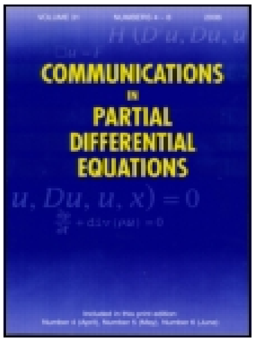

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Global Solutions for the Dirac–Klein–Gordon System in Two Space Dimensions

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The Cauchy problem for the classical Dirac–Klein–Gordon system in two space dimensions is globally well-posed for L^2 Schrödinger data and wave data in $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$. In the case of smooth data there exists a global smooth (classical) solution. The proof uses function spaces of Bourgain type based on Besov spaces – previously applied by Colliander, Kenig and Staffilani for generalized Benjamin-Ono equations and also by Bejenaru, Herr, Holmer and Tataru for the 2D Zakharov system – and the null structure of the system detected by d'Ancona, Foschi and Selberg, and a refined bilinear Strichartz estimate due to Selberg. The global existence proof uses an idea of Colliander, Holmer and Tzirakis for the 1D Zakharov system.

Keywords Dirac–Klein–Gordon system; Fourier restriction norm method; Well-posedness.

Mathematics Subject Classification 35Q55, 35L70.

1. Introduction and Main Results

Consider the Cauchy problem for the Dirac–Klein–Gordon equations in two space dimensions

$$i(\partial_t + \alpha \cdot \nabla)\psi + M\beta\psi = -\phi\beta\psi \tag{1}$$

$$\left(-\partial_t^2 + \Delta\right)\phi + m\phi = -\langle \beta\psi, \psi \rangle \tag{2}$$

with (large) initial data

$$\psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1. \tag{3}$$

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Here ψ is a two-spinor field, i.e., $\psi : \mathbf{R}^{1+2} \to \mathbf{C}^2$, and ϕ is a real-valued function, i.e., $\phi : \mathbf{R}^{1+2} \to \mathbf{R}$, $m, M \in \mathbf{R}$ and $\nabla = (\partial_{x_1}, \partial_{x_2})$, $\alpha \cdot \nabla = \alpha^1 \partial_{x_1} + \alpha^2 \partial_{x_2} \cdot \alpha^1, \alpha^2, \beta$ are hermitian (2×2)-matrices satisfying $\beta^2 = (\alpha^1)^2 = (\alpha^2)^2 = I$, $\alpha^j \beta + \beta \alpha^j = 0$, $\alpha^j \alpha^k + \alpha^k \alpha^j = 2\delta^{jk} I$.

$\langle \cdot, \cdot \rangle$ denotes the $\mathbf{C}^2$-scalar product. A particular representation is given by $\alpha^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\alpha^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\beta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

We consider Cauchy data in Sobolev spaces: $\psi_0 \in H^s$, $\phi_0 \in H^r$, $\phi_1 \in H^{r-1}$.

The fundamental conservation law is charge conservation $\|\psi(t)\|_{L^2} = \text{const}$.

In the $(1+1)$-dimensional case global well-posedness for smooth data was already established by Chadam [6] and also for much less regular data by Bournaveas [3], Fang [13], Bournaveas and Gibbeson [5], Machihara [15], Pecher [16], Selberg [18], Selberg and Tesfahun [19] and Tesfahun [20], the last two authors also for data $\psi_0 \notin L^2$. In the $(2+1)$-dimensional and $(3+1)$-dimensional case no global well-posedness results for large data were known so far. In $(2+1)$-dimensions local well-posedness was proven by Bournaveas [4], if $s > \frac{1}{4}$ and $r = s + \frac{1}{2}$, which was later improved by d'Ancona et al. [11] to the case $s > -\frac{1}{5}$ and $\max(\frac{1}{4} - \frac{s}{2}, \frac{1}{4} + \frac{s}{2}, s) < r < \min(\frac{3}{4} + 2s, \frac{3}{4} + \frac{3s}{2}, 1 + s)$. Their proof relied on the null structure of the system. This complete null structure was detected by d'Ancona et al. in their earlier paper [12], where it was applied to show an almost optimal local existence result in $(3+1)$-dimensions, namely if $s = \epsilon$, $r = \frac{1}{2} + \epsilon$ for any $\epsilon > 0$.

We now give the first global well-posedness result for large data in two space dimensions. It holds in the case $s = 0$, $r = \frac{1}{2}$, and more generally in the case $s \geq 0$, $r = s + \frac{1}{2}$, where local well-posedness was known to be true before already (by d'Ancona et al. [11]). Especially we show the existence of global classical solutions for smooth data. It is necessary to refine the local existence result by replacing Bourgain spaces $X_\pm^{s,b}$ and $X_\pm^{r,b}$ for $b > \frac{1}{2}$ constructed from Sobolev spaces by their analogue constructed from Besov spaces with respect to time, especially $X_\pm^{s,\frac{1}{2},1}$ and $X_\pm^{r,\frac{1}{2},1}$ (see the definition below). Spaces of this type were already successfully used to give a local well-posedness result for the 2D-Zakharov system by Bejenaru et al. [1] and Colliander et al. for generalized Benjamin-Ono equations [8]. The precise bound for the existence time then can be combined with the charge conservation to show global well-posedness for our 2D Dirac–Klein–Gordon system. A similar procedure was already used by Colliander et al. for the one-dimensional Zakharov system [7]. It turns out that the choice of the regularity parameters s and r in our case just allows to estimate both nonlinearities in a unified way. What one also needs are of course the Strichartz estimates for the wave equation, here also the Besov space version to avoid the endpoint Strichartz estimate in 2D. The Strichartz estimates however are not sufficient for a particularly delicate case where it is essential to use a bilinear refinement which was detected by Selberg [17] and can also be found in Foschi and Klainerman [14]. This version was already used by d'Ancona et al. [11] in their local well-posedness result.

We use the following function spaces. Let $\widehat{\ }$ denote the Fourier transform with respect to space or time and $\tilde{\ }$ the Fourier transform with respect to space and time simultaneously. Let $\varphi \in C_0^\infty(\mathbf{R^n})$ be a nonnegative function with $\operatorname{supp} \varphi \subset \{1/2 \leq |\xi| \leq 2\}$ and $\varphi(\xi) > 0$, if $\frac{1}{\sqrt{2}} \leq |\xi| \leq \sqrt{2}$. Setting $\hat{\rho}_k(\xi) := \varphi(2^{-k}\xi)$ $(k = 1, 2, \ldots)$, $\hat{\varphi}_k(\xi) := \frac{\hat{\rho}_k(\xi)}{\sum_{j=-\infty}^{+\infty} \varphi(2^{-j}\xi)}$ $(k = 1, 2, \ldots)$ and $\hat{\varphi}_0(\xi) := 1 - \sum_{k=1}^\infty \hat{\varphi}_k(\xi)$ we have $\operatorname{supp} \hat{\varphi}_k \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$, $\operatorname{supp} \hat{\varphi}_0 \subset \{|\xi| \leq 2\}$ and $\sum_{k=0}^\infty \hat{\varphi}_k = 1$. The Besov spaces are defined for $s \in \mathbf{R}$, $1 \le p, q \le \infty$ as follows:

$$B^s_{p,q} = \{f \in \mathcal{S}', \|f\|_{B^s_{p,q}} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}} = \left(\sum_{k=0}^{\infty}(2^{sk}\|\varphi_k * f\|_{L^p})^q\right)^{\frac{1}{q}} \quad \text{if } q < \infty,$$
$$\|f\|_{B^s_{p,\infty}} = \sup_{k\ge 0} 2^{sk}\|\varphi_k * f\|_{L^p}$$

(cf. e.g., Triebel [21, Section 2.3.1]).

Similarly the homogeneous Besov spaces are defined as the set of those $f \in \mathcal{S}'$, for which $\|f\|_{\dot{B}^s_{p,q}}$ is finite, where $\|f\|_{\dot{B}^s_{p,q}} = (\sum_{k=-\infty}^{+\infty}(2^{sk}\|\varphi'_k * f\|_{L^p})^q)^{\frac{1}{q}}$ with the usual modification for $q = \infty$ and $\hat{\varphi}'_k(\xi) := \frac{\hat{\rho}_k(\xi)}{\sum_{j=-\infty}^{+\infty}\varphi(2^{-j}\xi)}$ for $k \in \mathbf{Z}$. We also need the following Bourgain type spaces. The standard spaces belonging to the half waves are defined by the completion of $\mathcal{S}(\mathbf{R}\times\mathbf{R}^2)$ with respect to

$$\|f\|_{X^{s,b}_\pm} = \|U_\pm(-t)f\|_{H^b_t H^s_x} = \|\langle\xi\rangle^s\langle\tau \pm |\xi|\rangle^b \tilde{f}(\tau,\xi)\|_{L^2}$$

where

$$U_\pm(t) := e^{\mp it|D|} \quad \text{and} \quad \|g\|_{H^b_t H^s_x} = \|\langle\xi\rangle^s\langle\tau\rangle^b\tilde{g}(\xi,\tau)\|_{L^2_{\xi,\tau}}.$$

We also define $X^{s,b,q}_\pm$ as the space of all $u \in \mathcal{S}'(\mathbf{R}\times\mathbf{R}^2)$, where the following norms are finite:

$$\|f\|_{X^{s,b,q}_\pm} = \|U_\pm(-t)f\|_{B^b_{2,q}H^s_x} = \left(\sum_{k=0}^{\infty} 2^{qbk}\|\langle\xi\rangle^s\hat{\varphi}_k(\tau \pm |\xi|)\tilde{f}(\tau,\xi)\|^q_{L^2_{\tau,\xi}}\right)^{\frac{1}{q}}$$

for $1 \le q < \infty$, where

$$\|g\|_{B^b_{2,q}H^s_x} = \left(\sum_{k=0}^{\infty} 2^{qbk}\|\langle\xi\rangle^s\hat{\varphi}_k(\tau)\tilde{f}(\tau,\xi)\|^q_{L^2_{\tau,\xi}}\right)^{\frac{1}{q}}$$

and

$$\|f\|_{X^{s,b,\infty}_\pm} = \|U_\pm(-t)f\|_{B^b_{2,\infty}H^s_x} = \sup_{k\ge 0} 2^{bk}\|\langle\xi\rangle^s\hat{\varphi}_k(\tau \pm |\xi|)\tilde{f}(\tau,\xi)\|_{L^2_{\tau,\xi}}$$

for $q = \infty$, where

$$\|g\|_{B^b_{2,\infty}H^s_x} = \sup_{k\ge 0} 2^{bk}\|\langle\xi\rangle^s\hat{\varphi}_k(\tau)\tilde{g}(\tau,\xi)\|_{L^2_{\tau,\xi}}.$$

Note that $U_\pm(t) = e^{\mp it(-\Delta+1)^{1/2}}$ would lead to equivalent norms.

Spaces of type $X^{s,b,q}$ with various phase functions $\phi(\xi)$ instead of $\pm|\xi|$ have been used in the literature before, for example by Colliander et al. in their work on dispersion generalized Benjamin-Ono equations [8]. As was observed in [8, Proof of Lemma 5.1], they can be obtained by real interpolation from the standard $X^{s,b}$-spaces. In fact, by [2, Theorem 5.6.1] one has for $s \in \mathbf{R}$, $1 \leq q \leq \infty$, $b_0 \neq b_1$ and $b = (1-\theta)b_0 + \theta b_1$, $0 < \theta < 1$, that

$$(X^{s,b_0}, X^{s,b_1})_{\theta,q} = X^{s,b,q}.$$

Using the duality Theorem [2, Theorem 3.7.1] we see that for $1 \leq q < \infty$

$$(\overline{X}^{s,b,q})' = X^{-s,-b,q'},$$

where $\overline{X}$ denotes the space of complex conjugates of elements of X with norm $\|f\|_{\overline{X}} = \|\overline{f}\|_X$. In the proof of the crucial bilinear estimates for local well-posedness we will repeatedly make use of complex interpolation. To justify this we use the corresponding theorem on interpolation of spaces of vector valued sequences [2, Theorem 5.6.3] and take into account the considerations in [2, Section 6.4] to see that

$$(X^{s_0,b_0,q_0}, X^{s_1,b_1,q_1})_{[\theta]} = X^{s,b,q},$$

whenever $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $b = (1-\theta)b_0 + \theta b_1$, and $1 \leq q_0, q_1 \leq \infty$ as well as $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

The preceding remarks on duality and interpolation are completely independent of the specific phase function.

For $B \subset \mathcal{S}'(\mathbf{R} \times \mathbf{R}^2)$ we denote by $B(T)$ the space of restrictions of distributions in B to the set $(0, T) \times \mathbf{R}^2$ with induced norm.

We use the Strichartz estimates for the homogeneous wave equation in $\mathbf{R}^n \times \mathbf{R}$, which can be found e.g., in Ginibre and Velo [10, Proposition 2.1].

Proposition 1.1. *Let* $\gamma(r) = (n-1)(\frac{1}{2} - \frac{1}{r})$, $\delta(r) = n(\frac{1}{2} - \frac{1}{r})$, $n \geq 2$. *Let* $\rho, \mu \in \mathbf{R}$, $2 \leq q, r \leq \infty$ *satisfy* $0 \leq \frac{2}{q} \leq \min(\gamma(r), 1)$, $(\frac{2}{q}, \gamma(r)) \neq (1, 1)$, $\rho + \delta(r) - \frac{1}{q} = \mu$. *Then*

$$\|e^{\pm it|D|}u_0\|_{L^q(\mathbf{R}, \dot{B}^{\rho}_{r,2}(\mathbf{R}^n))} \leq c\|u_0\|_{\dot{H}^{\mu}(\mathbf{R}^n)}.$$

The same holds with $\dot{B}^{\rho}_{r,2}$ *replaced by* $\dot{H}^{\rho,r}$ *under the additional assumption* $r < \infty$.

The following consequence of estimates of Strichartz type is important for our considerations.

Proposition 1.2. *Let* $Y \subset \mathcal{S}'(\mathbf{R} \times \mathbf{R}^n)$ *be a set of functions of space and time with the property that*

$$\|hf\|_Y \leq c\|h\|_{L^\infty_t}\|f\|_Y$$

for all $h \in L^\infty_t$ *and* $f \in Y$. *Assume moreover the (Strichartz type) estimate*

$$\|U_\pm(t)u_0\|_Y \leq c\|u_0\|_{H^\mu},$$

where $U_\pm(t) = e^{\mp it|D|}$. *Then the following estimate holds*:

$$\|f\|_Y \leq c\|f\|_{X_\pm^{\mu,\frac{1}{2},1}}.$$

Proof. We combine Lemma 2.3 in [9] with the proof of the embedding $B^{1/2}_{2,1}(\mathbf{R}) \subset C^0(\mathbf{R})$. Let ψ be a $C_0^\infty(\mathbf{R})$-function with $\psi(\tau) = 1$ for $1/2 \le |\tau| \le 2$ and $\hat{\psi}_k(\tau) := \psi(2^{-k}\tau)$, so that $\hat{\psi}_k(\tau) = 1$ for $2^{k-1} \le |\tau| \le 2^{k+1}$. Furthermore we define $\hat{\psi}_0 \in C_0^\infty$ such that $\hat{\psi}_0(\tau) = 1$ for $|\tau| \le 2$. The functions φ_k are those which appear in the definition of the Besov norms (here in the 1-dimensional case). We thus have the property that $\widehat{\psi_k}(\tau) = 1$ for $\tau \in \operatorname{supp} \hat{\varphi}_k$ $(k = 0, 1, 2, \dots)$. We start from

$$f = \int e^{it\tau} U_\pm(t)(\mathscr{F}_t U_\pm(-\cdot)f)(\tau) d\tau.$$

Then we have with $h = e^{it\tau}$ (for fixed τ):

$$\begin{aligned}
\|f\|_Y &\le \int \|U_\pm(t)(\mathscr{F}_t U_\pm(-\cdot)f)(\tau)\|_Y d\tau \le c \int \|\mathscr{F}_t U_\pm(-\cdot)f\|_{H^\mu} d\tau \\
&\le c \sum_{k=0}^\infty \|\mathscr{F}_t(U_\pm(-\cdot)f)\hat{\varphi}_k\|_{L^1_\tau(H^\mu_x)} = c \sum_{k=0}^\infty \|\mathscr{F}_t(U_\pm(-\cdot)f)\hat{\varphi}_k \hat{\psi}_k\|_{L^1_\tau(H^\mu_x)} \\
&\le c \sum_{k=0}^\infty \|\mathscr{F}_t(U_\pm(-\cdot)f *_t \varphi_k)(\tau)\|_{L^2_\tau(H^\mu_x)} \|\hat{\psi}_k\|_{L^2_\tau} \\
&\le c \sum_{k=0}^\infty 2^{k/2} \|U_\pm(-\cdot)f *_t \varphi_k\|_{L^2_t(H^\mu_x)} \\
&= c\|U_\pm(-t)f\|_{B^{1/2}_{2,1}H^\mu_x} = c\|f\|_{X_\pm^{\mu,\frac{1}{2},1}}
\end{aligned}$$

where we used $\|\hat{\psi}_k\|_{L^2} = 2^{k/2}\|\psi\|_{L^2}$, $(k = 1, 2, \dots)$.

Similarly one can prove a bilinear version:

Proposition 1.3. *Let Y be as in Proposition* 1.2. *Assume*

$$\|U_{\pm 1}(t)u_0 U_{\pm 2}u_1\|_Y \le c\|u_0\|_{H^\mu}\|u_1\|_{H^\lambda}.$$

Then

$$\|f_0 f_1\|_Y \le c\|f_0\|_{X_{\pm 1}^{\mu,\frac{1}{2},1}} \|f_1\|_{X_{\pm 2}^{\lambda,\frac{1}{2},1}}.$$

Here $\pm_1$ and $\pm_2$ denote independent signs.

The main result reads as follows:

Theorem 1.1. *The Cauchy problem for the Dirac–Klein–Gordon system* (1)–(3) *is globally well-posed for data*

$$\psi_0 \in L^2(\mathbf{R}^2), \quad \phi_0 \in H^{1/2}(\mathbf{R}^2), \quad \phi_1 \in H^{-1/2}(\mathbf{R}^2).$$

More precisely there exists a unique global solution (ψ, ϕ) *such that for all* $T > 0$

$$\psi \in X_+^{0,\frac{1}{3},1}(T) + X_-^{0,\frac{1}{3},1}(T), \quad \phi \in X_+^{\frac{1}{2},\frac{1}{3},1}(T) + X_-^{\frac{1}{2},\frac{1}{3},1}(T),$$
$$\partial_t\phi \in X_+^{-\frac{1}{2},\frac{1}{3},1}(T) + X_-^{-\frac{1}{2},\frac{1}{3},1}(T).$$

This solution has the property

$$\psi \in C^0(\mathbf{R}^+, L^2(\mathbf{R}^2)), \quad \phi \in C^0(\mathbf{R}^+, H^{\frac{1}{2}}(\mathbf{R}^2)), \quad \partial_t\phi \in C^0(\mathbf{R}^+, H^{-\frac{1}{2}}(\mathbf{R}^2)).$$

For more regular data we also get the following result.

Theorem 1.2. *Let s be an arbitrary nonnegative number. If*

$$\psi_0 \in H^s(\mathbf{R}^2), \quad \phi_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^2), \quad \phi_1 \in H^{s-\frac{1}{2}}(\mathbf{R}^2),$$

the global solution of Theorem 1.1 *has the properties*: *For every* $T > 0$

$$\psi \in X_+^{s,\frac{1}{3},1}(T) + X_-^{s,\frac{1}{3},1}(T), \quad \phi \in X_+^{s+\frac{1}{2},\frac{1}{3},1}(T) + X_-^{s+\frac{1}{2},\frac{1}{3},1}(T),$$
$$\partial_t\phi \in X_+^{s-\frac{1}{2},\frac{1}{3},1}(T) + X_-^{s-\frac{1}{2},\frac{1}{3},1}(T)$$

and

$$\psi \in C^0(\mathbf{R}^+, H^s(\mathbf{R}^2)), \quad \phi \in C^0(\mathbf{R}^+, H^{s+\frac{1}{2}}(\mathbf{R}^2)), \quad \partial_t\phi \in C^0(\mathbf{R}^+, H^{s-\frac{1}{2}}(\mathbf{R}^2)).$$

If $s > \frac{5}{2}$, *this is a classical solution, i.e.,*

$$\psi \in C^1(\mathbf{R}^+ \times \mathbf{R}^2), \quad \phi \in C^2(\mathbf{R}^+ \times \mathbf{R}^2).$$

2. Proof of the Theorems

It is possible to simplify the system (1)–(3) by considering the projections onto the one-dimensional eigenspaces of the operator $-i\alpha \cdot \nabla$ belonging to the eigenvalues $\pm|\xi|$. These projections are given by $\Pi_\pm(D)$, where $D = \frac{\nabla}{i}$ and $\Pi_\pm(\xi) = \frac{1}{2}(I \pm \frac{\xi}{|\xi|} \cdot \alpha)$. Then $-i\alpha \cdot \nabla = |D|\Pi_+(D) - |D|\Pi_-(D)$ and $\Pi_\pm(\xi)\beta = \beta\Pi_\mp(\xi)$. Defining $\psi_\pm := \Pi_\pm(D)\psi$ and splitting the function ϕ into the sum $\phi = \frac{1}{2}(\phi_+ + \phi_-)$, where $\phi_\pm := \phi \pm iA^{-1/2}\partial_t\phi$, $A := -\Delta + 1$, the Dirac–Klein–Gordon system can be rewritten as

$$(-i\partial_t \pm |D|)\psi_\pm = -M\beta\psi_\mp + \Pi_\pm(\phi\beta(\psi_+ + \psi_-)) \tag{4}$$
$$(i\partial_t \mp A^{1/2})\phi_\pm = \mp A^{-1/2}\langle\beta(\psi_+ + \psi_-), \psi_+ + \psi_-\rangle \mp A^{-1/2}(m+1)(\phi_+ + \phi_-). \tag{5}$$

The initial conditions are transformed into

$$\psi_\pm(0) = \Pi_\pm(D)\psi_0, \quad \phi_\pm(0) = \phi_0 \pm iA^{-1/2}\phi_1 \tag{6}$$

In the following we consider the system of integral equations belonging to the Cauchy problem (4)–(6):

$$\psi_{\pm}(t) = e^{\mp it|D|}\psi_{\pm}(0) - i\int_0^t e^{\mp i(t-s)|D|}\Pi_{\pm}(D)\bigg(\frac{1}{2}(\phi_+(s)+\phi_-(s))\beta(\Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s))\bigg)ds + iM\int_0^t e^{\mp i(t-s)|D|}\beta\psi\mp(s)ds \tag{7}$$

$$\begin{aligned}\phi_{\pm}(t) = e^{\mp itA^{1/2}}\phi_{\pm}(0) \pm i\int_0^t e^{\mp i(t-s)A^{1/2}}A^{-1/2}\langle\beta(+\Pi_-(D)\psi_-(s)), +\Pi_-(D)\psi_-(s),\\ \Pi_+(D)\psi_+(s)+\Pi_-(D)\psi_-(s)\rangle ds\\ \pm i(m+1)\int_0^t e^{\mp i(t-s)A^{1/2}}A^{-1/2}(\phi_+(s)+\phi_-(s))ds\end{aligned} \tag{8}$$

We remark that any solution of this system automatically fulfills $\Pi_{\pm}(D)\psi_{\pm} = \psi_{\pm}$, because applying $\Pi_{\pm}(D)$ to the right hand side of (7) gives $\Pi_{\pm}(D)\psi_{\pm}(0) = \psi_{\pm}(0)$ and the integral terms also remain unchanged, because $\Pi_{\pm}(D)^2 = \Pi_{\pm}(D)$ and $\Pi_{\pm}(D)\beta\psi_{\mp}(s) = \beta\Pi_{\mp}(D)\psi_{\mp}(s) = \beta\psi_{\mp}(s)$. Thus $\Pi_{\pm}(D)\psi_{\pm}$ can be replaced by $\psi_{\pm}$, thus the system of integral equations reduces exactly to the one belonging to our Cauchy problem (4)–(6).

In order to construct solutions to this system of integral equations we use the following facts for the linear problem which are independent of the specific phase function. The following Proposition is closely related to the exposition in [1, Section 5], where slightly different function spaces are considered. For the moment let ψ denote a smooth time cut-off function and set $\psi_T(t) = \psi(\frac{t}{T})$, where $0 < T \leq 1$. The solution of the inhomogeneous linear equation

$$\partial_t v - i\phi(D)v = F \quad v(0) = 0$$

is denoted by $U_{*R}F$, defined by

$$U_{*R}F(t) = \int_0^t U(t-t')F(t')dt',$$

where $U(t)u_0 = e^{it\phi(D)}u_0$ solves the corresponding homogeneous equation with initial datum u_0 (cf. [9, Section 2]).

Proposition 2.1. *Let* $0 < T \leq 1$, $-\frac{1}{2} < b' < 0 < b \leq \frac{1}{2}$, $s \in \mathbf{R}$, $u_0 \in H^s$ *and* $F \in L_t^1(I, H^s)$ *for a time interval* $I \subset \mathbf{R}$. *Then*

i) $\|\psi_T U u_0\|_{X^{s,\frac{1}{2},1}} \leq c\|u_0\|_{H^s}$,

ii) $\|\psi_T U_{*R}F\|_{X^{s,\frac{1}{2},1}} \leq cT^{\frac{1}{2}+b'}\|F\|_{X^{s,b',\infty}}$,

iii) $\|\psi_T u\|_{X^{s,b,1}} \leq cT^{\frac{1}{2}-b}\|u\|_{X^{s,\frac{1}{2},1}}$.

Moreover $X^{s,\frac{1}{2},1} \subset C^0(\mathbf{R}, H^s)$ *with a continuous embedding.*

Proof. Without loss of generality we may assume $s = 0$. Then we consider the scaling transformations S_T and S^T defined by

$$S_T f(t,x) = f\bigg(\frac{t}{T}, x\bigg), \quad S^T f(t,x) = Tf(Tt, x),$$

which are formally adjoint to each other with respect to the inner product in L^2_{tx} (or merely in L^2_t, if f does not depend on x). An elementary calculation shows that for $b > 0$

$$\|S_T f\|_{B^b_{2,q}L^2_x} \le cT^{\frac{1}{2}-b}\|f\|_{B^b_{2,q}L^2_x}, \tag{9}$$

which is still true without the additional L^2_x-part of the norm. For $b = \frac{1}{2}$ and $q = 1$ we especially obtain i), when replacing f by ψu_0. To see ii), we write $Kf(t) = \int_0^t f(t')dt'$. Then $\psi_T Kf(Tt) = \psi KS^T f(t)$ (cf. [1, p. 20]), hence $\psi_T Kf = S_T(\psi KS^T f)$ and thus

$$\begin{aligned}\|\psi_T Kf\|_{B^{\frac{1}{2}}_{2,1}L^2_x} &= \|S_T(\psi KS^T f)\|_{B^{\frac{1}{2}}_{2,1}L^2_x} \le c\|\psi KS^T f\|_{B^{\frac{1}{2}}_{2,1}L^2_x}\\ &\le c\ \|\psi KS^T f\|_{H^{\frac{1}{2}+}L^2_x} \le c\|KS^T f\|_{H^{\frac{1}{2}+}L^2_x}\\ &\le c\ \|S^T f\|_{H^{-\frac{1}{2}+}L^2_x} \le c\|S^T f\|_{B^{b'}_{2,\infty}L^2_x},\end{aligned}$$

where $b' > -\frac{1}{2}$. Here we used (9), $H^{\frac{1}{2}+} \subset B^{\frac{1}{2}}_{2,1}$, the fact that $H^{\frac{1}{2}+}$ forms an algebra, Lemma 2.1 from [9], and $B^{b'}_{2,\infty} \subset H^{b'-}$. Dualizing (9) we see that the latter is bounded by $T^{\frac{1}{2}+b'}\|f\|_{B^{b'}_{2,\infty}L^2_x}$, which gives ii), when replacing f by $U(-\cdot)F$. Part iii) is a consequence of (9) and the multiplication law for 1-Besov-spaces below. The additional statement follows from the well-known embedding $B^{\frac{1}{2}}_{2,1} \subset C^0$.

Lemma 2.1 (One-Dimensional Besov-Multiplication-Law). *For $0 < b \le \frac{1}{2}$ we have*

$$\|\psi u\|_{B^b_{2,1}L^2_x} \le c\|\psi\|_{B^b_{2,1}}\|u\|_{B^{\frac{1}{2}}_{2,1}L^2_x}.$$

Proof. Let $P_k u = \varphi_k * u$, where φ_k are the defining functions of the Besov spaces, and $\widetilde{P}_k = P_{k-1} + P_k + P_{k+1}$. Then

$$\begin{aligned}\|\psi u\|_{B^b_{2,1}L^2_x} &= \sum_{l\ge 0} 2^{lb}\|P_l(\psi u)\|_{L^2_{tx}} \le \sum_{k,l\ge 0} 2^{lb}\|P_l((P_k\psi)u)\|_{L^2_{tx}}\\ &\le c\left(\sum_{l\le k+2} 2^{lb}\|(P_k\psi)u\|_{L^2_{tx}} + \sum_{l\ge k+3} 2^{lb}\|(P_k\psi)(\widetilde{P}_l u)\|_{L^2_{tx}}\right) =: \sum_1 + \sum_2\end{aligned}$$

with

$$\sum_1 \le c\sum_{k\ge 0} 2^{kb}\|P_k\psi\|_{L^2_t}\|u\|_{L^\infty_t L^2_x} \le c\|\psi\|_{B^b_{2,1}}\|u\|_{B^{\frac{1}{2}}_{2,1}L^2_x},$$

since $B^{\frac{1}{2}}_{2,1} \subset L^\infty$. To estimate $\sum_2$ we choose $\frac{1}{p} = \frac{1}{2} - b$, $\frac{1}{q} = b$ so that

$$\begin{aligned}\sum_2 &\le c\sum_{k\ge 0}\|P_k\psi\|_{L^p_t}\sum_{l\ge 0} 2^{lb}\|P_l u\|_{L^q_t L^2_x}\\ &\le c\sum_{k\ge 0}\|\varphi_k\|_{L^{q'}_t}\|\widetilde{P}_k\psi\|_{L^2_t}\sum_{l\ge 0} 2^{lb}\|\varphi_l\|_{L^{p'}_t}\|\widetilde{P}_l u\|_{L^2_{tx}},\end{aligned}$$

where we used Young's inequality. Since $\|\varphi_k\|_{L^{q'}_t} \sim 2^{\frac{k}{q}}$, we obtain the desired bound.

Proposition 2.2. *For* $0 \le b' < 1/2$ *and* $0 < T \le 1$ *we have*

$$\|f\|_{L^2(\mathbf{R}^2\times[0,T])} \le cT^{b'}\|f\|_{X^{0,b'}}$$

and

$$\|f\|_{X^{0,-b'}} \le cT^{b'}\|f\|_{L^2(\mathbf{R}^2\times[0,T])}.$$

Proof. By the embedding $H^{b'} \subset L^{\frac{2}{1-2b'}}$ $(0 \le b' < 1/2)$ we get

$$\|\psi_T g\|_{L^2[0,T]} \le \|\psi_T\|_{L^{\frac{1}{b'}}}\|g\|_{L^{\frac{2}{1-2b'}}} \le cT^{b'}\|\psi\|_{L^{\frac{1}{b'}}}\|g\|_{H^{b'}}.$$

From this we get:

$$\begin{aligned}\|\psi_T f\|_{L^2_{xt}} &= \|U(-\cdot)\psi_T f\|_{L^2_{xt}} = \|\psi_T U(-\cdot)f\|_{L^2_{xt}}\\ &\le cT^{b'}\|U(-\cdot)f\|_{H^{b'}_t L^2_x} = cT^{b'}\|f\|_{X^{0,b'}},\end{aligned}$$

The second claim follows by duality.

Concerning the nonlinearities we shall prove the following estimates in Section 3 below. Here and in the sequel the letter ψ is used again to denote the spinor field.

Proposition 2.3. *The following estimates are true*:

$$\|\langle\beta\Pi_{\pm 1}(D)\psi, \Pi_{\pm 2}(D)\psi'\rangle\|_{X_{\pm 3}^{-\frac{1}{2},-\frac{1}{3},\infty}} \le c\|\psi\|_{X_{\pm 1}^{0,\frac{1}{3},1}}\|\psi'\|_{X_{\pm 2}^{0,\frac{1}{3},1}} \tag{10}$$

and

$$\|\Pi_{\pm 2}(D)(\phi\beta\Pi_{\pm 1}(D)\psi)\|_{X_{\pm 2}^{0,-\frac{1}{3},\infty}} \le c\|\phi\|_{X_{\pm 3}^{\frac{1}{2},\frac{1}{3},1}}\|\psi\|_{X_{\pm 1}^{0,\frac{1}{3},1}}. \tag{11}$$

Here and in the following $\pm_1, \pm_2, \pm_3$ *denote independent signs.*

The following local existence result now is a consequence of these estimates.

Proposition 2.4. *Let* $\psi_\pm(0) \in L^2(\mathbf{R}^2)$, $\phi_\pm(0) \in H^{\frac{1}{2}}(\mathbf{R}^2)$. *Then there exists* $1 \ge T > 0$ *such that the system of integral equations* (7), (8) *has a unique solution*

$$\psi_\pm \in X_\pm^{0,\frac{1}{3},1}(T), \quad \phi_\pm \in X_\pm^{\frac{1}{2},\frac{1}{3},1}(T).$$

This solution has the following properties:

$$\psi_\pm \in C^0([0,T], L^2(\mathbf{R}^2)), \quad \phi_\pm \in C^0([0,T], H^{\frac{1}{2}}(\mathbf{R}^2)).$$

$\phi_\pm$ *fulfills*

$$\begin{aligned}&\|\phi_+(t)\|_{H_x^{\frac{1}{2}}} + \|\phi_-(t)\|_{H_x^{\frac{1}{2}}}\\ &\quad \le (\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}) + cT^{\frac{1}{2}}(\|\psi_+(0)\|^2_{L^2} + \|\psi_-(0)\|^2_{L^2}) + c_0T^{\frac{1}{2}},\end{aligned} \tag{12}$$

for $0 \le t \le T$, where c_0 is a fixed constant. T can be chosen such that

$$T^{\frac{1}{2}}(\|\psi_+(0)\|_{L^2} + \|\psi_-(0)\|_{L^2}) \le c, \tag{13}$$

$$T^{\frac{1}{2}}(\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}) \le c, \tag{14}$$

$$T^{\frac{1}{2}}(\|\psi_+(0)\|_{L^2}^2 + \|\psi_-(0)\|_{L^2}^2) \le c(\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}). \tag{15}$$

In addition, if T fulfills only (13) *and* (14) *we get the same result except estimate* (12).

Proof. Consider the transformation mapping the left hand side of our integral equations (7), (8) into the right hand sides. We construct a fixed point of it by the contraction mapping principle in the following set

$$\begin{aligned} M_T := \Big\{\psi_\pm \in X_\pm^{0,\frac{1}{3},1}(T),\ \phi_\pm \in X_\pm^{\frac{1}{2},\frac{1}{3},1}(T) :& \\ \|\psi_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_-\|_{X_-^{0,\frac{1}{3},1}} &\le cT^{\frac{1}{6}}(\|\psi_+(0)\|_{L^2} + \|\psi_-(0)\|_{L^2}) \\ \|\phi_+\|_{X_+^{\frac{1}{2},\frac{1}{3},1}} + \|\phi_-\|_{X_-^{\frac{1}{2},\frac{1}{3},1}} &\le cT^{\frac{1}{6}}(\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}})\Big\} \end{aligned}$$

Taking an element $(\psi_\pm, \phi_\pm) \in M_T$, the nonlinear term on the right hand side of (7) is estimated in the $X_\pm^{0,\frac{1}{3},1}(T)$-norm by use of Propositions 2.1 and 2.3 (we omit T here and in the following)

$$\begin{aligned} &\Big\| \int_0^t e^{\mp i(t-s)|D|}\Pi_\pm(D)\Big(\frac{1}{2}(\phi_+(s) + \phi_-(s))\beta(\Pi_+(D)\psi_+(s) \\ &\qquad + \Pi_-(D)\psi_-(s))\Big)ds\Big\|_{X_\pm^{0,\frac{1}{3},1}} \\ &\le cT^{\frac{1}{6}}\Big\| \int_0^t e^{\mp i(t-s)|D|}\Pi_\pm(D)\Big(\frac{1}{2}(\phi_+(s) + \phi_-(s))\beta(\Pi_+(D)\psi_+(s) \\ &\qquad\qquad + \Pi_-(D)\psi_-(s))\Big)ds\Big\|_{X_\pm^{0,\frac{1}{2},1}} \\ &\le cT^{\frac{1}{3}}\|\Pi_\pm(D)(\phi_+ + \phi_-)\beta(\Pi_\pm(D)\psi_+ + \Pi_-(D)\psi_-))\|_{X_\pm^{0,-\frac{1}{3},\infty}} \\ &\le cT^{\frac{1}{3}}(\|\phi_+\|_{X_+^{\frac{1}{2},\frac{1}{3},1}} + \|\phi_-\|_{X_-^{\frac{1}{2},\frac{1}{3},1}})(\|\psi_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_-\|_{X_+^{0,\frac{1}{3},1}}) \\ &\le cT^{\frac{1}{2}}T^{\frac{1}{6}}(\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}})(\|\psi_+(0)\|_{L^2} + \|\psi_-(0)\|_{L^2}) \\ &\le cT^{\frac{1}{6}}(\|\psi_+(0)\|_{L^2} + \|\psi_-(0)\|_{L^2}), \end{aligned}$$

where in the last line we used (14).

The linear terms on the right hand side of (7) are estimated as follows:

$$\|e^{\mp it|D|}\psi_\pm(0)\|_{X_\pm^{0,\frac{1}{3},1}} \le cT^{\frac{1}{6}}\|e^{\mp it|D|}\psi_\pm(0)\|_{X_\pm^{0,\frac{1}{2},1}} \le cT^{\frac{1}{6}}\|\psi_\pm(0)\|_{L^2},$$

and by Propositions 2.1 and 2.2:

$$\begin{aligned}
&\left\| \int_0^t e^{\mp i(t-s)|D|} \beta\psi_\mp(s)ds \right\|_{X_\pm^{0,\frac{1}{3},1}} \\
&\le cT^{\frac{1}{6}} \left\| \int_0^t e^{\mp i(t-s)|D|} \beta\psi_\mp(s)ds \right\|_{X_\pm^{0,\frac{1}{2},1}} \\
&\le cT^{\frac{1}{3}} \|\psi_\mp\|_{X_\pm^{0,-\frac{1}{3},\infty}} \le cT^{\frac{2}{3}} \|\psi_\mp\|_{L^2([0,T],L^2)} \\
&\le cT\|\psi_\mp\|_{X_\mp^{0,\frac{1}{3},1}} \le cT^{\frac{7}{6}}(\|\psi_+(0)\|_{L^2} + \|\psi_-(0)\|_{L^2}).
\end{aligned}$$

Next we consider the right hand side of (8).

$$\begin{aligned}
&\Bigg\| \int_0^t e^{\mp i(t-s)A^{\frac{1}{2}}} A^{-\frac{1}{2}} \langle \beta(\Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s)), \\
&\qquad \Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s)\rangle ds \Bigg\|_{X^{\frac{1}{2},\frac{1}{3},1}} \\
&\le cT^{\frac{1}{6}} \Bigg\| \int_0^t e^{\mp i(t-s)A^{\frac{1}{2}}} A^{-\frac{1}{2}} \langle \beta(\Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s)), \\
&\qquad\qquad \Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s)\rangle ds \Bigg\|_{X^{\frac{1}{2},\frac{1}{2},1}} \\
&\le cT^{\frac{1}{3}} \|\langle \beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-), \Pi_+(D)\psi_+ + \Pi_-(D)\psi_-\rangle\|_{X^{-\frac{1}{2},-\frac{1}{3},\infty}} \\
&\le cT^{\frac{1}{3}} \left(\|\psi_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_-\|_{X_-^{0,\frac{1}{3},1}}\right)^2 \\
&\le cT^{\frac{1}{2}} T^{\frac{1}{6}} (\|\psi_+(0)\|_{L^2} + \|\psi_-(0)\|_{L^2})^2 \\
&\le cT^{\frac{1}{6}} (\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}),
\end{aligned}$$

where we used (10) and also (15) in the last line.

The linear terms on the right hand side of (8) are handled as follows:

$$\|e^{\mp itA^{\frac{1}{2}}} \phi_\pm(0)\|_{X_\pm^{\frac{1}{2},\frac{1}{3},1}} \le cT^{\frac{1}{6}} \|e^{\mp itA^{\frac{1}{2}}} \phi_\pm(0)\|_{X_\pm^{\frac{1}{2},\frac{1}{2},1}} \le cT^{\frac{1}{6}} \|\phi_\pm(0)\|_{H^{\frac{1}{2}}}$$

and

$$\begin{aligned}
&\left\| \int_0^t e^{\mp i(t-s)A^{\frac{1}{2}}} A^{-\frac{1}{2}} (\phi_+(s) + \phi_-(s)) ds \right\|_{X_\pm^{\frac{1}{2},\frac{1}{3},1}} \\
&\le cT^{\frac{1}{6}} \left\| \int_0^t e^{\mp i(t-s)A^{\frac{1}{2}}} A^{-\frac{1}{2}} (\phi_+(s) + \phi_-(s)) ds \right\|_{X_\pm^{\frac{1}{2},\frac{1}{2},1}} \\
&\le cT^{\frac{7}{6}} \left(\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}\right).
\end{aligned}$$

Here we used the following estimate

$$\begin{aligned}
&\left\| \int_0^t e^{\mp i(t-s)A^{\frac{1}{2}}} A^{-\frac{1}{2}}(\phi_+(s) + \phi_-(s))ds \right\|_{X_\pm^{\frac{1}{2},\frac{1}{2},1}} \\
&\leq cT^{\frac{1}{6}}\big(\|\phi_+\|_{X_\pm^{-\frac{1}{2},-\frac{1}{3},\infty}} + \|\phi_-\|_{X_\pm^{-\frac{1}{2},-\frac{1}{3},\infty}}\big) \\
&\leq cT^{\frac{1}{2}}\big(\|\phi_+\|_{L^2_{xt}} + \|\phi_-\|_{L^2_{xt}}\big) \\
&\leq cT^{\frac{5}{6}}\big(\|\phi_+\|_{X_+^{0,\frac{1}{3},1}} + \|\phi_-\|_{X_-^{0,\frac{1}{3},1}}\big) \\
&\leq cT\big(\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}\big). \qquad (16)
\end{aligned}$$

Altogether we have shown that the set M_T is mapped into itself. Concerning the contraction property we get similarly for the difference of the right hand sides of (7) applied to functions $(\psi_\pm, \phi_\pm) \in M_T$ and $(\tilde{\psi}_\pm, \tilde{\psi}_\pm) \in M_T$ in the $X_\pm^{0,\frac{1}{3},1}$-norm an estimate by

$$\begin{aligned}
&cT^{\frac{1}{3}}\big(\|\phi_+ - \tilde{\phi}_+\|_{X_+^{\frac{1}{2},\frac{1}{3},1}} + \|\phi_- - \tilde{\phi}_-\|_{X_-^{\frac{1}{2},\frac{1}{3},1}}\big)\big(\|\psi_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_-\|_{X_-^{0,\frac{1}{3},1}} \\
&\quad + \|\tilde{\psi}_+\|_{X_+^{0,\frac{1}{3},1}} + \|\tilde{\psi}_-\|_{X_-^{0,\frac{1}{3},1}}\big) + \big(\|\psi_+ - \tilde{\psi}_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_- - \tilde{\psi}_-\|_{X_-^{0,\frac{1}{3},1}}\big) \\
&\quad \cdot \big(\|\phi_+\|_{X_+^{\frac{1}{2},\frac{1}{3},1}} + \|\phi_-\|_{X_-^{\frac{1}{2},\frac{1}{3},1}} + \|\tilde{\phi}_+\|_{X_+^{\frac{1}{2},\frac{1}{3},1}} + \|\tilde{\phi}_-\|_{X_-^{\frac{1}{2},\frac{1}{3},1}}\big) \\
&\quad \leq cT^{\frac{1}{2}}\big[\big(\|\phi_+ - \tilde{\phi}_+\|_{X_+^{\frac{1}{2},\frac{1}{3},1}} + \|\phi_- - \tilde{\phi}_-\|_{X_-^{\frac{1}{2},\frac{1}{3},1}}\big) \\
&\quad \cdot \big(\|\psi_+(0)\|_{L^2} + \|\tilde{\psi}_+(0)\|_{L^2} + \|\psi_-(0)\|_{L^2} + \|\tilde{\psi}_-(0)\|_{L^2}\big) \\
&\quad + \big(\|\psi_+ - \tilde{\psi}_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_- - \tilde{\psi}_-\|_{X_-^{0,\frac{1}{3},1}}\big) \\
&\quad \cdot \big(\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\tilde{\phi}_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}} + \|\tilde{\phi}_-(0)\|_{H^{\frac{1}{2}}}\big)\big] \\
&\quad \leq \frac{1}{2}\big(\|\phi_+ - \tilde{\phi}_+\|_{X_+^{\frac{1}{2},\frac{1}{3},1}} + \|\phi_- - \tilde{\phi}_-\|_{X_-^{\frac{1}{2},\frac{1}{3},1}} \\
&\quad + \|\psi_+ - \tilde{\psi}_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_- - \tilde{\psi}_-\|_{X_-^{0,\frac{1}{3},1}}\big),
\end{aligned}$$

using (13) and (14) in the last line. The linear integral term in (7) is treated easily, and the right hand side of (8) can also be estimated similarly. Thus the contraction property is proved leading to a unique (local) solution.

We now show that our local solution belongs to $C_t^0(L_x^2)$. From our integral equation we get

$$\begin{aligned}
&\|\psi_\pm\|_{C_t^0 L_x^2} \leq c\|\psi_\pm\|_{X_\pm^{0,\frac{1}{2},1}} \\
&\quad \leq c\bigg(\|\psi_\pm(0)\|_{L_x^2} + \left\| \int_0^t e^{\mp i(t-s)|D|}\Pi_\pm(D)(\phi\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-))(s)ds \right\|_{X_\pm^{0,\frac{1}{2},1}} \\
&\quad + |M| \left\| \int_0^t e^{\mp i(t-s)|D|}\beta\psi_\mp(s)ds \right\|_{X_\pm^{0,\frac{1}{2},1}}\bigg)
\end{aligned}$$

$$\leq c\big(\|\psi_\pm(0)\|_{L^2_x} + T^{\frac{1}{6}}\|\Pi_\pm(D)(\phi\beta(\Pi_+(D)\psi_+ + \Pi_-(D)\psi_-))\|_{X^{0,-\frac{1}{3},\infty}}$$
$$+ |M|T^{\frac{5}{6}}\big(\|\psi_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_-\|_{X_-^{0,\frac{1}{3},1}}\big)\big)$$
$$\leq c\big(\|\psi_\pm(0)\|_{L^2_x} + T^{\frac{1}{6}}\big(\|\phi_+\|_{X_+^{\frac{1}{2},\frac{1}{3},1}} + \|\phi_-\|_{X_-^{\frac{1}{2},\frac{1}{3},1}}\big)\big(\|\psi_+\|_{X_+^{0,\frac{1}{3},1}}\|\psi_-\|_{X_-^{0,\frac{1}{3},1}}\big)$$
$$+ |M|T^{\frac{5}{6}}\big(\|\psi_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_-\|_{X_-^{0,\frac{1}{3},1}}\big)\big).$$

Here we used the estimate

$$\left\|\int_0^t e^{\mp i(t-s)|D|}\beta\psi_\mp(s)ds\right\|_{X_\pm^{0,\frac{1}{2},1}} \leq cT^{\frac{1}{6}}\big(\|\psi_-\|_{X_\pm^{0,-\frac{1}{3},\infty}} + \|\psi_+\|_{X_\pm^{0,-\frac{1}{3},\infty}}\big)$$
$$\leq cT^{\frac{1}{2}}\big(\|\psi_-\|_{L^2_{xt}} + \|\psi_+\|_{L^2_{xt}}\big) \leq cT^{\frac{5}{6}}\big(\|\psi_+\|_{X_+^{0,\frac{1}{3},1}} + \|\psi_-\|_{X_-^{0,\frac{1}{3},1}}\big)$$

by Propositions 2.1 and 2.2. We have shown that $\psi_\pm \in C_t^0(L_x^2)$.

Next we estimate $\|\phi_\pm(t)\|_{H_x^{\frac{1}{2}}}$ for $0 \leq t \leq T$ by our integral equation (8).

$$\|\phi_\pm(t)\|_{H_x^{\frac{1}{2}}}$$
$$\leq \|\phi_\pm(0)\|_{H_x^{\frac{1}{2}}} + c\left\|\int_0^t e^{\mp i(t-s)A^{\frac{1}{2}}}A^{-\frac{1}{2}}\langle\beta(\Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s)),\right.$$
$$\left.\Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s))\rangle ds\right\|_{X_\pm^{\frac{1}{2},\frac{1}{2},1}}$$
$$+ c|m+1|\left\|\int_0^t e^{\mp i(t-s)A^{\frac{1}{2}}}A^{-\frac{1}{2}}\phi(s)ds\right\|_{X_\pm^{\frac{1}{2},\frac{1}{2},1}}$$
$$\leq \|\phi_\pm(0)\|_{H_x^{\frac{1}{2}}} + cT^{\frac{1}{6}}\|\langle\beta(\Pi_+(D)\psi_+(s) + \Pi_-(D)\psi_-(s)), \Pi_+(D)\psi_+(s)$$
$$+ \Pi_-(D)\psi_-(s))\rangle\|_{X_\pm^{-\frac{1}{2},-\frac{1}{3},\infty}} + c|m+1|\left\|\int_0^t e^{\mp i(t-s)A^{\frac{1}{2}}}A^{-\frac{1}{2}}\phi(s)ds\right\|_{X_\pm^{\frac{1}{2},\frac{1}{2},1}}$$
$$\leq \|\phi_\pm(0)\|_{H_x^{\frac{1}{2}}} + cT^{\frac{1}{6}}\big(\|\psi_+\|^2_{X_+^{0,\frac{1}{3},1}} + \|\psi_-\|^2_{X_-^{0,\frac{1}{3},1}}\big)$$
$$+ c|m+1|\left\|\int_0^t e^{\mp i(t-s)A^{\frac{1}{2}}}A^{-\frac{1}{2}}\phi(s)ds\right\|_{X_\pm^{\frac{1}{2},\frac{1}{2},1}}$$
$$\leq \|\phi_\pm(0)\|_{H_x^{\frac{1}{2}}} + cT^{\frac{1}{2}}\big(\|\psi_+(0)\|^2_{L^2} + \|\psi_-(0)\|^2_{L^2}\big)$$
$$+ c|m+1|T\big(\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}\big). \tag{17}$$

Here we used (16). By our choice (14) of T we arrive at (12). This proves that $\phi_\pm \in C^0([0,T], H^{\frac{1}{2}})$. For our global result it is important to notice that no implicit constant appears in front of the first term on the right hand side.

The additional claim of the proposition is easily proven by the contraction mapping principle in a similar way so that the proof of Proposition 2.4 is complete.

The proof of Theorem 1.1 can now be given along the lines of the paper of Colliander et al. [7] for the 1D Zakharov system.

Proof of Theorem 1.1. We start by using the addition in Proposition 2.4 leading to a local solution with the required regularity properties. Because $\|\psi(t)\|_{L^2}$ is conserved we get by iteration a global solution if also $\|\phi(t)\|_{H^{\frac{1}{2}}}$ remains bounded. Otherwise we use our Proposition 2.4 and remark first that

$$\|\psi(t)\|_{L^2}^2 = \|\psi_+(t)\|_{L^2}^2 + \|\psi_-(t)\|_{L^2}^2$$

is still conserved. This conservation law can be applied because $\psi_\pm \in C^0([0,T], L_x^2)$. Without loss of generality we can now suppose that at some time t we have

$$\|\phi_+(t)\|_{H^{\frac{1}{2}}} + \|\phi_-(t)\|_{H^{\frac{1}{2}}} \gg \|\psi_+(t)\|_{L^2}^2 + \|\psi_-(t)\|_{L^2}^2.$$

Take this time t as initial time $t = 0$ so that

$$\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}} \gg \|\psi_+(0)\|_{L^2}^2 + \|\psi_-(0)\|_{L^2}^2. \tag{18}$$

Then (15) is automatically satisfied. We define

$$T \sim \frac{1}{(\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}})^2},$$

so that (13) and (14) are fulfilled. From our estimate (12) we conclude that it is possible to use the local existence result l times with time intervals of length T, before the quantity $\|\phi_+(t)\|_{H^{\frac{1}{2}}} + \|\phi_-(t)\|_{H^{\frac{1}{2}}}$ doubles. Here we have

$$l \sim \frac{\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}}{T^{\frac{1}{2}}(\|\psi_+(0)\|_{L^2}^2 + \|\psi_-(0)\|_{L^2}^2 + 1)}.$$

After these l iterations we arrive at the time

$$lT \sim \frac{\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}}{\|\psi_+(0)\|_{L^2}^2 + \|\psi_-(0)\|_{L^2}^2 + 1} T^{\frac{1}{2}} \sim \frac{1}{\|\psi_+(0)\|_{L^2}^2 + \|\psi_-(0)\|_{L^2}^2 + 1}.$$

This quantity is independent of $\|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}$. Using conservation of $\|\psi_+(t)\|_{L^2}^2 + \|\psi_-(t)\|_{L^2}^2$ it is thus possible to repeat the whole procedure with time steps of equal length. This proves the global existence result.

Proof of Theorem 1.2. By the Leibniz rule for fractional derivatives from (10), (11) one easily gets the following estimates for the nonlinearities for arbitrary $s \geq 0$:

$$\begin{aligned}
&\|\langle \beta\Pi_{\pm 1}(D)\psi, \Pi_{\pm 2}(D)\psi'\rangle\|_{X_{\pm 3}^{s-\frac{1}{2},-\frac{1}{3},\infty}} \\
&\quad \leq c\Big(\|\psi\|_{X_{\pm 1}^{0,\frac{1}{3},1}}\|\psi'\|_{X_{\pm 2}^{s,\frac{1}{3},1}} + \|\psi\|_{X_{\pm 1}^{s,\frac{1}{3},1}}\|\psi'\|_{X_{\pm 2}^{0,\frac{1}{3},1}}\Big)
\end{aligned}$$

and

$$\|\Pi_{\pm 2}(D)(\phi\beta\Pi_{\pm 1}(D)\psi)\|_{X^{s,-\frac{1}{3},\infty}_{\pm 2}}$$
$$\leq c\Big(\|\phi\|_{X^{s+\frac{1}{2},\frac{1}{3},1}_{\pm 3}}\|\psi\|_{X^{0,\frac{1}{3},1}_{\pm 1}} + \|\phi\|_{X^{\frac{1}{2},\frac{1}{3},1}_{\pm 3}}\|\psi\|_{X^{s,\frac{1}{3},1}_{\pm 1}}\Big).$$

These estimates allow to construct a local solution with the required properties by the contraction mapping principle with an existence time

$$T^{\frac{1}{2}} \sim \frac{1}{\|\psi_+(0)\|_{L^2} + \|\psi_-(0)\|_{L^2} + \|\phi_+(0)\|_{H^{\frac{1}{2}}} + \|\phi_-(0)\|_{H^{\frac{1}{2}}}}$$

similarly as in the proof of Proposition 2.4. By uniqueness the global solution of Theorem 1.1 coincides locally with this solution. Thus the claim of Theorem 1.2 follows.

3. The Estimates for the Nonlinearities

In this section we give the proof of Proposition 2.3. We first show that the estimates (10) and (11) are completely equivalent to each other. By duality (11) is equivalent to the estimate

$$\left|\int \langle \Pi_{\pm 2}(D)(\phi\beta\Pi_{\pm 1}(D)\psi), \psi'\rangle dx\, dt\right| \leq c\|\phi\|_{X^{\frac{1}{2},\frac{1}{3},1}_{\pm 3}}\|\psi\|_{X^{0,\frac{1}{3},1}_{\pm 1}}\|\psi'\|_{X^{0,\frac{1}{3},1}_{\pm 2}}. \quad (19)$$

The left hand side equals

$$\left|\int \phi\langle \beta\Pi_{\pm 1}(D)\psi, \Pi_{\pm 2}(D)\psi'\rangle dx\, dt\right|. \quad (20)$$

Thus (19) is equivalent to (10).

The complete null structure of the system detected by d'Ancona et al. has the following consequences (cf. [12]). Denoting

$$\sigma_{\pm 1,\pm 2}(\eta,\zeta) := \Pi_{\pm 2}(\zeta)\beta\Pi_{\pm 1}(\eta) = \beta\Pi_{\mp 2}(\zeta)\Pi_{\pm 1}(\eta),$$

we remark that by orthogonality this quantity vanishes if $\pm_1\eta$ and $\pm_2\zeta$ line up in the same direction whereas in general (cf. [11, Lemma 1]):

Lemma 3.1.

$$\sigma_{\pm 1,\pm 2}(\eta,\zeta) = O(\angle(\pm_1\eta, \pm_2\zeta)),$$

where $\angle(\eta,\zeta)$ denotes the angle between the vectors η and ζ.

Consequently we get

$$|\langle \beta\Pi_{\pm 1}(D)\psi, \Pi_{\pm 2}(D)\psi'\rangle\tilde{}(\tau,\xi)|$$
$$\leq \int |\langle \beta\Pi_{\pm 1}(\eta)\tilde{\psi}(\lambda,\eta), \Pi_{\pm 2}(\eta-\xi)\tilde{\psi}'(\lambda-\tau,\eta-\xi)\rangle| d\lambda\, d\eta$$

$$= \int |\langle \Pi_{\pm 2}(\eta - \xi)\beta\Pi_{\pm 1}(\eta)\tilde{\psi}(\lambda, \eta), \tilde{\psi}'(\lambda - \tau, \eta - \xi)\rangle| d\lambda\, d\eta \tag{21}$$

$$\leq c \int \Theta_{\pm 1, \pm 2} |\tilde{\psi}(\lambda, \eta)|\, |\tilde{\psi}'(\lambda - \tau, \eta - \xi)| d\lambda\, d\eta,$$

where $\Theta_{\pm 1, \pm 2} = \angle(\pm_1 \eta, \pm_2(\eta - \xi))$.

We also need the following elementary estimates which can be found in [12, Section 5.1].

Lemma 3.2. *Denoting*

$$A_{\pm 1} = \tau \pm_1 |\xi|, \quad B = \lambda + |\eta|, \quad C_\pm = \lambda - \tau \pm |\eta - \xi|, \quad \Theta_\pm = \angle(\eta, \pm(\eta - \xi))$$

and

$$\rho_+ = |\xi| - |\,|\eta| - |\eta - \xi|\,|, \quad \rho_- = |\eta| + |\eta - \xi| - |\xi|$$

the following estimates hold:

$$\Theta_+^2 \sim \frac{|\xi|\rho_+}{|\eta||\eta - \xi|}, \quad \Theta_-^2 \sim \frac{(|\eta| + |\eta - \xi|)\rho_-}{|\eta||\eta - \xi|} \sim \frac{\rho_-}{\min(|\eta|, |\eta - \xi|)}$$

as well as

$$\rho_\pm \leq 2 \min(|\eta|, |\eta - \xi|)$$

and

$$\rho_\pm \leq |A_{\pm 1}| + |B| + |C_\pm|.$$

Proof. We only prove the last estimate. We have

$$\begin{aligned}\rho_+ &\leq |\xi| \pm |\eta| \mp |\eta - \xi| = |\xi| \mp \tau \pm \lambda \pm |\eta| \pm \tau \mp \lambda \mp |\eta - \xi| \\ &\leq |\,|\xi| \mp \tau| + |\lambda + |\eta|\,| + |\tau - \lambda - |\eta - \xi|\,| \\ &\leq |A_\mp| + |B| + |C_+|\end{aligned}$$

and

$$\begin{aligned}\rho_- &= (\lambda + |\eta|) + (\tau - \lambda + |\eta - \xi|) - (\tau + |\xi|) \\ &\leq |\lambda + |\eta|| + |\lambda - \tau - |\eta - \xi|| + |\tau + |\xi|| \\ &= |B| + |C_-| + |A_+|\end{aligned}$$

as well as for $\tau \geq 0$:

$$\rho_- \leq \lambda + |\eta| + \tau - \lambda + |\eta - \xi| \leq |B| + |C_-|$$

and for $\tau \leq 0$:

$$\rho_- \leq |\lambda + |\eta|| + |\tau - \lambda + |\eta - \xi|| + |\tau| + |\xi| \leq |B| + |C_-| + |A_-|.$$

Proof of Proposition 2.3. In order to prove (19) first for the signs $\pm_1 = +$ and $\pm_2 = \pm$ and taking into account (20) and (21) we have to show (recalling $\Theta_\pm := \angle(\eta, \pm(\eta - \xi))$):

$$\begin{aligned} I_\pm &:= \left| \iint \Theta_\pm \tilde{\psi}(\lambda, \eta)\tilde{\psi}'(\lambda - \tau, \eta - \xi) d\lambda\, d\eta \tilde{\phi}(\tau, \xi) d\tau\, d\xi \right| \\ &\leq c\|\psi'\|_{X_\pm^{0,\frac{1}{3},1}} \|\psi'\|_{X_+^{0,\frac{1}{3},1}} \|\phi\|_{X_{\pm 1}^{\frac{1}{2},\frac{1}{3},1}}. \end{aligned} \tag{22}$$

We may assume here without loss of generality that the Fourier transforms are nonnegative. Defining

$$\begin{aligned} \widetilde{F}(\lambda, \eta) &:= \langle \lambda + |\eta| \rangle^{\frac{1}{3}} \tilde{\psi}(\lambda, \eta) \\ \widetilde{G_\pm}(\lambda, \eta) &:= \langle \lambda \pm |\eta| \rangle^{\frac{1}{3}} \tilde{\psi}'(\lambda, \eta) \\ \widetilde{H}_\pm(\tau, \xi) &:= \langle \tau \pm |\xi| \rangle^{\frac{1}{3}} \langle \xi \rangle^{\frac{1}{2}} \tilde{\phi}(\tau, \xi) \end{aligned}$$

we thus have to show

$$\begin{aligned} J_\pm &:= \left| \iint \Theta_\pm \frac{\widetilde{F}(\lambda, \eta)}{\langle B \rangle^{\frac{1}{3}}} \frac{\widetilde{G}_\pm(\lambda - \tau, \eta - \xi)}{\langle C_\pm \rangle^{\frac{1}{3}}} \frac{\widetilde{H}_{\pm 1}(\tau, \xi)}{\langle A_{\pm 1} \rangle^{\frac{1}{3}} \langle \xi \rangle^{\frac{1}{2}}} d\lambda\, d\eta\, d\tau\, d\xi \right| \\ &\leq c\|F\|_{X_+^{0,0,1}} \|G_\pm\|_{X_\pm^{0,0,1}} \|H_{\pm 1}\|_{X_{\pm 1}^{0,0,1}}. \end{aligned}$$

Let us first consider the low-frequency case, where $\min(|\eta|, |\eta - \xi|) \leq 1$. Assuming without loss of generality (by symmetry) $|\eta| \leq 1$ we estimate

$$\begin{aligned} I_\pm &\leq \|\psi\|_{L_t^4(L_x^\infty)} \|\psi'\|_{L_t^4(L_x^2)} \|\phi\|_{L_t^2 L_x^2} \\ &\leq \|\psi\|_{X_+^{2,\frac{1}{4}+}} \|\psi'\|_{X_\pm^{0,\frac{1}{4}+}} \|\phi\|_{L_t^2 L_x^2} \\ &\leq \|\psi\|_{X_+^{0,\frac{1}{4}+}} \|\psi'\|_{X_\pm^{0,\frac{1}{4}+}} \|\phi\|_{X^{0,0}}, \end{aligned}$$

which implies the desired estimate. From now on we assume $|\eta|, |\eta - \xi| \geq 1$.

Estimate for J_+: We use

$$\Theta_+ \leq \frac{|\xi|^{\frac{1}{2}} \rho_+^{\frac{1}{2}}}{|\eta|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}}} \leq c \frac{\langle \xi \rangle^{\frac{1}{2}} \rho_+^{\frac{1}{6}}}{\langle \eta \rangle^{\frac{1}{2}} \langle \eta - \xi \rangle^{\frac{1}{2}}} (\langle A_\pm \rangle^{\frac{1}{3}} + \langle B \rangle^{\frac{1}{3}} + \langle C_+ \rangle^{\frac{1}{3}})$$

and also

$$\rho_+ \leq 2 \min(|\eta|, |\eta - \xi|).$$

We thus get

$$J_+ \leq c\big(I_1^+ + I_2^+ + I_3^+\big),$$

where

$$I_1^+ = \int \frac{\widetilde{F}(\lambda,\eta)}{\langle \eta\rangle^{\frac{5}{12}}\langle B\rangle^{\frac{1}{3}}} \frac{\widetilde{G}_+(\lambda-\tau,\eta-\xi)}{\langle \eta-\xi\rangle^{\frac{5}{12}}\langle C_+\rangle^{\frac{1}{3}}} \widetilde{H}_{\pm 1}(\tau,\xi) d\lambda\, d\eta\, d\tau\, d\xi,$$

$$I_2^+ = \int \frac{\widetilde{F}(\lambda,\eta)}{\langle \eta\rangle^{\frac{1}{3}}} \frac{\widetilde{G}_+(\lambda-\tau,\eta-\xi)}{\langle \eta-\xi\rangle^{\frac{1}{2}}\langle C_+\rangle^{\frac{1}{3}}} \frac{\widetilde{H}_{\pm 1}(\tau,\xi)}{\langle A\rangle^{\frac{1}{3}}} d\lambda\, d\eta\, d\tau\, d\xi,$$

$$I_3^+ = \int \frac{\widetilde{F}(\lambda,\eta)}{\langle \eta\rangle^{\frac{1}{2}}\langle B\rangle^{\frac{1}{3}}} \frac{\widetilde{G}_+(\lambda-\tau,\eta-\xi)}{\langle \eta-\xi\rangle^{\frac{1}{3}}} \frac{\widetilde{H}_{\pm 1}(\tau,\xi)}{\langle A\rangle^{\frac{1}{3}}} d\lambda\, d\eta\, d\tau\, d\xi.$$

We only consider I_1^+ and I_2^+, because I_3^+ is similar to I_2^+.

Estimate for I_1^+: Hölder's inequality and Parseval's identity give

$$I_1^+ \le c\|H_{\pm 1}\|_{L^2_{xt}} \left\| \mathscr{F}^{-1}\left(\frac{\widetilde{F}(\lambda,\eta)}{\langle \eta\rangle^{\frac{5}{12}}\langle B\rangle^{\frac{1}{3}}} \right) \right\|_{L^4_{xt}} \left\| \mathscr{F}^{-1}\left(\frac{\widetilde{G_+}(\lambda-\tau,\eta-\xi)}{\langle \eta-\xi\rangle^{\frac{5}{12}}\langle C_+\rangle^{\frac{1}{3}}} \right) \right\|_{L^4_{xt}}.$$

Concerning the last two factors we use Strichartz' inequality for the wave equation which gives for $U(t) = e^{it|D|}$:

$$\|U(t)u_0\|_{L^8_t(H_x^{-\frac{5}{8},8})} \le c\|u_0\|_{L^2}.$$

This implies by Proposition 1.2:

$$\|f\|_{L^8_t(H_x^{-\frac{5}{8},8})} \le c\|U(-t)f\|_{B^{\frac{1}{2}}_{2,1}L^2_x} = c\|f\|_{X_+^{0,\frac{1}{2},1}}.$$

Moreover we have

$$\|f\|_{L^2_tL^2_x} = \|U(-t)f\|_{L^2_tL^2_x} \le c\|U(-t)f\|_{B^0_{2,1}L^2_x} = c\|f\|_{X_+^{0,0,1}}.$$

Complex interpolation gives by [2, Theorem 6.4.5]:

$$\|f\|_{L^4_t(H_x^{-\frac{5}{12},4})} \le c\|U(-t)f\|_{B^{\frac{1}{3}}_{2,1}L^2_x} = c\|f\|_{X_+^{0,\frac{1}{3},1}}.$$

This is equivalent to

$$\|f\|_{L^4_tL^4_x} \le c\|U(-t)f\|_{B^{\frac{1}{3}}_{2,1}H_x^{\frac{5}{12}}} = c\|f\|_{X_+^{\frac{5}{12},\frac{1}{3},1}}.$$

Thus we get

$$I_1^+ \le c\|H_{\pm 1}\|_{X_{\pm 1}^{0,0,1}}\|F\|_{X_+^{0,0,1}}\|G_+\|_{X_+^{0,0,1}},$$

where we used the embedding $X_\pm^{0,0,1} \subset L^2_{xt}$.

Estimate for I_2^+: Using Parseval's identity and Hölder's inequality we get

$$I_2^+ \le c\left\| \mathscr{F}^{-1}\left(\frac{\widetilde{F}(\lambda,\eta)}{\langle \eta\rangle^{\frac{1}{3}}} \right) \right\|_{L^2_t(L^3_x)} \left\| \mathscr{F}^{-1}\left(\frac{\widetilde{G}_+(\lambda-\tau,\eta-\xi)}{\langle \eta-\xi\rangle^{\frac{1}{2}}\langle C_+\rangle^{\frac{1}{3}}} \right) \right\|_{L^3_t(L^6_x)} \left\| \mathscr{F}^{-1}\left(\frac{\widetilde{H}_{\pm 1}}{\langle A\rangle^{\frac{1}{3}}} \right) \right\|_{L^6_t(L^2_x)}.$$

The first factor is estimated using Sobolev's embedding theorem by $\|F\|_{L^2_{xt}}$.

Concerning the last factor we estimate by Sobolev and Minkowski's inequality as follows:

$$\begin{aligned}\|f\|_{L_t^6(L_x^2)} &= \|U(\mp t)f\|_{L_t^6(L_x^2)} \le \|U(\mp t)f\|_{L_x^2(L_t^6)} \\ &\le c\|U(\mp t)f\|_{L_x^2 H_t^{\frac{1}{3}}} = \|U(\mp t)f\|_{H_t^{\frac{1}{3}} L_x^2} = \|f\|_{X_\pm^{0,\frac{1}{3}}}.\end{aligned} \tag{23}$$

Thus the last factor can be estimated by $c\|H_{\pm 1}\|_{L^2_{xt}} \le c\|H_{\pm 1}\|_{X^{0,0,1}_{\pm 1}}$.

Concerning the second factor we start with Strichartz' estimate

$$\|U(t)u_0\|_{L_t^4(B^{-\frac{3}{4}}_{\infty,2})} \le c\|u_0\|_{L_x^2},$$

which implies by Proposition 1.2

$$\|f\|_{L_t^4(B^{-\frac{3}{4}}_{\infty,2})} \le c\|U(-t)f\|_{B^{\frac{1}{2}}_{2,1} L_x^2} = c\|f\|_{X_+^{0,\frac{1}{2},1}}.$$

Moreover we have

$$\|f\|_{L_t^2(B^0_{2,2})} = \|f\|_{X^{0,0}} = \|U(-t)f\|_{L_t^2 L_x^2} \le c\|U(-t)f\|_{B^0_{2,1} L_x^2} = \|f\|_{{}^{0,0,1}_{+}}.$$

We now use the complex interpolation method. By [2, Theorem 6.4.5] we have

$$\left(B^{-\frac{3}{4}}_{\infty,2}, B^0_{2,2}\right)_{[\frac{2}{3}]} = B^{-\frac{1}{2}}_{6,2} \quad \text{and also } \left(X_+^{0,\frac{1}{2},1}, X_+^{0,0,1}\right)_{[\frac{2}{3}]} = X_+^{0,\frac{1}{3},1},$$

so that we get with $B^{-\frac{1}{2}}_{6,2} \subset H^{-\frac{1}{2},6}$ ([2, Theorem 6.4.4])

$$\|f\|_{L_t^3(H^{-\frac{1}{2},6})} \le c\|f\|_{L_t^3(B^{-\frac{1}{2}}_{6,2})} \le c\|U(-t)f\|_{B^{\frac{1}{3}}_{2,1} L_x^2} = \|f\|_{X_+^{0,\frac{1}{3},1}},$$

which implies

$$\|f\|_{L_t^3(L_x^6)} \le c\|f\|_{X_+^{\frac{1}{2},\frac{1}{3},1}}. \tag{24}$$

Thus the second factor is estimated by $\|G_+\|_{X_+^{0,0,1}}$.

Estimate for J_-: If $|\eta| \ll |\eta - \xi|$ we have $|\xi| \sim |\eta - \xi|$, thus by Lemma 3.2:

$$\Theta_-^2 \sim \frac{\rho_-}{\min(|\eta|, |\eta - \xi|)} \sim \frac{|\xi|\rho_-}{|\eta||\eta - \xi|},$$

so that

$$\Theta_- \le c\frac{\langle\xi\rangle^{\frac{1}{2}}\rho_-^{\frac{1}{6}}}{\langle\eta\rangle^{\frac{1}{2}}\langle\eta - \xi\rangle^{\frac{1}{2}}}(\langle A_\pm\rangle^{\frac{1}{3}} + \langle B\rangle^{\frac{1}{3}} + \langle C_-\rangle^{\frac{1}{3}}).$$

Because also $\rho_- \le 2\min(|\eta|, |\eta - \xi|)$ the same estimates as for J_+ can be given. If $|\eta| >> |\eta - \xi|$, we have $|\xi| \ge ||\eta| - |\eta - \xi|| \sim |\eta|$ and the same estimate for Θ_- holds. This is also true if $|\xi| \sim |\eta| \sim |\eta - \xi|$.

It remains to consider J_- in the case $|\xi| << |\eta| \sim |\eta - \xi|$, which we assume from now on. We then have

$$\Theta_- \leq \frac{\rho_-^{\frac{1}{2}}}{\langle\eta\rangle^{\frac{1}{4}}\langle\eta-\xi\rangle^{\frac{1}{4}}}$$

and thus

$$J_- \leq c\left|\iint \rho_-^{\frac{1}{2}} \frac{\widetilde{F}(\lambda,\eta)}{\langle\eta\rangle^{\frac{1}{4}}\langle B\rangle^{\frac{1}{3}}} \frac{\widetilde{G}_-(\lambda-\tau,\eta-\xi)}{\langle\eta-\xi\rangle^{\frac{1}{4}}\langle C_-\rangle^{\frac{1}{3}}} \frac{\widetilde{H}_{\pm1}(\tau,\xi)}{\langle\xi\rangle^{\frac{1}{2}}\langle A_{\pm1}\rangle^{\frac{1}{3}}} d\lambda\, d\eta\, d\tau\, d\xi\right|.$$

Using the estimates $\rho_- \leq 2\min(|\eta|, |\eta-\xi|)$ and $\rho_- \leq |A_{\pm1}| + |B| + |C_-|$ (cf. Lemma 3.2) we get

$$J_- \leq c(I_1^- + I_2^- + I_3^-),$$

where

$$I_1^- = \iint \frac{\widetilde{F}(\lambda,\eta)}{\langle\eta\rangle^{\frac{1}{6}}\langle B\rangle^{\frac{1}{3}}} \frac{\widetilde{G}_-(\lambda-\tau,\eta-\xi)}{\langle\eta-\xi\rangle^{\frac{1}{6}}\langle C_-\rangle^{\frac{1}{3}}} \frac{\widetilde{H}_{\pm1}(\tau,\xi)}{\langle\xi\rangle^{\frac{1}{2}}} d\lambda\, d\eta\, d\tau\, d\xi$$

$$I_2^- = \iint \frac{\widetilde{F}(\lambda,\eta)}{\langle\eta\rangle^{\frac{1}{12}}} \frac{\widetilde{G}_-(\lambda-\tau,\eta-\xi)}{\langle\eta-\xi\rangle^{\frac{1}{4}}\langle C_-\rangle^{\frac{1}{3}}} \frac{\widetilde{H}_{\pm1}(\tau,\xi)}{\langle\xi\rangle^{\frac{1}{2}}\langle A_{\pm1}\rangle^{\frac{1}{3}}} d\lambda\, d\eta\, d\tau\, d\xi$$

$$I_3^- = \iint \frac{\widetilde{F}(\lambda,\eta)}{\langle\eta\rangle^{\frac{1}{4}}\langle B\rangle^{\frac{1}{3}}} \frac{\widetilde{G}_-(\lambda-\tau,\eta-\xi)}{\langle\eta-\xi\rangle^{\frac{1}{12}}} \frac{\widetilde{H}_{\pm1}(\tau,\xi)}{\langle\xi\rangle^{\frac{1}{2}}\langle A_{\pm1}\rangle^{\frac{1}{3}}} d\lambda\, d\eta\, d\tau\, d\xi.$$

The terms I_2^- and I_3^- are similar, so that we concentrate on I_1^- and I_2^-.

Estimate for I_1^-: We have

$$\begin{aligned} I_1^- &\leq \|\widetilde{H}_{\pm1}\|_{L^2_{\xi\tau}} \left\| \int \langle\xi\rangle^{-\frac{1}{2}} \frac{\widetilde{F}(\lambda,\eta)}{\langle\eta\rangle^{\frac{1}{6}}\langle B\rangle^{\frac{1}{3}}} \frac{\widetilde{G}_-(\lambda-\tau,\eta-\xi)}{\langle\eta-\xi\rangle^{\frac{1}{6}}\langle C_-\rangle^{\frac{1}{3}}} d\lambda\, d\eta \right\|_{L^2_{\xi\tau}} \\ &= \|H_{\pm1}\|_{L^2_{xt}} \left\| \int \langle\xi\rangle^{-\frac{1}{2}} \frac{\widetilde{F}(\lambda,\eta)}{\langle\eta\rangle^{\frac{1}{6}}\langle\lambda+|\eta|\rangle^{\frac{1}{3}}} \frac{\widetilde{G}'(\tau-\lambda,\xi-\eta)}{\langle\xi-\eta\rangle^{\frac{1}{6}}\langle\tau-\lambda+|\xi-\eta|\rangle^{\frac{1}{3}}} d\lambda\, d\eta \right\|_{L^2_{\xi\tau}}, \end{aligned}$$

where $\widetilde{G}'(\lambda,\eta) := \widetilde{G}_-(-\lambda,-\eta)$. This shows that we in fact are in the $(+,+)$-case. We also remark that we assumed $|\xi| \ll |\eta| \sim |\xi-\eta|$. Using Proposition 4.2 we arrive at

$$I_1^- \leq c\|H_{\pm1}\|_{L^2_{xt}}\|F\|_{X_+^{0,0,1}}\|G'\|_{X_+^{0,0,1}} \leq c\|H_{\pm1}\|_{X_{\pm1}^{0,0,1}}\|F\|_{X_+^{0,0,1}}\|G_-\|_{X_-^{0,0,1}}.$$

Estimate for I_2^-: Parseval's identity and Hölder's inequality imply

$$\begin{aligned} I_2^- \leq c&\left\|\mathscr{F}^{-1}\left(\frac{\widetilde{F}}{\langle\eta\rangle^{\frac{1}{12}}}\right)\right\|_{L^2_t(L_x^{\frac{24}{11}})} \left\|\mathscr{F}^{-1}\left(\frac{\widetilde{G}_-}{\langle\eta-\xi\rangle^{\frac{1}{4}}\langle C_-\rangle^{\frac{1}{3}}}\right)\right\|_{L^6_t(L_x^{\frac{8}{3}})} \\ &\left\|\mathscr{F}^{-1}\left(\frac{\widetilde{H}_{\pm1}}{\langle\xi\rangle^{\frac{1}{2}}\langle A_{\pm1}\rangle^{\frac{1}{3}}}\right)\right\|_{L^3_t(L^6_x)}. \end{aligned}$$

The first factor is controlled using Sobolev's embedding $H^{\frac{1}{12}} \subset L^{\frac{24}{11}}$ by $\|F\|_{L^2_{xt}}$, the last factor is handled as before using the estimate (24), and the second one similarly as before as follows. First, Sobolev's embedding in x gives

$$\left\|\mathcal{F}^{-1}\left(\frac{\widetilde{G}_-}{\langle\eta-\xi\rangle^{\frac{1}{4}}\langle C_-\rangle^{\frac{1}{3}}}\right)\right\|_{L^6_t(L_x^{\frac{8}{3}})} \le c\left\|\mathcal{F}_t^{-1}\left(\frac{\widehat{G}_-}{\langle C_-\rangle^{\frac{1}{3}}}\right)\right\|_{L^6_t(L^2_x)}.$$

Now we use (23) so that the second factor is estimated by $\|G_-\|_{L^2_{xt}} \le c\|G_-\|_{X_-^{0,0,1}}$.

This completes the proof of estimate (22).

The remaining cases $\pm_1 = -$ and $\pm_2 = \pm$ in (19) and (20) can be treated in the same way. Using $\Pi_\mp(\eta) = \Pi_\pm(-\eta)$ we in fact get by (21)

$$\begin{aligned}
&\left|\iint \phi\langle\beta\Pi_-(D)\psi, \Pi_\pm(D)\psi'\rangle dx\, dt\right| \\
&= \left|\iint \tilde{\phi}\langle\beta\Pi_-(\eta)\tilde{\psi}(\lambda,\eta), \Pi_\pm(\eta-\xi)\tilde{\psi}'(\lambda-\tau,\eta-\xi)\rangle d\lambda\, d\eta\, d\xi\, d\tau\right| \\
&= \left|\iint \tilde{\phi}\langle\Pi_\pm(\eta-\xi)\beta\Pi_+(-\eta)\tilde{\psi}(\lambda,\eta), \tilde{\psi}'(\lambda-\tau,\eta-\xi)\rangle d\lambda\, d\eta\, d\xi\, d\tau\right| \\
&\le c\iint \Theta_\mp|\tilde{\psi}(\lambda,\eta)||\tilde{\psi}'(\lambda-\tau,\eta-\xi)|d\lambda\, d\eta|\tilde{\phi}(\tau,\xi)|d\tau\, d\xi \\
&= I_\mp,
\end{aligned}$$

because by Lemma 3.1

$$\begin{aligned}
\Pi_\pm(\eta-\xi)\beta\Pi_+(-\eta) &= \sigma_{+,\pm}(-\eta,\eta-\xi) \\
&= O(\angle(-\eta,\pm(\eta-\xi))) = O(\angle(\eta,\mp(\eta-\xi))) = O(\Theta_\mp),
\end{aligned}$$

which can be handled like $I_\pm$ above, namely as follows. Our aim is to show

$$I_\pm \le c\|\psi\|_{X_-^{0,\frac{1}{3},1}}\|\psi'\|_{X_\mp^{0,\frac{1}{3},1}}\|\phi\|_{X_{\pm 1}^{\frac{1}{2},\frac{1}{3},1}}.$$

This can be handled in the same way as before, provided the following Lemma holds.

Lemma 3.3. *Denoting*

$$A_{\pm 1} = \tau \pm_1 |\xi|, \quad B_- = \lambda - |\eta|, \quad C_\pm = \lambda - \tau \pm |\eta-\xi|$$

we have

$$\rho_\pm \le |A_{\pm 1}| + |B_-| + |C_\mp|$$

where

$$\rho_+ = |\xi| - ||\eta| - |\eta-\xi||, \quad \rho_- = |\eta| + |\eta-\xi| - |\xi|.$$

Proof.

$$\begin{aligned}\rho_+ &\le |\xi| \pm |\eta| \mp |\eta-\xi| = |\xi| \pm \tau \mp \lambda \pm |\eta| \mp \tau \pm \lambda \mp |\eta-\xi| \\ &\le ||\xi| \pm \tau| + |\lambda - |\eta|| + |\lambda - \tau - |\eta-\xi|| = |A_\pm| + |B_-| + |C_-|\end{aligned}$$

and

$$\begin{aligned}\rho_- &= |\eta| + |\eta-\xi| - |\xi| \\ &= -\lambda + |\eta| + \lambda - \tau + |\eta-\xi| + \tau - |\xi| \le |B_-| + |C_+| + |A_-|\end{aligned}$$

as well as for $\tau \le 0$:

$$\begin{aligned}\rho_- &= |\eta| + |\eta-\xi| - |\xi| = |\eta| - \lambda + \lambda - \tau + |\eta-\xi| + \tau - |\xi| \\ &\le ||\eta| - \lambda| + |\lambda - \tau + |\eta-\xi|| \\ &\le |B_-| + |C_+|\end{aligned}$$

and for $\tau \ge 0$

$$\begin{aligned}\rho_- &= |\eta| + |\eta-\xi| - |\xi| = |\eta| - \lambda + \lambda - \tau + |\eta-\xi| + \tau - |\xi| \\ &\le ||\eta| - \lambda| + |\lambda - \tau + |\eta-\xi|| + \tau + |\xi| \le |B_-| + |C_+| + |A_+|.\end{aligned}$$

This completes the proof of the Lemma and Proposition 2.3.

4. A Bilinear Strichartz Type Estimate

The following bilinear refinement is crucial for the estimate of the term I_1^-. It follows from the following proposition, which can be found in [12].

Defining

$$[(f,g)_{HH\to L}]\widehat{\ }(\xi) := \int_{\mathbf{R}^2} \chi_{\{|\xi|<<|\eta|+|\xi-\eta|\}} \hat{f}(\eta)\hat{g}(\xi-\eta)d\eta,$$

where χ_A is the characteristic function of the set A.

Proposition 4.1 ([11, Theorem 6]). *Let*

$$u_\pm(t) := e^{\mp it|D|}f$$

and

$$v_\pm(t) := e^{\mp it|D|}g.$$

Then we have

$$\||D|^{-s_3}(u_\pm, v_\pm)_{HH\to L}\|_{L^2_{xt}} \le c\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}},$$

where $s_1 + s_2 + s_3 = \frac{1}{2}$, $s_1, s_2 < \frac{5}{8}$, $s_1 + s_2 > 0$.

Using Proposition 1.3 we get

Corollary 4.1. *Under the assumptions of Proposition* 4.1 *the following estimate holds*

$$\||D|^{-s_3}(u,v)_{HH\to L}\|_{L^2_{xt}} \le c\|u\|_{X_\pm^{s_1,\frac{1}{2},1}}\|v\|_{X_\pm^{s_2,\frac{1}{2},1}}$$

where it is essential that the two signs on the right hand side are equal.

The following consequence is exactly what we need in order to control I_1^- in a suitable way.

Proposition 4.2.

$$\|\langle D\rangle^{-\frac{1}{2}}(u,v)_{HH\to L}\|_{L^2_{xt}} \le c\|u\|_{X_\pm^{\frac{1}{6},\frac{1}{3},1}}\|v\|_{X_\pm^{\frac{1}{6},\frac{1}{3},1}}.$$

Proof. The previous corollary is applied with $s_1 = s_2 = s_3 = \frac{1}{6}$ leading to

$$\|\langle D\rangle^{-\frac{1}{6}}(u,v)_{HH\to L}\|_{L^2_{xt}} \le c\|u\|_{X_\pm^{\frac{1}{6},\frac{1}{2},1}}\|v\|_{X_\pm^{\frac{1}{6},\frac{1}{2},1}}. \quad (25)$$

It is interpolated with the following estimate which follows from Sobolev and the estimate $\|f\|_{L_t^4(L_x^2)} \le c\|f\|_{X_\pm^{0,\frac{1}{4}}}$, which is proven like (23).

$$\begin{aligned}\|\langle D\rangle^{-\frac{2}{3}}(u,v)_{HH\to L}\|_{L^2_{xt}} &\le c\|uv\|_{L_t^2(L_x^{\frac{6}{5}})} \le c\|u\|_{L_t^4(L_x^{\frac{12}{5}})}\|v\|_{L_t^4(L_x^{\frac{12}{5}})}\\ &\le c\|u\|_{L_t^4(H_x^{\frac{1}{6}})}\|v\|_{L_t^4(H_x^{\frac{1}{6}})} \le c\|u\|_{X_\pm^{\frac{1}{6},\frac{1}{4}}}\|v\|_{X_\pm^{\frac{1}{6},\frac{1}{4}}}\\ &\le c\|u\|_{X_\pm^{\frac{1}{6},\frac{1}{4},1}}\|v\|_{X_\pm^{\frac{1}{6},\frac{1}{4},1}}. \end{aligned} \quad (26)$$

Complex bilinear interpolation between (25) and (26) gives the result, using

$$\left(X_\pm^{\frac{1}{6},\frac{1}{2},1}, X_\pm^{\frac{1}{6},\frac{1}{4},1}\right)_{[\frac{1}{3}]} = X_\pm^{\frac{1}{6},\frac{1}{3},1}.$$

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