where $I_{\beta}=\sum_{j \in \beta^{c}} \alpha^{2} W_{1 j}\left(x_{j}\right)+\sum_{\substack{i \in \beta, j \in \beta^{c} c \\ i \in \beta^{c}, j \in \beta}} \alpha^{2} W_{i j}\left(x_{i}-x_{j}\right)$. Hence, (4Y) and (50]) imply

$$
k(P) \geq E_{0}(k(P)) j_{a}^{2}+\sum_{\beta \subsetneq a} j_{\beta}\left(k_{\beta} \otimes \mathbb{1}+\mathbb{1} \otimes \bar{k}_{\beta^{c}}+I_{\beta}\right) j_{\beta}+o(\mathbb{1}) .
$$

Note that $j_{a}^{2}$ and $I_{\beta} j_{\beta}^{2}$ are relatively compact with respect to $k(P)$. Thus we have

$$
\inf \sigma_{\text {ess }}(k(P)) \geq \max \left\{E_{0}\left(k_{\beta}\right)+E_{0}\left(\bar{k}_{\beta^{c}}\right) \mid \beta \subsetneq a\right\}
$$

For all $\beta \subsetneq a$ it holds that
(51) $\lim _{\alpha \rightarrow \infty} \frac{E_{0}(k(P))}{\alpha^{2}}=\sum_{i<j} W_{i j}(0)<\sum_{j \in \beta} W_{1 j}(0)+\sum_{\substack{i, j \in \beta \\ i<j}} W_{i j}(0)+\sum_{\substack{i, j \in \beta^{c} \\ i<j}} W_{i j}(0)$

$$
=\lim _{\alpha \rightarrow \infty} \frac{E_{0}\left(k_{\beta}\right)+E_{0}\left(\bar{k}_{\beta^{c}}\right)}{\alpha^{2}}
$$

Therefore there exist $\alpha_{2}(P)$ such that for all $\alpha>\alpha_{2}(P), \inf \sigma_{\text {eff }}(k(P))>E_{0}(k(P))$.

Lemma 3.13. Let $|\alpha|>\alpha_{2}$, where $\alpha_{2}$ is given in Lemma [.]., and $u_{\alpha}$ be a normalized ground state of $k(0)$. Then $\left|u_{\alpha}\left(x_{2}, \ldots, x_{N}\right)\right|^{2} \rightarrow \delta\left(x_{2}\right) \cdots \delta\left(x_{N}\right)$ as $\alpha \rightarrow$ $\infty$ in the sense of distributions.

Proof. It suffices to show that for all $\epsilon>0$,

$$
\begin{equation*}
\lim _{\alpha \rightarrow \infty} \int_{|\mathbf{X}|>\epsilon}\left|u_{\alpha}(\mathbf{X})\right|^{2} d \mathbf{X}=0 \tag{52}
\end{equation*}
$$

where $\mathbf{X}=\left(x_{2}, \cdots, x_{N}\right)$, since (52) implies that

$$
\lim _{\alpha \rightarrow 0} \int_{\mathbb{R}^{d(N-1)}} f(\mathbf{X})\left|u_{\alpha}(\mathbf{X})\right|^{2} d \mathbf{X}=f(0)
$$

for all $f \in C_{0}^{\infty}\left(\mathbb{R}^{d(N-1)}\right)$. We write $k_{\alpha}(0)$ to emphasize the $\alpha$ dependence of $k(0)$. Since $k_{\alpha}(0) / \alpha^{2} \geq \sum_{i<j} W_{i j}(0)$ and $\lim _{\alpha \rightarrow \infty} \inf \sigma\left(k_{\alpha}(0)\right) / \alpha^{2}=\sum_{i<j} W_{i j}(0)$, we have

$$
\begin{aligned}
& \sum_{i<j} W_{i j}(0)=\lim _{\alpha \rightarrow 0} \alpha^{-2}\left(u_{\alpha}, k_{\alpha}(0) u_{\alpha}\right) \\
& \geq \liminf _{\alpha \rightarrow \infty}\left(u_{\alpha},\left(\sum_{j \geq 2} W_{1 j}\left(x_{j}\right)+\sum_{2 \leq i<j \leq N} W_{i j}\left(x_{i}-x_{j}\right)\right) u_{\alpha}\right) \geq \sum_{i<j} W_{i j}(0) .
\end{aligned}
$$

Then

$$
\begin{equation*}
\sum_{i<j} W_{i j}(0)=\liminf _{\alpha \rightarrow \infty}\left(u_{\alpha},\left(\sum_{j \geq 2} W_{1 j}\left(x_{j}\right)+\sum_{2 \leq i<j \leq N} W_{i j}\left(x_{i}-x_{j}\right)\right) u_{\alpha}\right) \tag{53}
\end{equation*}
$$

follows. Suppose that $c_{\epsilon}=\liminf _{\alpha \rightarrow \infty} \int_{|\mathbf{X}|>\epsilon}\left|u_{\alpha}(\mathbf{X})\right|^{2} \mathbf{X}>0$. Then

$$
\begin{aligned}
& \liminf _{\alpha \rightarrow \infty} \int_{\mathbb{R}^{d(N-1)}} \sum_{j \geq 2}\left(W_{1 j}\left(x_{j}\right)-W_{1 j}(0)\right)\left|u_{\alpha}(\mathbf{X})\right|^{2} d \mathbf{X} \\
& >c_{\epsilon} \sum_{j \geq 2} \sup _{|\mathbf{X}|>\epsilon}\left(W_{1 j}\left(x_{j}\right)-W_{1 j}(0)\right)>0
\end{aligned}
$$

which contradicts (53]). Therefore (52) holds.
Proof of Theorem [2.3:
First we assume that $V \in C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$. It is enough to show $\mathcal{E}^{V}<\Xi^{V}$, since the uniform exponential decay $\left\|\Phi_{\sigma}(x)\right\| \leq C_{\delta} e^{-\delta|x|}$ is established in Lemma 3.8 for sufficiently small $|\alpha / \kappa|$. Assume $|\alpha|>\max \left\{\alpha_{1}, \alpha_{2}\right\}>0$. Let $u_{\alpha}$ be a normalized ground state of $k(0)$. By $\Omega_{1}(a+b) \leq|a|+\Omega_{1}(b)$ for $a, b \in \mathbb{R}^{d}$, we have

$$
\begin{equation*}
\mathscr{U} h_{\mathrm{eff}}^{0} \mathscr{U}^{-1} \leq \sqrt{-\Delta_{1}}+k(0) \tag{54}
\end{equation*}
$$

By (2) of (EN), there exists a normalized vector $v \in C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$ such that

$$
\begin{equation*}
(v,(\sqrt{-\Delta}+N V) v)<0 \tag{55}
\end{equation*}
$$

We set $\Psi\left(x_{1}, \cdots, x_{N}\right)=v\left(x_{1}\right) u_{\alpha}\left(x_{2}, \cdots, x_{N}\right)$. Then, by (54)

$$
\begin{equation*}
\mathcal{E}^{V} \leq\left(\Psi, \mathscr{U} h_{\mathrm{eff}}^{V} \mathscr{U}^{-1} \Psi\right) \leq(v,(\sqrt{-\Delta}+V) v)+\mathcal{E}^{0}+\left(\Psi, \sum_{j=2}^{N} V\left(x_{1}+x_{j}\right) \Psi\right) \tag{56}
\end{equation*}
$$

Let $V_{j, \text { smeared }}^{\alpha}\left(x_{1}\right)=\int_{\mathbb{R}^{d(N-1)}} V\left(x_{j}+x_{1}\right)\left|u_{\alpha}(\mathbf{X})\right|^{2} d \mathbf{X}$. By Lemma [3.13, we have

$$
\lim _{\alpha \rightarrow \infty}\left(\Psi, V\left(x_{j}+x_{1}\right) \Psi\right)=\lim _{\alpha \rightarrow \infty}\left(v, V_{j, \text { smeared }}^{\alpha} v\right)=(v, V v)
$$

and then by (5.5) and (56),

$$
\begin{equation*}
\mathcal{E}^{V} \leq(v,(\sqrt{-\Delta}+N V) v)+\mathcal{E}^{0}<\mathcal{E}^{0} \tag{57}
\end{equation*}
$$

follows for $\alpha>\alpha_{3}$ with some $\alpha_{3}>0$. By this inequality and Lemma B.IT, we conclude that for $\alpha$ with $|\alpha|>\bar{\alpha}=\max \left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$,

$$
\begin{aligned}
\Sigma^{V}-E^{V} & \geq \Xi^{V}-\mathcal{E}^{V}-\mathcal{G}(\alpha / \kappa) \geq \mathcal{E}^{0}-\mathcal{E}^{V}-\mathcal{G}(\alpha / \kappa) \\
& >-(v,(\sqrt{-\Delta}+N V) v)-\mathcal{G}(\alpha / \kappa)
\end{aligned}
$$

Notice that $\mathcal{G}(\alpha / \kappa) \rightarrow 0$ as $\kappa \rightarrow \infty$ and $-(v,(\sqrt{-\Delta}+N V) v)>0$. Then the righthand side above is positive for sufficiently small $|\alpha| / \kappa$. Since $\mathcal{G}$ is monotonously increasing, it is trivial to see that $\kappa_{0}=\bar{\alpha} / \mathcal{G}^{-1}(a)$, where $a=-(v,(\sqrt{-\Delta}+N V) v)$
and $\bar{\alpha}(\kappa)=\mathcal{G}^{-1}(a) \kappa$. Then the theorem follows for $V \in C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$. For general $V$ we can prove the theorem by the same limiting argument as [HS08, Appendix]. See Appendix ${ }^{5}$

## Appendix

## §4. Stability condition:relativistic version

In this section we shall prove Proposition [2.7. We only show an outline of the proof. The detail is left to the reader.

## §4.1. Case $\sigma>0$

Since the scaling parameter $\kappa$ does not play any role in this section we set $\kappa=1$. Let $\sigma>0$. We decompose the single boson Hilbert space into high energy part and low energy part as $L^{2}\left(\mathbb{R}^{d}\right) \cong \mathcal{K}_{>\sigma} \oplus \mathcal{K}_{\leq \sigma}$, where $\mathcal{K}_{\leq \sigma}=L^{2}\left(\left\{k \in \mathbb{R}^{d} \mid \omega(k) \leq \sigma\right\}\right)$ and $\mathcal{K}_{>\sigma}=L^{2}\left(\left\{k \in \mathbb{R}^{d} \mid \omega(k)>\sigma\right\}\right)$. Correspondingly, we have the identification:

$$
\begin{equation*}
\mathscr{H} \cong \mathscr{H}_{>\sigma} \otimes \mathscr{F}\left(\mathcal{K}_{\leq \sigma}\right), \tag{58}
\end{equation*}
$$

where $\mathscr{H}_{>\sigma}=L^{2}\left(\mathbb{R}^{d N}\right) \otimes \mathscr{F}\left(\mathcal{K}_{>\sigma}\right)$. We define the regularized Hamiltonian by

$$
\begin{equation*}
H_{\sigma}^{V}=H_{0}+H_{\mathrm{I}, \sigma} \tag{59}
\end{equation*}
$$

Here $H_{\mathrm{I}, \sigma}$ is the regularized interaction defined by

$$
H_{\mathrm{I}, \sigma}=\sum_{j=1}^{N} \alpha_{j} \int_{\mathbb{R}^{d N}}^{\oplus} \phi_{j, \sigma}\left(x_{j}\right) d X
$$

and $\phi_{j, \sigma}(x)$ is given by $\phi_{j}(x)$ with cutoff $\lambda_{j}(k)$ replaced by $\lambda_{j}(k) \mathbb{1}_{\omega(k)>\sigma}(k)$. Then $H_{\sigma}^{V}$ approximates $H^{V}$ in the following sense:

Lemma 4.1. $H_{\sigma}^{V}$ converges to $H^{V}$ as $\sigma \rightarrow 0$ in the norm resolvent sense.
Let $E_{\sigma}^{V}=\inf \sigma\left(H_{\sigma}^{V}\right)$ and $\Sigma_{\sigma}^{V}$ be a lowest two cluster threshold for $H_{\sigma}^{V}$, which is defined in the same way as $\Sigma^{V}$. From Lemma $4 . \square$, we can show that $E_{\sigma}^{V}$ and $\Sigma_{\sigma}^{V}$ converges to $E^{V}$ as $\Sigma^{V}$ as $\sigma \rightarrow 0$, respectively. Therefore for sufficiently small $\sigma>0$, it follows that

$$
\begin{equation*}
\Sigma_{\sigma}^{V}>E_{\sigma}^{V} \tag{60}
\end{equation*}
$$

Under the identification (58), $H_{\sigma}^{V}$ can be decomposed as

$$
H_{\sigma}^{V} \cong H_{\sigma}^{V}\left\lceil\mathscr{H}_{>\sigma} \otimes \mathbb{1}_{\mathscr{F}\left(\mathcal{K}_{\leq \sigma}\right)}+\mathbb{1}_{\mathscr{H}_{>\sigma}} \otimes H_{\mathrm{f}}\left\lceil\mathscr{F}\left(\mathcal{K}_{\leq \sigma}\right)\right.\right.
$$

Since $H_{\mathrm{f}}\left\lceil_{\mathscr{F}\left(\mathcal{K}_{<\sigma}\right)}\right.$ has a ground state, $H_{\sigma}^{V}$ also may have a ground state if and only if $H_{\sigma}^{V}\left\lceil_{\mathscr{H}_{>\sigma}}\right.$ does. We shall prove the existence of a ground state of $H_{\sigma}^{V}\left\lceil_{\mathscr{H}_{>\sigma}}\right.$ for sufficiently small $\sigma>0$ in what follows. For $\sigma>0$, we truncate $\omega$ as

$$
\omega_{\sigma}(k)=\left\{\begin{array}{lll}
|k| & \text { for } & |k|>\sigma \\
\sigma & \text { for } & |k| \leq \sigma
\end{array}\right.
$$

and we set $H_{\mathrm{f}, \sigma}=\mathrm{d} \Gamma\left(\omega_{\sigma}\right)$. Then

$$
H_{\sigma}^{V}\left\lceil_{\mathscr{H}>_{\sigma}}=H_{0, \sigma}+H_{\mathrm{I}, \sigma}\right.
$$

with $H_{0, \sigma}=H_{\mathrm{p}} \otimes \mathbb{1}+\mathbb{1} \otimes H_{\mathrm{f}, \sigma}$. We denote the Fourier transformation from $L^{2}\left(\mathbb{R}_{y}^{d}\right)$ to $L^{2}\left(\mathbb{R}_{k}^{d}\right)$ by $F$. We set $\check{\mathcal{K}}_{>\sigma}=\left\{\check{f}=F^{-1} f \in L^{2}\left(\mathbb{R}_{y}^{d}\right) \mid f \in \mathcal{K}_{>\sigma}\right\}$. We introduce a notation. Let $T: \mathcal{K}_{1} \rightarrow \mathcal{K}_{2}$ be a contraction operator from a Hilbert space $\mathcal{K}_{1}$ to another one $\mathcal{K}_{2}$. Then we define $\Gamma(T)=\oplus_{n=0}^{\infty} \otimes^{n} T$ with $\otimes^{0} T=\mathbb{1}$, which is also a contraction operator from $\mathscr{F}\left(\mathcal{K}_{1}\right)$ to $\mathscr{F}\left(\mathcal{K}_{2}\right)$. Let

$$
\check{H}_{\sigma}^{V}=\Gamma\left(F^{-1}\right) H_{\sigma}^{V}\left\lceil\mathscr{H}_{>\sigma} \Gamma(F),\right.
$$

which is defined on $\check{\mathscr{H}}_{>\sigma}=L^{2}\left(\mathbb{R}^{d N}\right) \otimes \mathscr{F}\left(\check{\mathcal{K}}_{>\sigma}\right)$. Let $\chi, \bar{\chi} \in C^{\infty}\left(\mathbb{R}^{d N}\right)$ be a cutoff function such that $\chi(X)^{2}+\bar{\chi}(X)^{2}=1$ with $\chi(X)=1$ for $|X| \leq 1$ and $\chi(X)=0$ for $|X| \geq 2$. Then the following statement holds: For $R>0$, we set $\chi_{R}(X)=\chi(X / R)$, $\bar{\chi}_{R}(X)=\bar{\chi}(X / R)$.

Lemma 4.2. It follows that

$$
\check{H}_{\sigma}^{V}=\chi_{R} \check{H}_{\sigma}^{V} \chi_{R}+\bar{\chi}_{R} \check{H}_{\sigma}^{V} \bar{\chi}_{R}+\hat{O}\left(R^{-1}\right)
$$

where $\hat{O}\left(R^{-1}\right)$ is an operator such that $\left\|\hat{O}\left(R^{-1}\right)\right\| \leq C / R$ for some constant $C>0$.
Proof. The operator equality
(61)

$$
\check{H}_{\sigma}^{V}=\chi_{R} \check{H}_{\sigma}^{V} \chi_{R}+\bar{\chi}_{R} \check{H}_{\sigma}^{V} \bar{\chi}_{R}+\frac{1}{2} \sum_{j=1}^{N}\left[\chi_{R},\left[\chi_{R}, \Omega_{j}\left(p_{j}\right)\right]\right]+\frac{1}{2} \sum_{j=1}^{N}\left[\bar{\chi}_{R},\left[\bar{\chi}_{R}, \Omega_{j}\left(p_{j}\right)\right]\right] .
$$

holds. By the Fourier transformation, we have

$$
\left[\chi_{R}, \Omega_{j}\left(p_{j}\right)\right]=(2 \pi)^{-d N / 2} \int_{\mathbb{R}^{d N}} \hat{\chi}(K) e^{i K \cdot X / R}\left(\Omega_{j}\left(p_{j}\right)-\Omega_{j}\left(p_{j}-k_{j} / R\right)\right) d K
$$

where $K=\left(k_{1}, \cdots, k_{N}\right) \in \mathbb{R}^{d N}$. By the triangle inequality, we have

$$
\begin{aligned}
\left|\Omega_{j}\left(p_{j}\right)-\Omega_{j}\left(p_{j}-\frac{k_{j}}{R}\right)\right|= & \left|\left\|\left(p_{j}, m_{j}\right)\right\|_{\mathbb{C}^{4}}-\left\|\left(p_{j}-\frac{k_{j}}{R}, m_{j}\right)\right\|_{\mathbb{C}^{4}} \|\right| \\
& \leq\left\|\left(\frac{k_{j}}{R}, 0\right)\right\|_{\mathbb{C}^{4}}=\frac{1}{R}\left|k_{j}\right|
\end{aligned}
$$

Hence, $\left[\chi_{R}, \Omega_{j}\left(p_{j}\right)\right]$ is a bounded operator with the bound

$$
\begin{equation*}
\left\|\left[\chi_{R}, \Omega_{j}\left(p_{j}\right)\right]\right\| \leq \frac{1}{R}(2 \pi)^{-d N / 2} \int_{\mathbb{R}^{d N}}|\hat{\chi}(K)| \cdot\left|k_{j}\right| d K \tag{62}
\end{equation*}
$$

Similarly, by noting that $\mathbb{1}-\bar{\chi} \in C_{0}^{\infty}\left(\mathbb{R}^{d N}\right)$ and $\left[\bar{\chi}_{R}, \Omega_{j}\left(p_{j}\right)\right]=\left[\mathbb{1}-\bar{\chi}_{R}, \Omega_{j}\left(p_{j}\right)\right]$, we have

$$
\left.\left.\left\|\left[\bar{\chi}_{R}, \Omega_{j}\left(p_{j}\right)\right]\right\| \leq \frac{1}{R}(2 \pi)^{-d N / 2} \int_{\mathbb{R}^{d N}} \right\rvert\,(\mathbb{1} \widehat{-\bar{\chi}(K})\right)|\cdot| k_{j} \mid d K .
$$

Then the lemma follows.
Let $j, \bar{j} \in C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$ be another cutoff function such that $j(y)^{2}+\bar{j}(y)^{2}=1$ for every $y \in \mathbb{R}^{d}$ with $j(y)=1$ for $|y| \leq 1$ and $j(y)=0$ for $|y| \geq 2$. We set $j_{P}(y)=j(y / P), \bar{j}_{P}(y)=\bar{j}(y / P)$ for $P>0$. The map

$$
u_{P}: \check{\mathcal{K}}_{>\sigma} \rightarrow L^{2}\left(\mathbb{R}_{y}^{d}\right) \oplus L^{2}\left(\mathbb{R}_{y}^{d}\right), \quad f \mapsto j_{P} f \oplus \bar{j}_{P} f
$$

is isometry, since $\left\|j_{P} f \oplus \bar{j}_{P} f\right\|^{2}=\|f\|^{2}$. We also note that adjoint $u_{P}^{*}$ maps $f \oplus g \in L^{2}\left(\mathbb{R}^{d}{ }_{y}\right) \oplus L^{2}\left(\mathbb{R}^{d}{ }_{y}\right)$ to $j_{P} f+\bar{j}_{P} g \in L^{2}\left(\mathbb{R}^{d}\right)$. The operator

$$
U_{P}=\mathbb{1}_{L^{2}\left(\mathbb{R}^{d N}\right)} \otimes \Gamma\left(u_{P}\right): \check{\mathscr{H}}_{>\sigma} \rightarrow \check{\mathscr{H}} \otimes \mathscr{F}\left(L^{2}\left(\mathbb{R}_{y}^{d}\right)\right)
$$

is also an isometry, where $\check{\mathscr{H}}=L^{2}\left(\mathbb{R}^{d N}\right) \otimes \mathscr{F}\left(L^{2}\left(\mathbb{R}_{y}^{d}\right)\right)$. Let

$$
\check{H}_{0, \sigma}=\Gamma\left(F^{-1}\right) H_{0, \sigma} \Gamma(F), \quad \check{H}_{\mathrm{f}, \sigma}=\Gamma\left(F^{-1}\right) H_{\mathrm{f}, \sigma} \Gamma(F)
$$

Lemma 4.3. (1) For every $\sigma>0$, we have

$$
\chi_{R} \check{H}_{\sigma}^{V} \chi_{R}=\chi_{R} U_{P}^{*}\left\{\check{H}_{\sigma}^{V} \otimes \mathbb{1}+\mathbb{1} \otimes \check{H}_{\mathrm{f}, \sigma}\right\} U_{P} \chi_{R}+\hat{o}(\mathbb{1}),
$$

as operators in $\mathscr{H}_{>\sigma}$, where $\hat{o}(\mathbb{1})$ denotes an operator such that $\hat{o}(\mathbb{1})\left(\check{H}_{0, \sigma}+\mathbb{1}\right)^{-1}$ is bounded and $\lim _{P \rightarrow \infty} \lim _{R \rightarrow \infty}\left\|\hat{o}(\mathbb{1})\left(\check{H}_{0, \sigma}+1\right)^{-1}\right\|=0$. (2) We have

$$
\bar{\chi}_{R} \check{H}_{\sigma}^{V} \bar{\chi}_{R} \geq \Sigma_{\sigma}^{V} \bar{\chi}_{R}^{2}+o\left(R^{0}\right)
$$

where $o\left(R^{0}\right)$ is a number such that $\lim _{R \rightarrow \infty} o\left(R^{0}\right)=0$.
Proof. See [GLLOT1, Lemma A.1].
Proposition 4.4. There exists a ground state of $H_{\sigma}^{V}$.
Proof. By Lemma 4.2 and Lemma 4.3,

$$
\check{H}_{\sigma}^{V}=\chi_{R} U_{P}^{*}\left\{\check{H}_{\sigma}^{V} \otimes \mathbb{1}+\mathbb{1} \otimes \check{H}_{\mathrm{f}, \sigma}\right\} U_{P} \chi_{R}+\bar{\chi}_{R} \check{H}_{\sigma}^{V} \bar{\chi}_{R}+\hat{o}(\mathbb{1}) .
$$

Since $\omega_{\sigma} \geq \sigma$, we have $\check{H}_{\mathrm{f}, \sigma} \geq \sigma\left(\mathbb{1}-P_{\Omega}\right)$, where $P_{\Omega}$ denotes the orthogonal projection on the vacuum space $\{\mathbb{C} \Omega\}$. By this inequality and Lemma 4.3 ,

$$
\check{H}_{\sigma}^{V} \geq\left(E_{\sigma}^{V}+\sigma\right) \chi_{R}^{2}+\Sigma_{\sigma}^{V} \bar{\chi}_{R}^{2}-K+\hat{o}(\mathbb{1})
$$

where $K=\sigma \chi_{R} U_{P}^{-1}\left(\mathbb{1} \otimes P_{\Omega}\right) U_{P} \chi_{R}=\chi_{R}^{2} \otimes \Gamma\left(j_{P}\right)$. $K$ is relatively compact with respect to $\sum_{j=1}^{N} \Omega_{j}+\check{H}_{\mathrm{f}, \sigma}$. Since, by (V), $\sum_{j=1}^{N} \Omega_{j}+\check{H}_{\mathrm{f}, \sigma}$ is also relatively bounded with respect to $\check{H}_{\sigma}^{V}, K$ is then relatively compact with respect to $\check{H}_{\sigma}^{V}$. By the definition of $\hat{o}(\mathbb{1})$, there is a constant $C$ independent of $P$ and $R$ such that $\hat{o}(\mathbb{1}) \geq$ $-o(\mathbb{1})\left(\check{H}_{\sigma}^{V}+C\right)$. Thus, we have the operator inequality

$$
(1+o(\mathbb{1})) \check{H}_{\sigma}^{V}-E_{\sigma}^{V}+o(\mathbb{1})-K \geq \sigma \chi_{R}^{2}+\left(\Sigma_{\sigma}^{V}-E_{\sigma}^{V}\right) \bar{\chi}_{R}^{2} \geq \min \left\{\sigma, \Sigma_{\sigma}^{V}-E_{\sigma}^{V}\right\}
$$

Since $K$ does not change the essential spectrum of $\check{H}_{\sigma}^{V}$, for all $P$ and $R$, we have

$$
(1+o(\mathbb{1})) \inf \left(\sigma_{\mathrm{ess}}\left(H_{\sigma}^{V}\right)\right)-E_{\sigma}^{V}+o(\mathbb{1}) \geq \min \left\{\sigma, \Sigma_{\sigma}^{V}-E_{\sigma}^{V}\right\}
$$

Hence, by (601),

$$
\inf \sigma_{\mathrm{ess}}\left(H_{\sigma}^{V}\right)-E_{\sigma}^{V} \geq \min \left\{\sigma, \Sigma_{\sigma}^{V}-E_{\sigma}^{V}\right\}>0
$$

Therefore $\sigma\left(\check{H}_{\sigma}^{V}\right) \cap\left[E_{\sigma}^{V}, E_{\sigma}^{V}+\min \left\{\sigma, \Sigma_{\sigma}^{V}-E_{\sigma}^{V}\right\}\right)$ is purely discrete spectrum. In particular $H_{\sigma}^{V}$ has a ground state.

## §4.2. Case $\sigma=0$

Next we prove the existence of ground state of $H^{V}$. For $\sigma>0$, let $\Phi_{\sigma} \in \mathscr{H}$ be a normalized ground state of $H_{\sigma}^{V}$. Let $\left\{\sigma_{n}\right\}$ be a sequence such that $\lim _{n \rightarrow \infty} \sigma_{n}=0$ and $\Phi_{\sigma_{n}}$ converges weakly to some vector $\Phi \in \mathscr{H}$. It is well known in [AH97] that if $\Phi \neq 0$ then $\Phi$ is a ground state of $H^{V}$. In the following we prove that a subsequence of $\left\{\Phi_{\sigma}\right\}_{\sigma}$ converges to some non-zero vector $\Phi$.

Lemma 4.5. The energy bound $\sup _{0<\sigma \ll 1}\left(\Phi_{\sigma}, H_{0} \Phi_{\sigma}\right)<\infty$ holds. In addition we suppose $E^{V}<\Sigma^{V}$. Then $\sup _{0<\sigma \ll 1}\left(\Phi_{\sigma}, N \Phi_{\sigma}\right)<\infty$.

Proof. The former follows from the definition of $\Phi_{\sigma}$, and the later from [Ger00, Lemma IV2].

We denote the set of bounded operator on a Hilbert space $\mathcal{K}$ by $B(\mathcal{K})$. For each $k \in \mathbb{R}^{d}$, let

$$
v(k)=\sum_{j=1}^{N} \frac{\alpha_{j}}{\sqrt{2}} \hat{\lambda}_{j}(-k) e^{-i k x_{j}} .
$$

Then $v(k) \in B\left(L^{2}\left(\mathbb{R}_{X}^{d N}\right)\right)$. For each $k \in \mathbb{R}^{d}$, we set

$$
T(k)=\left(H^{V}-E^{V}+\omega(k)\right)^{-1}\left(v(k) \otimes \mathbb{1}_{\mathscr{F}}\right) .
$$

Then $T(k) \in B(\mathscr{H})$ for every $k \in \mathbb{R}^{d},(\Psi, T(k) \Phi)$ is measurable for all $\Phi, \Psi \in \mathscr{H}$, and $\int_{\mathbb{R}^{d}}\|T(k)\|_{B(\mathscr{H})}^{2} d k<\infty$. Hence $T(\cdot)$ can be regarded as a vector in the Banach space $L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)$. Since $\Phi_{\sigma} \in D\left(N^{1 / 2}\right), a(k) \Phi_{\sigma}$ is well defined for almost every $k \in \mathbb{R}^{d}$. Let $\theta_{s}, s \in \mathbb{R}^{d}$, be the shift on $L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)$, i.e., for $B \in L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)$,

$$
\left(\theta_{s} B\right)(k)=B(k-s), \quad \text { a.e. } k \in \mathbb{R}^{d} .
$$

Lemma 4.6. The map $\mathbb{R}^{d} \ni s \mapsto\left\|\theta_{s} T e^{-\delta|x|}\right\|_{L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)} \in \mathbb{R}$ is continuous.
Proof. Since $\theta_{s}$ is a translation, it is enough to show that $\left\|\theta_{s} T e^{\delta|x|}\right\|$ is continuous at $s=0$, i.e., $\left\|\theta_{s} T e^{-\delta|x|}-T e^{-\delta|x|}\right\|_{L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)}$ converges to 0 as $s \rightarrow 0$. We have

$$
\begin{align*}
& \left\|\theta_{s} T e^{-\delta|x|}-T e^{-\delta|x|}\right\|_{L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)}  \tag{63}\\
& \leq\left(\int_{|k| \leq C_{1}}+\int_{|k| \geq C_{2}}+\int_{C_{1}<|k|<C_{2}}\right)\left\|T(k-s) e^{-\delta|x|}-T(k) e^{-\delta|x|}\right\|_{B(\mathscr{H})}^{2}
\end{align*}
$$

for $0<C_{1}<C_{2}$. For $C_{1}<|k|<C_{2}$, we write

$$
\begin{aligned}
& T(k-s) e^{-\delta|x|}-T(k) e^{-\delta|x|} \\
& =\left(H^{V}-E^{V}+\omega(k)\right)^{-1}\left(\sum_{j=1}^{N} \Omega_{j}+\mathbb{1}\right)\left(\sum_{j=1}^{N} \Omega_{j}+\mathbb{1}\right)^{-1}(v(k-s)-v(k)) e^{-\delta|x|} \\
& +\left(H^{V}-E^{V}+\omega(k)\right)^{-1}\left(H^{V}-E^{V}+\omega(k-s)\right)^{-1} v(k-s)(\omega(k-s)-\omega(k)) e^{-\delta|x|} .
\end{aligned}
$$

Since for all $k$ with $C_{1}<|k|<C_{2}$

$$
\sup _{C_{1} \leq|k|}\left\|\left(H^{V}-E^{V}+\omega(k)\right)^{-1}\left(\sum_{j=1}^{N} \Omega_{j}+\mathbb{1}\right)\right\|<\infty
$$

we have

$$
\begin{aligned}
& \left\|T(k-s) e^{-\delta|x|}-T(k) e^{-\delta|x|}\right\|_{B(\mathscr{H})} \\
& \leq C\left\|\left(\sum_{j=1}^{N} \Omega_{j}+\mathbb{1}\right)^{-1} e^{-\delta|x|}(v(k-s)-v(k))\right\|_{B(\mathscr{H})}+C\left\|e^{-\delta|x|} v(k-s)\right\|_{B(\mathscr{H})}
\end{aligned}
$$

for some constant $C>0$ depending on $C_{1}$ and $C_{2}$. Note that

$$
\left(\sum_{j=1}^{N} \Omega_{j}+\mathbb{1}\right)^{-1} e^{-\delta|x|}
$$

is compact. By Proposition 4.7 below, we have

$$
\begin{equation*}
\lim _{s \rightarrow 0} \int_{C_{1}<|k|<C_{2}}\left\|\left(\sum_{j=1}^{N} \Omega_{j}+\mathbb{1}\right)^{-1} e^{-\delta|x|}(v(k-s)-v(k))\right\|_{B(\mathscr{H})}^{2} d k=0 \tag{64}
\end{equation*}
$$

Next we see that

$$
\begin{aligned}
& \lim _{s \rightarrow 0} \int_{|k| \leq C_{1}}\left\|T(k-s) e^{-\delta|x|}-T(k) e^{-\delta|x|}\right\|_{B(\mathscr{H})}^{2} d k \\
& \leq 2 \lim _{s \rightarrow 0} \int_{|k| \leq C_{1}}\left(\frac{|\hat{\lambda}(-k)|^{2}}{|\omega(k)|^{2}}+\frac{|\hat{\lambda}(-k+s)|^{2}}{|\omega(-k+s)|^{2}}\right) d k \leq 4 \int_{k \leq C_{1}} \frac{|\hat{\lambda}(-k)|^{2}}{\omega(k)^{2}} d k
\end{aligned}
$$

and the right-hand side above converges to zero as $C_{1} \rightarrow 0$. Similarly,

$$
\begin{equation*}
\lim _{C_{2} \rightarrow \infty} \lim _{s \rightarrow 0} \int_{|k| \geq C_{2}}\left\|T(k-s) e^{-\delta|x|}-T(k) e^{-\delta|x|}\right\|_{B(\mathscr{H})}^{2} d k=0 . \tag{65}
\end{equation*}
$$

Therefore, by combining (64) - (65), we complete the proof.
Proposition 4.7. [Ger06, proof of Lemma 3.2] Suppose that map $\mathbb{R}^{d} \ni k \mapsto$ $m(k) \in B\left(L^{2}\left(\mathbb{R}^{d N}\right)\right)$ is a weakly measurable map such that for all $0<C_{1}<C_{2}$,

$$
\int_{C_{1} \leq|k| \leq C_{2}}\|m(k)\|_{B\left(L^{2}\left(\mathbb{R}^{d N}\right)\right)}^{2} d k<\infty
$$

and $R$ be a compact operator on $L^{2}\left(\mathbb{R}^{d N}\right)$. Then for all $0<C_{1}<C_{2}$,

$$
\lim _{s \rightarrow 0} \int_{C_{1}<|k|<C_{2}}\|R(m(k-s)-m(k))\|_{B\left(L^{2}\left(\mathbb{R}^{d N}\right)\right)}^{2} d k=0
$$

Lemma 4.8. Let $F \in C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$ be a cutoff function with $0 \leq F \leq 1, F(s)=1$ for $|s| \leq 1 / 2, F(s)=0$ for $|s| \geq 1$. Let $F_{R}=F_{R}\left(-i \nabla_{k}\right)=F\left(-i \nabla_{k} / R\right)$. Then

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \sup _{0<\sigma \ll 1}\left(\Phi_{\sigma}, \mathrm{d} \Gamma\left(\mathbb{1}-F_{R}\right) \Phi_{\sigma}\right)=0 \tag{66}
\end{equation*}
$$

Proof. It is shown in [Ger00, proof of Proposition IV.3] that

$$
\lim _{\sigma \rightarrow 0} \int_{\mathbb{R}^{d}}\left\|a(k) \Phi_{\sigma}-T(k) \Phi_{\sigma}\right\|_{\mathscr{H}}^{2} d k=0
$$

Then

$$
\left(\Phi_{\sigma}, \mathrm{d} \Gamma\left(\mathbb{1}-F_{R}\right) \Phi_{\sigma}\right)_{\mathscr{H}}=\int_{\mathbb{R}^{d}}\left(T(k) \Phi_{\sigma},\left(\mathbb{1}-F_{R}\right) T(k) \Phi_{\sigma}\right)_{\mathscr{H}} d k+o\left(\sigma^{0}\right),
$$

where $o\left(\sigma^{0}\right)$ denotes a constant converges to 0 as $\sigma \rightarrow 0$. By Cauchy-Schwarz inequality yields that the right-hand side above has the upper bound by

$$
\begin{equation*}
\|T\|_{L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)} \cdot\left\|\left(\mathbb{1}-F_{R}\right) T(k) e^{-\delta|x|}\right\|_{L^{2}\left(\mathbb{R}_{k}^{d} ; B(\mathscr{H})\right)} \cdot\left\|e^{\delta|x|} \Phi_{\sigma}\right\|_{\mathscr{H}}+o\left(\sigma^{0}\right) \tag{67}
\end{equation*}
$$

Note that $\sup _{0<\sigma \ll 1}\left\|e^{\delta|x|} \Phi_{\sigma}\right\|_{\mathscr{H}}<\infty$ for some $\delta>0$ by assumption. By the Fourier transformation, we have

$$
\begin{align*}
& \left\|\left(\mathbb{1}-F_{R}\right) T(k) e^{-\delta|x|}\right\|_{L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)}^{2}  \tag{68}\\
& =\int_{\mathbb{R}^{d}}\left\|(2 \pi)^{-d / 2} \int_{\mathbb{R}^{d}} d s \hat{F}(s)\left(\mathbb{1}-\theta_{-s / R}\right) T(k) e^{-\delta|x|}\right\|_{B(\mathscr{H})}^{2} d k \\
& \leq(2 \pi)^{-d} \int_{\mathbb{R}^{d}}|\hat{F}(s)|^{2} \cdot\left\|\left(\mathbb{1}-\theta_{-s / R}\right) T e^{-\delta|x|}\right\|_{L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)} d s .
\end{align*}
$$

Notice that

$$
|\hat{F}(s)|^{2} \cdot\left\|\left(\mathbb{1}-\theta_{-s / R}\right) T e^{-\delta|x|}\right\|_{L^{2}\left(\mathbb{R}^{d} ; B(\mathscr{H})\right)} \leq|\hat{F}(s)|^{2} \cdot 2\left\|\hat{\lambda}_{j} / \omega\right\|
$$

and the right-hand-side above is integrable in $s$ and independent of $R$. Moreover, Lemma 4.6 implies that the integrand of the last term in ( $\overline{68)}$ ) converges to 0 as $R \rightarrow \infty$. Therefore, by the Lebesgue dominated convergence theorem, ( 68 ) converges to 0 as $R \rightarrow \infty$, and hence (66) holds.

Proposition 4.9. (Proof of Proposition [2.7) $H^{V}$ has a ground state.
Proof. The proof is parallel with that of [Ger00, Lemma IV.5]. By $\left(\mathbb{1}-\Gamma\left(F_{R}\right)\right)^{2} \leq$ $\mathrm{d} \Gamma\left(\mathbb{1 1}-F_{R}\right)$ and Lemma 4.8 , we have

$$
\begin{equation*}
\left\|\left(\mathbb{1}-\Gamma\left(F_{R}\right)\right) \Phi_{\sigma}\right\| \leq o\left(R^{0}\right)+o\left(\sigma^{0}\right) . \tag{69}
\end{equation*}
$$

Let $\left\{\sigma_{n}\right\}_{n}$ be the subsequence such that $\lim _{n \rightarrow \infty} \sigma_{n}=0$ and $\Phi=\mathrm{w}-\lim _{n \rightarrow \infty} \Phi_{\sigma_{n}}$. By Lemmas 4.5 and [3.8, ( $\mathbf{6 . 9}$ ), for all $\varepsilon>0$, there exist $R_{0}>0, \lambda_{0}>0, n_{0}>0$ such that for all $R>R_{0}, \lambda_{0}>\lambda$ and $n \geq n_{0}$,

$$
\begin{aligned}
& \left\|\left(\mathbb{1}-\chi\left(H_{0} \leq \lambda\right)\right) \Phi_{\sigma_{n}}\right\|<\varepsilon, \quad\left\|(\mathbb{1}-\chi(N \leq \lambda)) \Phi_{\sigma_{n}}\right\|<\varepsilon, \\
& \left\|(\mathbb{1}-\chi(|X| \leq \lambda)) \Phi_{\sigma_{n}}\right\|<\varepsilon, \quad\left\|\left(\mathbb{1}-\Gamma\left(F_{R}\right)\right) \Phi_{\sigma_{n}}\right\|<\varepsilon,
\end{aligned}
$$

where $\chi(s \leq \lambda)$ denotes a characteristic function of support $\{s \in \mathbb{R} \mid s<\lambda\}$. Note that $K=\chi\left(H_{0} \leq \lambda\right) \chi(N \leq \lambda) \chi(|X| \leq \lambda) \Gamma\left(F_{R}\right)$ is a compact operator. For all
large $R>0, \lambda>0$, we have

$$
\begin{aligned}
\|\Phi\| & \geq\|K \Phi\|-\|(\mathbb{1}-K) \Phi\| \\
& \geq \lim _{n \rightarrow \infty}\left\|K \Phi_{\sigma_{n}}\right\|-\|(\mathbb{1}-K) \Phi\| \\
& \geq \liminf _{n \rightarrow \infty}\left(\left\|\Phi_{\sigma_{n}}\right\|-\left\|(\mathbb{1}-K) \Phi_{\sigma_{n}}\right\|\right)-\|(\mathbb{1}-K) \Phi\| \\
& \geq 1-4 \varepsilon-\|(\mathbb{1}-K) \Phi\| .
\end{aligned}
$$

Clearly $\mathbb{1}-K$ strongly converges to 0 when $R$ and $\lambda$ goes to infinity. Since $\varepsilon>0$ is arbitrary, we have $\|\Phi\|=1$. Therefore $H^{V}$ has a normalized ground state $\Phi$.

## §5. Essential spectrum

We give general lemmas given in [HSO8] without proofs.
Lemma 5.1. Let $K_{\epsilon}, \epsilon>0$, and $K$ be self-adjoint operators on a Hilbert space $\mathcal{K}$ and $\sigma_{\text {ess }}\left(K_{\epsilon}\right)=\left[\xi_{\epsilon}, \infty\right)$. Suppose that $\lim _{\epsilon \rightarrow 0} K_{\epsilon}=K$ in the uniform resolvent sense, and $\lim _{\epsilon \rightarrow 0} \xi_{\epsilon}=\xi$. Then $\sigma_{\text {ess }}(K)=[\xi, \infty)$. In particular $\lim _{\epsilon \rightarrow 0} \inf \sigma_{\text {ess }}\left(K_{\epsilon}\right)=\inf \sigma_{\text {ess }}(K)$.

Lemma 5.2. Let $\Delta$ be the $d$-dimensional Laplacian. Assume that $V(-\Delta+1)^{-1 / 2}$ is a compact operator. Then there exists a sequence $\left\{V^{\epsilon}\right\}_{\epsilon>0}$ such that $V^{\epsilon} \in$ $C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$ and $\lim _{\epsilon \rightarrow 0} V^{\epsilon}(-\Delta+1)^{-1 / 2}=V(-\Delta+1)^{-1 / 2}$ uniformly.

Set

$$
k_{0}(\beta)=-\sum_{j \in \beta} \sqrt{-\Delta_{j}}+\sum_{i, j \in \beta} V_{i j}, \quad k_{V}(\beta)=h_{0}(\beta)+\sum_{j \in \beta} V_{j}
$$

with $V_{i}, V_{i j} \in L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$ such that $V_{i}(-\Delta+1)^{-1 / 2}$ and $V_{i j}(-\Delta+1)^{-1 / 2}$ are compact operators. We define $K=k_{V}\left(C_{N}\right)$. Let

$$
\begin{equation*}
\Xi_{V}=\min _{\beta \subsetneq C_{N}}\left\{\inf \sigma\left(k_{0}(\beta)\right)+\inf \sigma\left(k_{V}(\beta)\right)\right\} \tag{70}
\end{equation*}
$$

be the lowest two cluster threshold of $K$.
Lemma 5.3. There exist sequences $\left\{V_{i}^{\epsilon}\right\}_{\epsilon},\left\{V_{i j}^{\epsilon}\right\}_{\epsilon} \subset C_{0}^{\infty}\left(\mathbb{R}^{d}\right), i, j=1, \ldots, N$, such that

$$
\text { (1) } \lim _{\epsilon \rightarrow 0} \Xi_{V}(\epsilon)=\Xi_{V}, \quad \text { (2) } \lim _{\epsilon \rightarrow 0} \inf \sigma_{\mathrm{ess}}(K(\epsilon))=\inf \sigma_{\mathrm{ess}}(K) \text {, }
$$

where $\Xi_{V}(\epsilon)($ resp. $K(\epsilon))$ is $\Xi_{V}$ (resp. $K$ ) with $V_{i}$ and $V_{i j}$ replaced by $V_{i}^{\epsilon}$ and $V_{i j}^{\epsilon}$, respectively.
§6. Functional integration and energy comparison inequality
In this Appendix we shall show Lemma 5.2 and Proposition 3.4 by functional integrations. In order to do that we take a Schrödinger representation instead of the Fock representation. We quickly review the Schrödinger representation.

Let $\mathscr{Q}=\mathscr{S}_{\mathbb{R}}^{\prime}\left(\mathbb{R}^{d}\right)$ be the set of real-valued Schwartz distributions on $\mathbb{R}^{d}$. The boson Fock space $\mathscr{F}$ can be identified with $L^{2}(\mathscr{Q}, \mu)$ with some Gaussian measure $\mu$ such that

$$
\mathbb{E}_{\mu}[\phi(f)]=0, \quad \mathbb{E}_{\mu}[\phi(f) \phi(g)]=\frac{1}{2}(f, g)
$$

for $f, g \in L_{\mathbb{R}}^{2}\left(\mathbb{R}^{d}\right)$. Then the scalar field operator in $\mathscr{F}$ is unitarily equivalent to the Gaussian random variable $\phi(f)$ in $L^{2}(\mathscr{Q})$ :

$$
\phi(f) \sim \frac{1}{\sqrt{2}} \int\left(a^{*}(k) \hat{f}(-k)+a(k) \hat{f}(k)\right) d k
$$

for $f \in L_{\mathbb{R}}^{2}\left(\mathbb{R}^{d}\right)$. Moreover $H_{\mathrm{f}}$ can be unitarily transformed to the self-adjoint operator in $L^{2}(\mathscr{Q})$. We denote it by the same notation, $H_{\mathrm{f}}$.

Furthermore we need the Euclidean quantum field to construct the functional integral representation of the one-parameter semigroup generated by the Nelson Hamiltonian $H^{V}$. Set $\mathscr{Q}_{E}=\mathscr{S}_{\mathbb{R}}^{\prime}\left(\mathbb{R}^{d+1}\right)$. Thus $L^{2}\left(\mathscr{Q}_{E}, \mu_{E}\right)$ be the $L^{2}$ space endowed with a Gaussian measure such that

$$
\mathbb{E}_{\mu_{E}}\left[\phi_{E}(F)\right]=0, \quad \mathbb{E}_{\mu_{E}}\left[\phi_{E}(F) \phi_{E}(G)\right]=\frac{1}{2}(F, G)_{L^{2}\left(\mathbb{R}^{d+1}\right)}
$$

Let $j_{t}: L_{\mathbb{R}}^{2}\left(\mathbb{R}^{d}\right) \rightarrow L_{\mathbb{R}}^{2}\left(\mathbb{R}^{d+1}\right)$ be the family of isometries connecting $L^{2}(\mathscr{Q})$ and $L^{2}\left(\mathscr{Q}_{E}\right)$, which satisfies that

$$
j_{s}^{*} j_{t}=e^{-|t-s| \omega(-i \nabla)}
$$

for all $s, t \in \mathbb{R}$. Let $J_{s}=\Gamma\left(j_{s}\right)$ be the second quantization of $j_{s}$. Then $J_{s}: L^{2}(\mathscr{Q}) \rightarrow$ $L^{2}\left(\mathscr{Q}_{E}\right)$ is also the family of isometries such that $J_{s}^{*} J_{t}=e^{-|t-s| H_{\mathrm{f}}}$ for all $s, t \in \mathbb{R}$. We identify $\mathscr{H}$ with the set of $L^{2}(\mathscr{Q})$-valued $L^{2}$ function on $\mathbb{R}^{d N}, \int_{\mathbb{R}^{d N}}^{\oplus} L^{2}(\mathscr{Q}) d X$, and $H^{V}$ can be expressed as

$$
\begin{equation*}
H_{\mathrm{p}} \otimes \mathbb{1}+\kappa^{2} \mathbb{1} \otimes H_{\mathrm{f}}+\kappa \alpha \sum_{j=1}^{N} \int_{\mathbb{R}^{d N}}^{\oplus} \phi\left(\lambda\left(\cdot-x_{j}\right)\right) d X \tag{71}
\end{equation*}
$$

in the Schrödinger representation.
Next we prepare a probabilistic description of the self-adjoint operator $H_{\mathrm{p}}$. Let $\left(X_{t}\right)_{t \geq 0}=\left(X_{t}^{1}, \ldots, X_{t}^{N}\right)_{t \geq 0}$ be the $\mathbb{R}^{d N}$-valued Lévy processes on a probability space ( $\mathcal{D}, B, \mathrm{P}^{x}$ ) starting from $x=0$ with the characteristic function (34). Set
$W\left(x_{1}, \ldots, x_{N}\right)=\sum_{j=1}^{N} V\left(x_{j}\right)$. Then we have the Feynman-Kac formula:

$$
\left(f, e^{-H_{\mathrm{p}}} g\right)=\int_{\mathbb{R}^{d N}} \mathbb{E}_{\mathrm{P}}^{x}\left[\bar{f}\left(X_{0}\right) g\left(X_{t}\right) e^{-\int_{0}^{t} W\left(X_{s}\right) d s}\right]
$$

The functional integral representation of $e^{-t H^{V}}$ can be obtained in the same way as the standard Nelson model. Only the difference is the process associated with kinetic term. Instead of the Brownian motion the Lévy process $\left(X_{t}^{j}\right)_{t \geq 0}$ is entered for $e^{-t H^{V}}$. The Feynman-Kac type formula of $e^{-t H^{V}}$ is then given by

$$
\begin{aligned}
& \left(F, e^{-t H^{V}} G\right)_{\mathscr{H}}= \\
& \int_{\mathbb{R}^{d N}} d x \mathbb{E}_{\mathrm{P}}^{x}\left[e^{-\int_{0}^{t} W\left(X_{s}\right) d s}\left(J_{0} F\left(X_{0}\right), e^{-\kappa \phi_{E}\left(\sum_{j=1}^{N} \int_{0}^{t} j_{\kappa^{2} s} \lambda_{j}\left(--X_{s}\right) d s\right)} J_{\kappa^{2} t} G\left(X_{t}\right)\right)_{L^{2}\left(\mathscr{Q}_{E}\right)}\right] .
\end{aligned}
$$

Next we also consider the Feynman-Kac formula of

$$
\exp \left(-t e^{-i T} H^{V} e^{i T}\right)
$$

It is given by the composition of $d N$ dimensional Brownian motion $\left(B_{t}^{1}, \ldots, B_{t}^{N}\right)_{t \geq 0}$ on a probability space $\left(\mathcal{C}, \mathcal{B}, \mathrm{W}^{x}\right)$ and $N$ independent subordinators $\left(T_{t}^{j}\right)_{t \geq 0}, j=$ $1, \ldots, N$, on $\left(\Omega_{\mu}, \mathcal{B}_{\mu}, \mu\right)$ such that $B_{T_{t}^{j}}^{j}$ has the same distribution of $X_{t}^{j}$. Set $B_{T_{t}}=$ $\left(B_{T_{t}^{j}}^{j}\right)_{t \geq 0, j=1, \ldots, N}$. We have the proposition below:
Proposition 6.1. Let $F, G \in \mathscr{H}$. Then

$$
\begin{aligned}
& \left(F, e^{-t e^{-i T} H^{V} e^{i T}} G\right) \\
& \begin{aligned}
&=e^{t E_{\text {diag }}} \int_{\mathbb{R}^{d N}} d x \mathbb{E}_{\mathrm{W} \times \mu}^{x, 0}\left[e^{-\int_{0}^{t}\left(W+V_{\text {eff }}\right)\left(B_{T_{s}}\right) d s}\right. \\
&\left.\times\left(J_{0} F\left(B_{T_{0}}\right), e^{-i \kappa^{-1} \phi_{E}\left(K_{t}\right)} J_{\kappa^{2} t} G\left(B_{T_{t}}\right)\right)_{L^{2}\left(\mathscr{Q}_{E}\right)}\right]
\end{aligned}
\end{aligned}
$$

Here $K_{t}=\sum_{j=1}^{N} \int_{0}^{T_{t}^{j}} j_{\left(T^{j-1}\right)_{\kappa^{2} s}} \lambda_{j}\left(\cdot-B_{s}^{j}\right) \circ d B_{s}^{j}$ denotes the $L^{2}\left(\mathbb{R}^{d+1}\right)$-valued Stratonovich integral and $j_{\left(T^{j-1}\right)_{t}}$ is some isometries defined by $\left(T_{t}^{j}\right)_{t \geq 0}$.

Proof. See [Hirl4, Theorem 3.15].
By using Proposition we can compute the scaling limit of $e^{-i T} H^{V} e^{i T}$ as $\kappa \rightarrow \infty$. Note that $\left(J_{0} \Phi, J_{\kappa^{2} t} \Psi\right) \rightarrow\left(\Phi, P_{\Omega} \Phi\right)$ as $\kappa \rightarrow \infty$ for $t \neq 0$. Then by the functional integral representation Proposition 6.1 we immediately see that

$$
\begin{equation*}
\lim _{\kappa \rightarrow \infty}\left(F, e^{-t e^{-i T} H^{V} e^{i T}} G\right)=\left(F, e^{-t\left(h_{\mathrm{eff}}^{V}-E_{\mathrm{diag}}\right)} \otimes P_{\Omega} G\right) \tag{72}
\end{equation*}
$$

Since $h_{\text {eff }}^{V}$ has a ground state, this suggests that $H^{V}$ also has a ground state for sufficiently large $\kappa$. This has been indeed done in Section 3.

By functional integral representation we have the energy comparison bound.

Proposition 6.2. It follows that $\inf \sigma\left(H^{V}\right) \leq \inf \sigma\left(h_{\text {eff }}^{V}\right)+E_{\text {diag }}$.
Proof. By Proposition [.] we have

$$
\left|\left(F, e^{-t e^{-i T} H^{V} e^{i T}} G\right)\right| \leq e^{t E_{\mathrm{diag}}}\left(|F|, e^{-t\left(h_{\mathrm{eff}}^{V}+H_{\mathrm{f}}\right)}|G|\right) .
$$

Then the proposition follows.
In the same way as Proposition 6.2 but $H^{V}$ is replaced by $H^{V}(\beta)$ or $H^{0}(\beta)$ we have the lemma below.

Proposition 6.3. (Lemma 5.2) It follows that

$$
\begin{equation*}
\inf \sigma\left(H^{\#}(\beta)\right) \leq \inf \sigma\left(h_{\mathrm{eff}}^{\#}(\beta)\right)+\sum_{j \in \beta} \frac{\alpha^{2}}{2}\left\|\hat{\lambda}_{j} / \sqrt{\omega}\right\|^{2}, \quad \#=0, V \tag{73}
\end{equation*}
$$

Next we show Proposition [3.4. We can also construct the functional integral representation of $e^{-t H_{\sigma}^{V}}$ in the quite same as that of $e^{-t H^{V}}$. Only the difference is to replace $\hat{\lambda}_{j}$ with $\hat{\lambda}_{j}\left\lceil_{\omega(k)>\sigma}\right.$.

Proposition 6.4. Proposition $\sqrt{3.4}$ follows.
Proof. Notice that $\Phi_{\sigma}=e^{-t\left(e^{-i T} H_{\sigma}^{V} e^{i T}-E_{\sigma}^{V}\right)} \Phi_{\sigma}$. Then by Proposition we can see that

$$
\Phi_{\sigma}(x)=e^{t\left(E_{\sigma}^{V}+E_{\text {diag }}\right)} \mathbb{E}_{\mathrm{W} \times \mu}^{x, 0}\left[e^{-\int_{0}^{t} W_{\text {eff }}\left(B_{T_{s}}\right) d s} J_{0}^{*} e^{-i \kappa^{-1} \phi_{E}\left(K_{t}\right)} J_{\kappa^{2} t} \Phi_{\sigma}\left(B_{T_{t}}\right)\right]
$$

Thus it is straightforward to see by the Schwartz inequality that

$$
\left\|\Phi_{\sigma}(x)\right\|_{\mathscr{F}} \leq e^{t\left(E_{\sigma}^{V}+E_{\mathrm{diag}}\right)}\left(\mathbb{E}_{\mathrm{W} \times \mu}^{x, 0}\left[e^{-2 \int_{0}^{t} W_{\mathrm{eff}}\left(B_{T_{s}}\right) d s}\right]\right)^{1 / 2}\left\|\Phi_{\sigma}\right\|_{\mathscr{H}}
$$

Note that $\lim _{\sigma \rightarrow 0} E_{\sigma}^{V}=E^{V}$. Then the proposition follows, since $B_{T_{t}}$ has the same distribution with $X_{t}$.
$\S 7$. Bound $E(0) \leq E(P)$ and continuity of $E(\cdot)$
We next consider a fiber decomposition of the translation invariant relativistic Schrödinger operator $H_{\mathrm{p}}=\sum_{j=1}^{N} \Omega_{j}+V_{\text {eff }}$ in $L^{2}\left(\mathbb{R}^{d N}\right)$.

For notational convenience and generalizations, we consider the Schrödinger operator of the form $H_{\mathrm{p}}=\sum_{j=0}^{N} \Omega_{j}+v$ in $L^{2}\left(\mathbb{R}^{d(N+1)}\right)$, where

$$
v=\sum_{j=0}^{N} v_{i j}\left(x_{i}-x_{j}\right)
$$

and we assume that $v$ is relativistic Kato-class. Let $X_{t}=\left(X_{t}^{j}\right)_{t \geq 0}, j=0, \ldots, N$, be $N+1$ independent Lévy processes with $\mathbb{E}_{\mathrm{P}}^{x}\left[e^{i u \cdot X_{t}^{j}}\right]=e^{-t \Omega_{j}(u)}$, and set $\mathbf{X}_{t}=$
$\left(X_{t}^{j}\right)_{t \geq 0, j=1, \ldots, N}$. Let $P_{\text {tot }}=\sum_{j=0}^{N} p_{j}$ be the total momentum. Then $H_{\mathrm{p}}$ commutes with $P_{\text {tot }}$, and then $H_{\mathrm{p}} \cong \int_{\mathbb{R}^{d}}^{\oplus} k(P) d P$, where $k(P)$ is a self-adjoint operator on $L^{2}\left(\mathbb{R}^{d N}\right)$. Let $E(P)=\inf \sigma(k(P))$.
Theorem 7.1. (1) $E(0) \leq E(P)$ for all $P \in \mathbb{R}^{d}$. (2) $\mathbb{R}^{d} \ni P \mapsto E(P) \in \mathbb{R}$ is continuous.

We shall prove this theorem by making use of a path integral representation. Let us set $x=\left(x_{0}, \mathbf{x}\right) \in \mathbb{R}^{d} \times \mathbb{R}^{d N}$. Let $U=F e^{i x_{0} \cdot \sum_{j=1}^{N} p_{j}}: L^{2}\left(\mathbb{R}^{d(N+1)}\right) \rightarrow$ $L^{2}\left(\mathbb{R}^{d(N+1)}\right)$ be the unitary operator, where $F$ denotes the Fourier transformation with respect to $x_{0}$ variable, i.e., $F f(k, \mathbf{x})=(2 \pi)^{-d / 2} \int f\left(x_{0}, \mathbf{x}\right) e^{-i k \cdot x_{0}} d x_{0}$. We have

$$
(U f)(k, \mathbf{x})=(2 \pi)^{-d / 2} \int_{\mathbb{R}^{d}} e^{-i k \cdot x_{0}} f\left(x_{0}, x_{1}+x_{0}, \cdots, x_{N}+x_{0}\right) d x_{0}
$$

Thus we can directly see that $\left(U P_{\text {tot }} U^{-1} f\right)(k, \mathbf{x})=k f(k, \mathbf{x})$. Hence $U$ diagonalize $P_{\text {tot }}$, and thus $U H_{\mathrm{p}} U^{-1}=\int_{\mathbb{R}^{d}} k(P) d P$. We have

$$
\begin{equation*}
\left(f, e^{-t H_{\mathrm{P}}} g\right)_{L^{2}\left(\mathbb{R}^{d(N+1)}\right)}=\int_{\mathbb{R}^{d(N+1)}} d x \mathbb{E}_{\mathrm{P}}^{\left(x_{0}, \mathbf{x}\right)}\left[\overline{f\left(X_{0}\right)} g\left(X_{t}\right) e^{-\int_{0}^{t} v\left(X_{s}\right) d s}\right] \tag{74}
\end{equation*}
$$

We construct the Feynman-Kac formula of $\left(f, e^{-t k(P)} g\right)_{L^{2}\left(\mathbb{R}^{d N}\right)}$. Let $v=0$. Then

$$
k(P)=\Omega_{0}\left(P-\sum_{j=1}^{N} p_{j}\right)+\sum_{j=1}^{N} \Omega_{j}\left(p_{j}\right)
$$

Since $\mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[e^{i X_{t}^{0}\left(P-\sum_{j=1}^{N} p_{j}\right)}\right]=e^{-t \Omega_{0}\left(P-\sum_{j=1}^{N} p_{j}\right)}$, we intuitively see that

$$
\left(f, e^{-t k(P)} g\right)_{L^{2}\left(\mathbb{R}^{d N}\right)}=\int_{\mathbb{R}^{d N}} d \mathbf{x} \mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[\overline{f\left(\mathbf{X}_{0}\right)} e^{i X_{t}^{0} \cdot\left(P-\sum_{j=1}^{N} p_{j}\right)} g\left(\mathbf{X}_{t}\right)\right]
$$

Note that $e^{-i X_{t}^{0} \cdot \sum_{j=1}^{N} p_{j}}$ denotes a translation, i.e.,

$$
\left(e^{-i X_{t}^{0} \cdot \sum_{j=1}^{N} p_{j}} g\right)\left(\mathbf{X}_{t}\right)=g\left(X_{t}^{1}-X_{t}^{0}, \cdots, X_{t}^{N}-X_{t}^{0}\right)
$$

In the next proposition we see the Feynman-Kac formula with potential.
Proposition 7.2. Let $F, G \in L^{2}\left(\mathbb{R}^{d N}\right)$ and $P \in \mathbb{R}^{d}$. Then

$$
\begin{equation*}
\left(F, e^{-t k(P)} G\right)_{L^{2}\left(\mathbb{R}^{d N}\right)}=\int_{\mathbb{R}^{d N}} d \mathbf{x} \mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[\overline{F\left(\mathbf{X}_{0}\right)} e^{-\int_{0}^{t} v\left(X_{s}\right) d s} e^{i X_{t}^{0} \cdot\left(P-\sum_{j=1}^{N} p_{j}\right)} G\left(\mathbf{X}_{t}\right)\right] \tag{75}
\end{equation*}
$$

Proof. Let $\xi \in \mathbb{R}^{d}$. First we see that

$$
\begin{equation*}
\left(f, e^{-t H_{\mathrm{p}}} e^{i \xi \cdot P_{\mathrm{tot}}} g\right)_{L^{2}\left(\mathbb{R}^{d(N+1)}\right)}=\int_{\mathbb{R}^{d}} d P e^{i \xi \cdot P}\left(f(P), e^{-t k(P)} g(P)\right)_{L^{2}\left(\mathbb{R}^{d N}\right)} \tag{76}
\end{equation*}
$$

where

$$
f(P)=(U f)(P, \mathbf{x})=(2 \pi)^{-d / 2} \int_{\mathbb{R}^{d}} e^{-i P \cdot X} f\left(X, x_{1}+X, \cdots, x_{N}+X\right) d X
$$

and $g(P)$ is similarly given. Now we put $f=f_{s}=p_{s} \otimes F$ and $g=g_{r}=p_{r} \otimes G$, where $F, G \in \mathscr{S}\left(\mathbb{R}^{3 N}\right)$ and $p_{s}(X)=(2 \pi s)^{-d} \exp \left(-|X|^{2} /(2 s)\right)$ is the heat kernel. Note that $f_{s} \rightarrow \delta\left(x_{0}\right) \otimes F$ as $s \downarrow 0$. We have

$$
\begin{aligned}
& \left.\lim _{s \downarrow 0} \int_{\mathbb{R}^{d}} d P e^{i \xi \cdot P}\left(f_{s}(P), e^{-t k(P)} g_{r}(P)\right)_{L^{2}\left(\mathbb{R}^{d N}\right.}\right) \\
& =(2 \pi)^{-d / 2} \int_{\mathbb{R}^{d}} d P e^{i \xi \cdot P}\left(F, e^{-t k(P)} g_{r}(P)\right)_{L^{2}\left(\mathbb{R}^{d N}\right)}
\end{aligned}
$$

The right hand side above is the inverse Fourier transform of the function $h: P \rightarrow$ $\left(F, e^{-t k(P)} g_{r}(P)\right)_{L^{2}\left(\mathbb{R}^{d N}\right)}$ and

$$
\begin{equation*}
\lim _{r \downarrow 0} h(P)=\left(F, e^{-k(P)} G\right)_{L^{2}\left(\mathbb{R}^{d N}\right)}(2 \pi)^{-d / 2} \tag{77}
\end{equation*}
$$

On the other hand the left hand side of ( $\mathbf{\pi 6 )}$ ) can be represented by the FeynmanKac formula:

$$
\begin{align*}
& \left(f_{s}, e^{-t H_{\mathrm{p}}} e^{i \xi \cdot P_{\mathrm{tot}}} g_{r}\right) \\
& =\int_{\mathbb{R}^{d(N+1)}} d x \mathbb{E}_{\mathrm{P}}^{\left(x_{0}, \mathbf{x}\right)}\left[\overline{f_{s}\left(X_{0}\right)} e^{-\int_{0}^{t} v\left(X_{s}\right) d s} g_{r}\left(X_{t}^{0}+\xi, \cdots, X_{t}^{N}+\xi\right)\right] \tag{78}
\end{align*}
$$

Taking $s \downarrow 0$, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^{d(N+1)}} d x \mathbb{E}_{\mathrm{P}}^{\left(x_{0}, \mathbf{x}\right)}\left[\overline{f_{s}\left(X_{0}\right)} e^{-\int_{0}^{t} v\left(X_{s}\right) d s} g_{r}\left(X_{t}^{0}+\xi, \cdots, X_{t}^{N}+\xi\right)\right] \rightarrow \\
& \mathbb{E}_{\mathrm{P}}^{(0, \mathbf{0})}\left[\int_{\mathbb{R}^{d N}} d \mathbf{x} \overline{F(\mathbf{x})} e^{-\int_{0}^{t} v\left(X_{s}+(0, \mathbf{x})\right) d s} g_{r}\left(X_{t}^{0}+\xi, X_{t}^{1}+\xi+x_{1} \cdots, X_{t}^{N}+\xi+x_{N}\right)\right]
\end{aligned}
$$

The right hand side is the function with respect to $\xi$. We take the Fourier transform with respect to $\xi$. Then

$$
\begin{aligned}
\mathbb{E}_{\mathrm{P}}^{(0,0)} & {\left[\int_{\mathbb{R}^{d N}} d \mathbf{x} \overline{F(\mathbf{x})} e^{-\int_{0}^{t} v\left(X_{s}+(0, \mathbf{x})\right) d s}\right.} \\
& \left.\times(2 \pi)^{-d / 2} \int_{\mathbb{R}^{d}} d \xi e^{-i \xi \cdot P} g_{r}\left(X_{t}^{0}+\xi, X_{t}^{1}+\xi+x_{1}, \cdots, X_{t}^{N}+\xi+x_{N}\right)\right]
\end{aligned}
$$

Take $r \downarrow 0$. We have

$$
\begin{aligned}
& \mathbb{E}_{\mathrm{P}}^{(0,0)}\left[\int_{\mathbb{R}^{d N}} d \mathbf{x} \overline{F(\mathbf{x})} e^{-\int_{0}^{t} v\left(X_{s}+(0, \mathbf{x})\right) d s} e^{i X_{t}^{0} \cdot P}\right. \\
& \left.\times G\left(X_{t}^{1}-X_{t}^{0}+x_{1}, \cdots, X_{t}^{N}-X_{t}^{0}+x_{N}\right)\right] \\
= & \mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[\int_{\mathbb{R}^{d N}} d \mathbf{x} \overline{F\left(X_{0}\right)} e^{-\int_{0}^{t} v\left(X_{s}\right) d s} e^{i X_{t}^{0}\left(P-\sum_{j=1}^{N} p_{j}\right)} G\left(\mathbf{X}_{t}\right)\right] .
\end{aligned}
$$

Comparing ( $\mathbb{\pi 7}$ ) with the right hand side above, we conclude the theorem for $F, G \in \mathscr{S}$. By a limiting argument the theorem is valid for all $f, g \in L^{2}\left(\mathbb{R}^{d N}\right)$.

Proof of Theorem [7.1): By Proposition [7.2 we have

$$
\begin{equation*}
\left|\left(f, e^{-t k(P)} g\right)\right| \leq \int_{\mathbb{R}^{d N}} d x \mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[\left|f\left(\mathbf{X}_{0}\right)\right| e^{-\int_{0}^{t} v\left(X_{s}\right) d s}\left|e^{-i X_{t}^{0} \cdot \sum_{j=1}^{N} p_{j}} g\left(\mathbf{X}_{t}\right)\right|\right] \tag{79}
\end{equation*}
$$

Since $e^{-i X_{t}^{0} \cdot \sum_{j=1}^{N} p_{j}}$ is the shift operator,

$$
\left|e^{-i X_{t}^{0} \cdot \sum_{j=1}^{N} p_{j}} g\left(\mathbf{X}_{t}\right)\right| \leq e^{-i X_{t}^{0} \cdot \sum_{j=1}^{N} p_{j}}\left|g\left(\mathbf{X}_{t}\right)\right|
$$

follows. Then we obtain $\left|\left(f, e^{-t k(P)} g\right)\right| \leq\left(|f|, e^{-t k(0)}|g|\right)$ which yields (1).
Next we show (2). By Feynman Kac formula it is immediate to see that

$$
\begin{aligned}
& \left(F,\left(e^{-t k(P)}-e^{-t k(Q)}\right) G\right) \\
& =\int_{\mathbb{R}^{d N}} d \mathbf{x} \mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[\overline{F\left(\mathbf{X}_{0}\right)} e^{-\int_{0}^{t} v\left(X_{s}\right) d s} e^{-i X_{t}^{0} \cdot \sum_{j=1}^{N} p_{j}}\left(i \int_{X_{t}^{0} \cdot Q}^{X_{t}^{0} \cdot P} e^{i \theta} d \theta\right) G\left(\mathbf{X}_{t}\right)\right]
\end{aligned}
$$

Then

$$
\frac{\left|\left(F,\left(e^{-t k(P)}-e^{-t k(Q)}\right) G\right)\right|}{\|F\|\|G\|} \leq|P-Q| \sup _{\mathbf{x} \in \mathbb{R}^{d N}}\left(\mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[\left|X_{t}^{0}\right|^{2} e^{-2 \int_{0}^{t} v\left(X_{s}\right) d s}\right]\right)^{1 / 2}
$$

Since $v$ is relativistic Kato-class,

$$
\begin{aligned}
& \sup _{\mathbf{x} \in \mathbb{R}^{d N}} \mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[\left|X_{t}^{0}\right|^{2} e^{-2 \int_{0}^{t} v\left(X_{s}\right) d s}\right] \\
& \leq \sup _{\mathbf{x} \in \mathbb{R}^{d N}} \mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[\left|X_{t}^{0}\right|^{4}\right]^{1 / 2} \sup _{\mathbf{x} \in \mathbb{R}^{d N}}\left(\mathbb{E}_{\mathrm{P}}^{(0, \mathbf{x})}\left[e^{-4 \int_{0}^{t} v\left(X_{s}\right) d s}\right]\right)^{1 / 2}<\infty .
\end{aligned}
$$

Then we conclude that $e^{-t k(P)}$ uniformly converges to $e^{-t k(Q)}$ as $|P-Q| \rightarrow 0$. Then (2) follows.

## Acknowledgment

FH acknowledges support of Grant-in-Aid for Science Research (B) 20340032 from JSPS, Grant-in-Aid for Challenging Exploratory Research 22654018 and Grant-in-Aid for Challenging Exploratory Research 15K13445 from JSPS. FH thanks a
hospitality of ICMS Edinburgh, where this work is partially done. IS acknowledges support of Grant-in-Aid for Young Scientists (B) 22740087 from JSPS. IS thanks a hospitality of Orsay University, where this work is partially done. IS's work is supported by the program for dissemination of tenure-track system funded by the ministry of education and science, Japan.

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[^0]:    Communicated by T. Kumagai. Received January 9, 2015. Revised May 23, 2015.
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