FUNCTIONAL INTEGRAL REPRESENTATIONS OF A HEAT SEMIGROUP IN QED

Department of Mathematics Faculty of Science Hokkaido Univ.
Sapporo 060 Japan
Fumio HIROSHIMA

Abstract

This note presents functional integral representations for heat semigroups with infinitesimal generators given by self-adjoint Hamiltonians describing an interaction of a non-relativistic charged particle and a quantized radiation field in the Coulomb gauge without the dipole approximation.

1 INTRODUCTION

Until now many authors give functional integral representations for heat semi groups with infinitesimal generators given by self-adjoint Hamiltonians describing quantum systems. Using functional integral representations, they analyze quantum systems. In this note we concern with quantum systems which describe interactions of a non-relativistic charged particle and a quantized radiation field in the Coulomb gauge without the dipole approximation, which is the so called "Pauli-Fierz model"([1,2,3,4,5,6]).

Mathematically, the set of state vectors $\mathcal{M}_B$ of this quantum system can be described by the tensor product of $L^2(\mathbb{R}^d)$ and the Boson Fock space $\mathcal{F}(\mathcal{W})$ over $\mathcal{W} = \underbrace{L^2(\mathbb{R}^d) \oplus ... \oplus L^2(\mathbb{R}^d)}_{d-1}$. Pauli-Fierz Hamiltonians $\mathbf{H}_{\rho,B}$ are formally defined as operators acting in $\mathcal{M}$ as follows ((2.2)):

$$\mathbf{H}_{\rho,B} = \frac{1}{2}\sum_{\mu=1}^{d}(-iD_\mu \otimes -A_\mu(\rho))^2 + I \otimes \mathbf{H}_{0,B},$$

where D_μ is the generalized derivative in the μ-th direction, $\mathbf{H}_{0,B}$ the free Hamiltonian in the Boson Fock space $\mathcal{F}(\mathcal{W})$ and $A_\mu(\rho)$ the time-zero radiation field in the μ-th direction

((2.1)). Many researches of the Pauli-Fierz model have been devoted to dealing with Hamiltonians with the dipole approximation ([1,2,3]). Without the dipole approximation, it is not known even essential self-adjointness of $\mathbf{H}_{\rho,B}$ except for special ρ's ([4,5,6]). Then analysis of Hamiltonians without the dipole approximation is crucial.

In this note, constructing a Hilbert space $L^2(Q_{-1}, d\mu_{-1})$ (see section 2), we define Hamiltonians acting in $\mathcal{M} = L^2(\mathbb{R}^d) \otimes L^2(Q_{-1}, d\mu_{-1})$ by the quadratic form sum of the generators $\mathbf{H}_{\rho,0}$ of strongly continuous 1-parameter semigroups and a self-adjoint operator $\mathbf{H}_0$ in $L^2(Q_{-1}, d\mu_{-1})$ ((2.3)). We shall show that $\mathbf{H}_{\rho,0} \dot{+} I \otimes \mathbf{H}_0$ is unitarily equivalent to $\mathbf{H}_{f,B}$ on some domains with some ρ's and f's (Theorem 2.1). Again we define Pauli-Fierz Hamiltonians by $\mathbf{H}_\rho = \mathbf{H}_{\rho,0} \dot{+} I \otimes \mathbf{H}_0$.

The Wiener path integral method is useful to get path integral representations of heat semigroups with generators:

$$\mathbf{H}_{cl} = \frac{1}{2} \sum_{\mu=1}^{d} (-iD_\mu - A_\mu)^2 + V,$$

where A_μ is a vector potential and V a scalar potential. It is known as the Feynman-Kac-Itô formula ([7]). In connection with construction of quantum field models from markoff fields, E.Nelson introduced a "generalized path space"(functional space). He also introduced a natural embedding of a Boson Fock space in d space dimension into a constant time subspace in the L^2 space over the "generalized path space" in $d + 1$ dimensions. The natural embedding gives us a functional integral representation of a heat semigroup with the free Hamiltonian in the Boson Fock space as the generator. It is called the "Feynman-Kac-Nelson formula ([8]).

The purpose of this note is to give functional integral representations to the expectation $\left\langle F, e^{-t\mathbf{H}_\rho} G \right\rangle_{\mathcal{M}}$. In order to do so, we shall use the FKI and the FKN formulas simultaneously. And we shall need to define the "time-ordered Hilbert space-valued stochastic integral associated with a family of isometry $\{[\widetilde{j}_t]\}_{t \in \mathbb{R}^d}$ from a Hilbert space $[\widetilde{\mathcal{H}}_{-1}]$ to another one $[\widetilde{\mathcal{H}}_{-2}]$"(see Theorem 3.7).

2 PAULI-FIERZ HAMILTONIAN

In this section, we define Pauli-Fierz Hamiltonians and give their probabilistic descriptions. For mathematical generality, we consider the situation where a charged particle moves in $\mathbb{R}^d$ and a quantized radiation field is over $\mathbb{R}^d$. We define polarization vectors $e^r(r = 1, ..., d-1)$ as measurable functions $e^r : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$e^r(k) \cdot e^s(k) = \delta_{rs}, \quad k \cdot e^r(k) = 0, \quad a.e. k \in \mathbb{R}^d.$$

Put

$$d_{\mu\nu}(k) \equiv \sum_{r=1}^{d-1} e^r_\mu(k) e^r_\nu(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}.$$

The Boson Fock space $\mathcal{F}(\mathcal{W})$ over $\mathcal{W} = \underbrace{L^2(\mathbb{R}^d) \oplus ... \oplus L^2(\mathbb{R}^d)}_{d-1}$ is defined by

$$\mathcal{F}(\mathcal{W}) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{W}), \quad \mathcal{F}_n(\mathcal{W}) = \otimes_s^n \mathcal{W}, \ n \geq 1, \quad \mathcal{F}_0 = \mathbb{C},$$

where $\otimes_s^n$ denotes the n-fold symmetric tensor product. Put $\Omega = \{1, 0, 0, ...\}$. Let

$$\mathcal{F}^N(\mathcal{W}) = \bigoplus_{n=0}^{N} \mathcal{F}_n(\mathcal{W}) \bigoplus_{n>N+1} \{0\}, \quad \mathcal{F}^\infty(\mathcal{W}) = \bigcup_{N=0}^{\infty} \mathcal{F}^N(\mathcal{W}).$$

The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ $(f \in \mathcal{W})$ act on $\mathcal{F}^\infty(\mathcal{W})$ and leave it invariant with the canonical commutation relations (CCR): for $f, g \in \mathcal{W}$

$$[a(f), a^\dagger(g)] = \left\langle \bar{f}, g \right\rangle_{\mathcal{W}}, \quad [a^\sharp(f), a^\sharp(g)] = 0,$$

where $[A, B] = AB - BA$, $a^\sharp$ denotes either a or $a^\dagger$. It is well known that

$$\mathcal{F}(\mathcal{W}) = \mathcal{L}\left\{a^\dagger(f_1)...a^\dagger(f_n)\Omega, \Omega | f_j \in \mathcal{W}, j = 1, ..., n, n \geq 1\right\}^{-}.$$

We recall here second quantizations of operators. For any contraction operator T on $\mathcal{W}$, the"second quantization of T", $\Gamma_B(T) : \mathcal{F}(\mathcal{W}) \to \mathcal{F}(\mathcal{W})$, is a contraction operator uniquely determined by

$$\Gamma_B(T)\Omega = 0, \ \ \Gamma_B(T) a^\dagger(f_1) a^\dagger(f_2) ... a^\dagger(f_n)\Omega = a^\dagger(Tf_1) a^\dagger(Tf_2) ... a^\dagger(Tf_n)\Omega, n \geq 1.$$

For a nonnegative self-adjoint operator A in $\mathcal{W}$, the "second quantization of A", $d\Gamma_B(A)$, is defined by the infinitesimal generator of the strongly continuous 1-parameter semigroup

$$\Gamma_B(e^{-tA}) = e^{-td\Gamma_B(A)}.$$

We define a maximal multiplication operator ω_B in $L^2(\mathbb{R}^d)$ by

$$(\omega_B f)(k) = h(k)f(k),$$

where $h(k) = |k|$. Put $\widetilde{\omega}_B = \underbrace{\omega_B \oplus ... \oplus \omega_B}_{d-1}$. Then $\mathbf{H}_{0,B} = d\Gamma_B(\widetilde{\omega}_B)$ shall be the free Hamiltonian in $\mathcal{F}(\mathcal{W})$. We define the μ-th direction time-zero radiation field by

$$A_\mu(x,f) = \frac{1}{\sqrt{2}}\left\{ a^\dagger\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r \hat{f} e^{-i\cdot k}}{\sqrt{h}}\right) + a\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r \tilde{\hat{f}} e^{i\cdot k}}{\sqrt{h}}\right)\right\}, \mu = 1,...,d, \tag{2. 1}$$

where $\hat{f}$ is the Fourier transformation of f ($\check{f}$ the inverse Fourier transformation of f in what follows) and $\tilde{g}(k) = g(-k)$. A Hilbert space of state vectors in a system of the non-relativistic charged particle interacting with the quantized radiation field is given by $\mathcal{M}_B = L^2(\mathbb{R}^d) \otimes \mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^d; \mathcal{F}(\mathcal{W}))$. We shall use this identification without notice. Then interaction Hamiltonians (Pauli-Fierz Hamiltonians) of the non-relativistic charged particle with mass one and the quantized radiation field is "formally" defined as an operator acting in $\mathcal{M}_B$ by

$$\mathbf{H}_\rho = \frac{1}{2}\sum_{\mu=1}^{d}\left(-iD_\mu \otimes I - A_\mu(\rho)\right)^2 + I \otimes \mathbf{H}_{0,B}, \tag{2. 2}$$

where we take the natural unit $c = \hbar = 1$ and

$$A_\mu(\rho) = \int_{\mathbb{R}^d}^{\oplus} A_\mu(x,\rho)dx.$$

Generally, it is crucial whether Hamiltonians defined on some domains have unique self-adjoint extensions, since the unique extensions lead to the uniqueness of time evolutions of state vectors in quantum systems. Nevertheless it is not known whether the formally defined Hamiltonians $\mathbf{H}_\rho$ restricted to some concrete domains have unique self-adjoint extensions. Then we must construct self-adjoint extensions of $\mathbf{H}_\rho$ in some way.

We have to give probabilistic descriptions to the Hamiltonian $\mathbf{H}_\rho$. First we define two real Hilbert spaces $\mathcal{H}_{-1}$ and $\mathcal{H}_{-2}$ by

$$\mathcal{H}_{-1} \equiv \left\{ f \in \mathcal{S}_r'(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d} \frac{|\hat{f}(k)|^2}{|k|}dk < \infty \right\}, \mathcal{H}_{-2} \equiv \left\{ f \in \mathcal{S}_r'(\mathbb{R}^{d+1}) \,\middle|\, \int_{\mathbb{R}^{d+1}} \frac{|\hat{f}(k)|^2}{|k|^2}dk < \infty \right\},$$

where $\mathcal{S}_r'(\mathbb{R}^n)$ denotes the set of the real tempered distributions on $\mathbb{R}^n (n = d, d+1)$. Put

$$\widetilde{\mathcal{H}}_{-1} = \underbrace{\mathcal{H}_{-1} \oplus ... \oplus \mathcal{H}_{-1}}_{d}, \quad \widetilde{\mathcal{H}}_{-2} = \underbrace{\mathcal{H}_{-2} \oplus ... \oplus \mathcal{H}_{-2}}_{d}.$$

We introduce bilinear forms $(\cdot,\cdot)_{-1}$ and $(\cdot,\cdot)_{-2}$ in $\widetilde{\mathcal{H}}_{-1}$ and $\widetilde{\mathcal{H}}_{-2}$ by

$$(f,g)_{-1}=\sum_{\mu,\nu=1}^{d}\int_{\mathbb{R}^d}\frac{d_{\mu\nu}(k)\bar{\hat{f}}_\mu(k)\hat{g}_\nu(k)}{|k|}dk,\quad (f,g)_{-2}=2\sum_{\mu,\nu=1}^{d}\int_{\mathbb{R}^{d+1}}\frac{d_{\mu\nu}(k)\bar{\hat{f}}_\mu(k)\hat{g}_\nu(k)}{|k|^2}dk.$$

We denote the associated semi-norms by $|\cdot|_{-1}$ and $|\cdot|_{-2}$, respectively and put

$$N_{-1}=\left\{f\in\widetilde{\mathcal{H}}_{-1}\Big|\,|f|_{-1}=0\right\},\quad N_{-2}=\left\{f\in\widetilde{\mathcal{H}}_{-2}\Big|\,|f|_{-2}=0\right\}.$$

Then we define pre-Hilbert spaces by the quotient spaces

$$[\widetilde{\mathcal{H}}_{-1}]=\widetilde{\mathcal{H}}_{-1}/N_{-1},\quad [\widetilde{\mathcal{H}}_{-2}]=\widetilde{\mathcal{H}}_{-2}/N_{-2},$$

with inner products $<\cdot,\cdot>_{-1}$ and $<\cdot,\cdot>_{-2}$ defined by

$$\langle\pi_{-1}(f),\pi_{-1}(g)\rangle_{-1}\equiv(f,g)_{-1},\quad \langle\pi_{-2}(f),\pi_{-2}(g)\rangle_{-2}\equiv(f,g)_{-2}.$$

Here $\pi_{-1}(f)$ and $\pi_{-2}(f)$ denote the equivalence classes of f in $\widetilde{\mathcal{H}}_{-1}$ and $\widetilde{\mathcal{H}}_{-2}$, respectively. We denote the norms associated with the inner products $<\cdot,\cdot>_{-1}$ and $<\cdot,\cdot>_{-2}$ by $||\cdot||_{-1}$ and $||\cdot||_{-2}$, respectively. The Hilbert spaces constructed by the completions of $[\widetilde{\mathcal{H}}_{-1}]$ and $[\widetilde{\mathcal{H}}_{-2}]$ with respect to $||\cdot||_{-1}$ and $||\cdot||_{-2}$ are denoted by the same symbols.
Let $\{\phi_{-1}(\pi_{-1}(f))|f\in\widetilde{\mathcal{H}}_{-1}\}$ and $\{\phi_{-2}(\pi_{-2}(f))|f\in\widetilde{\mathcal{H}}_{-2}\}$ be the Gaussian random processes indexed by the Hilbert spaces $[\widetilde{\mathcal{H}}_{-1}]$ and $[\widetilde{\mathcal{H}}_{-2}]$ such that the characteristic functions are given by

$$\int_{Q_j}e^{i\phi_j(\pi_j(f))}d\mu_j=e^{-\frac{1}{4}||\pi_j(f)||_j^2},\quad j=-1,-2,$$

where $(Q_j,d\mu_j), j=-1,-2$ denote the underlying measure spaces of these processes. It is well known that $L^2(Q_j,d\mu_j)$ has the orthogonal decomposition

$$L^2(Q_j,d\mu_j)=\bigoplus_{n=0}^{\infty}\Gamma_n([\widetilde{\mathcal{H}}_j]),\quad j=-1,-2,$$

with

$$\Gamma_0([\widetilde{\mathcal{H}}_j])=\mathbb{C},$$
$$\Gamma_n([\widetilde{\mathcal{H}}_j])=\mathcal{L}\{:\phi_j(\pi_j(f_1))\phi_j(\pi_j(f_2))...\phi_j(\pi_j(f_n)):|f_k\in\widetilde{\mathcal{H}}_j,k=1,..,n\}^-,\quad n\geq 1.$$

Here $:\cdot:$ means the Wick product and $\mathcal{L}$ the linear span of the vectors in $\{...\}$ over $\mathbb{C}$. We denote the complexifications of $[\widetilde{\mathcal{H}}_j]$ by $[\widetilde{\mathcal{H}}_j]_{\mathbb{C}}$. Suppose that T is a contraction operator from $[\widetilde{\mathcal{H}_i}]_{\mathbb{C}}$ to $[\widetilde{\mathcal{H}_j}]_{\mathbb{C}}$. Corresponding to each such T we can define a contraction operator $\Gamma(T) : L^2(Q_i; d\mu_i) \longrightarrow L^2(Q_j; d\mu_j)$ by

$$\begin{aligned} \Gamma(T)\Omega_i &= 0, \\ \Gamma(T) : \phi_i(\pi_i(f_1))...\phi_i(\pi_i(f_n)) : &= : \phi_j(T\pi_j(f_1))\phi_j(T\pi_j(f_2))...\phi_j(T\pi_j(f_n)) :, \end{aligned}$$

where Ω_i denotes the constant function 1 in $L^2(Q_i, d\mu_i)$. For a nonnegative self-adjoint operator $A : [\widetilde{\mathcal{H}_i}]_{\mathbb{C}} \longrightarrow [\widetilde{\mathcal{H}_i}]_{\mathbb{C}}$ $(i = -1, -2)$ we define $d\Gamma(A)$ by

$$\begin{aligned} & d\Gamma(A)\Omega_i = 0, \\ & d\Gamma(A) : \phi_i(\pi_i(f_1))...\phi_i(\pi_i(f_n)) : \\ & =: \phi_i(A\pi_i(f_1))\phi_i(\pi_i(f_2))...\phi_i(\pi_i(f_n)) : + : \phi_i(\pi_i(f_1))\phi_i(A\pi_i(f_2))...\phi_i(\pi_i(f_n)) : \\ & +...+ : \phi_i(\pi_i(f_1))\phi_i(\pi_i(f_2))...\phi_i(A\pi_i(f_n)) :, \quad \pi_i(f_k) \in D(A), k = 1, ..., n. \end{aligned}$$

It is well known that $d\Gamma(A)$ has the unique self-adjoint extension in $L^2(Q_i; d\mu_i)$. We denote it by the same symbol $d\Gamma(A)$. We define an operator ω in $\mathcal{H}_{-1}$ by

$$\widehat{\omega f}(k) = h(k)\hat{f}(k),$$

and put $\tilde{\omega} = \underbrace{\omega \oplus ... \oplus \omega}_{d}$. Furthermore, $[\tilde{\omega}] : [\widetilde{\mathcal{H}_{-1}}] \to [\widetilde{\mathcal{H}_{-1}}]$ is defined by

$$[\tilde{\omega}]\pi_{-1}(f) = \pi_{-1}(\tilde{\omega}f), \quad D([\tilde{\omega}]) = \{\pi_{-1}(f) \in [\widetilde{\mathcal{H}_{-1}}] | \tilde{\omega}f \in \widetilde{\mathcal{H}_{-1}}\}.$$

Set $d\Gamma([\tilde{\omega}]) = \mathbf{H}_0$, $L^2(Q_{-1}, d\mu_{-1}) = \mathcal{F}$, $L^2(Q_{-2}, d\mu_{-2}) = \mathcal{E}$, $\phi_{-1}(\cdot) = \phi_{\mathcal{F}}(\cdot)$ and $\phi_{-2}(\cdot) = \phi_{\mathcal{E}}(\cdot)$. Similarly to the Boson Fock space $\mathcal{F}(\mathcal{W})$, we put

$$\mathcal{F}^N = \bigoplus_{n=0}^{N} \Gamma_n([\widetilde{\mathcal{H}_{-1}}]) \bigoplus_{n>N+1} \{0\}, \quad \mathcal{F}^\infty = \bigcup_{N=0}^{\infty} \mathcal{F}^N.$$

For an $\mathcal{H}_{-1}$-valued function on $\mathbb{R}^d$, $\rho(\cdot) : \mathbb{R}^d \to \mathcal{H}_{-1}$, we put $\tilde{\rho}_\mu(\cdot) = \overbrace{(0, ..., \underbrace{\rho(\cdot)}_{the\ \mu-th}, ..., 0)}^{d}$.

Then we define an operator in $\mathcal{M} = L^2(\mathbb{R}^d) \otimes \mathcal{F} \cong L^2(\mathbb{R}^d; \mathcal{F})$ by

$$\phi^\rho_{\mathcal{F},\mu} = \int^{\oplus}_{\mathbb{R}^d} \phi_{\mathcal{F}}\left(\pi_{-1}\left(\tilde{\rho}_\mu(x)\right)\right) dx.$$

Define operators in $\mathcal{M}$ by

$$\mathbf{H}_\rho = \frac{1}{2}\sum_{\mu=1}^{d} \left(-iD_\mu \otimes I - \phi^\rho_{\mathcal{F},\mu}\right)^2 + I \otimes \mathbf{H}_0 \equiv \mathbf{H}_{\rho,0} + I \otimes \mathbf{H}_0. \tag{2. 3}$$

Theorem 2.1 ([4,Theorem 3.1]) *Set* $\mathcal{D}_B = C_0^\infty(\mathbb{R}^d)\hat{\otimes}\mathcal{F}^\infty(\mathcal{W}) \cap D(I \otimes \mathbf{H}_{0,B})$ *and* $\mathcal{D} = C_0^\infty(\mathbb{R}^d)\hat{\otimes}\mathcal{F}^\infty \cap D(I \otimes \mathbf{H}_0)$. *Let* $\rho = \left(\hat{f}(\cdot)e^{i\cdot x}\right)^\vee$, $f \in \mathcal{H}_{-1}$. *Then there exists a unitary operator* $\mathcal{U}$ *from* $\mathcal{F}(\mathcal{W})$ *to* $\mathcal{F}$ *such that* $\mathcal{U}$ *maps* $\mathcal{D}_B$ *onto* $\mathcal{D}$ *and*

$$\mathcal{U}^{-1}\mathbf{H}_\rho \mathcal{U}\Big|_{\mathcal{D}} = \mathbf{H}_{f,B}|_{\mathcal{D}}.$$

By Theorem 2.1, we call $\mathbf{H}_\rho$ the Pauli-Fierz Hamiltonian again. We can give connection between $\mathcal{F}$ and $\mathcal{E}$. For $t \in \mathbb{R}$ we define an operator j_t by

$$j_t : \mathcal{H}_{-1} \longrightarrow \mathcal{H}_{-2}, \quad j_t f = \delta_t \otimes f,$$

where δ_t is the one-dimensional delta function with mass at $\{t\}$. We put $\widetilde{j}_t = j_t \oplus ... \oplus j_t$ and define

$$[\widetilde{j}_t]\pi_{-1}(f) = \pi_{-2}(\widetilde{j}_t f).$$

It can be easily seen that $[\widetilde{j}_t]$ is a linear isometry. Hence the range of $[\widetilde{j}_t]$ is a closed subspace of $[\widetilde{\mathcal{H}}_{-2}]$. We denote the projection onto $Ran([\widetilde{j}_t])$ by $[e_t]$. We denote the projection onto $Ran([\widetilde{j}_t])$ by $[e_t]$. Let

$$U_{[a,b]} \equiv L\left\{\pi_{-2}(f) \in [\widetilde{\mathcal{H}}_{-2}] \,\Big|\, \pi_{-2}(f) \in Ran([\widetilde{j}_t]), a \le t \le b\right\}.$$

We denote the projection onto the closure $\overline{U_{[a,b]}}$ by $[e_{[a,b]}]$.

Proposition 2.2 ([9,Propositions III.3 and III.4])
(a) $[\widetilde{j}_t][\widetilde{j}_t]^* = [e_t]$.
(b) $[\widetilde{j}_t]^*[\widetilde{j}_s] = e^{-|t-s|[\widetilde{\omega}]}$.
(c) Let $a \le b \le c$. *Then* $[e_a][e_b][e_c] = [e_a][e_c]$.
(d) Let $a \le b \le t \le c \le d$. *Then* $[e_{[a,b]}][e_t][e_{[c,d]}] = [e_{[a,b]}][e_{[c,d]}]$.

Proof: (a) is straightforwardly seen. Since we have

$$\begin{aligned}\left\langle [\widetilde{j}_t]^*[\widetilde{j}_s]\pi_{-1}(f), \pi_{-1}(g)\right\rangle_{-1} &= \left\langle \pi_{-2}(\widetilde{j}_s f), \pi_{-2}(\widetilde{j}_t g)\right\rangle_{-2} \\ &= \frac{1}{\pi}\sum_{\mu,\nu=1}^d \int_{\mathbb{R}^{d+1}} \frac{\overline{\hat{f}}_\mu(\vec{k})\hat{g}_\nu(\vec{k})d_{\mu\nu}(\vec{k})e^{i(t-s)k_0}}{|\vec{k}|^2 + k_0^2} d\vec{k}dk_0 \\ &= \sum_{\mu,\nu=1}^d \int_{\mathbb{R}^d} \frac{\overline{\hat{f}}_\mu(\vec{k})\hat{g}_\nu(\vec{k})d_{\mu\nu}(\vec{k})e^{-|t-s||\vec{k}|}}{|\vec{k}|} d\vec{k},\end{aligned}$$

the statement (b) holds. Since

$$[e_a][e_b][e_c] = [\widetilde{j}_a][\widetilde{j}_a]^*[\widetilde{j}_b][\widetilde{j}_b]^*[\widetilde{j}_c][\widetilde{j}_c]^* = [\widetilde{j}_a]e^{-(c-a)[\widetilde{\omega}]}[\widetilde{j}_c]^* = [e_a][e_c],$$

the statement (c) follows. For any $\pi_{-2}(f)$ and $\pi_{-2}(g)$, by the definition of $[e_{[a,b]}]$ and $[e_{[c,d]}]$, they can be presented as follows

$$\begin{aligned} [e_{[c,d]}]\pi_{-2}(f) &= \lim_{n\to\infty}\sum_{\alpha=1}^{N_n} f_{n_\alpha}, \quad f_{n_\alpha} \in Ran([e_{t_{n_\alpha}}]), t_{n_\alpha} \in [c,d], \\ [e_{[a,b]}]\pi_{-2}(g) &= \lim_{m\to\infty}\sum_{\beta=1}^{M_m} f_{m_\beta}, \quad g_{m_\beta} \in vRan([e_{t_{m_\beta}}]), t_{m_\beta} \in [a,b]. \end{aligned}$$

Hence by (c) we have

$$\begin{aligned} \left\langle [e_{[a,b]}][e_t][e_{[c,d]}]\pi_{-2}(f), \pi_{-2}(g)\right\rangle_{-2} &= \lim_{n,m\to\infty}\sum_{\alpha,\beta=1}^{N_n,M_m} \left\langle [e_t]f_{n_\alpha}, g_{m_\beta}\right\rangle_{-2} \\ &= \lim_{n,m\to\infty}\sum_{\alpha,\beta=1}^{N_n,M_m} \left\langle f_{n_\alpha}, g_{m_\beta}\right\rangle_{-2} \\ &= \left\langle [e_{[a,b]}][e_{[c,d]}]\pi_{-2}(f), \pi_{-2}(g)\right\rangle_{-2}. \end{aligned}$$

Then (d) follows. □

We introduce notations; $\Gamma([e_{[a,b]}]) \equiv E_{[a,b]}, \Gamma([\widetilde{j}_t]) \equiv J_t, \Gamma([e_t]) \equiv E_t$.

Proposition 2.3 ([9,Theorem III.5])

(a) J_t is a linear isometry from $\mathcal{F}$ to $\mathcal{E}$.

(b) $J_t J_t^* = E_t$.

(c) $J_t^* J_s = e^{-|t-s|\mathbf{H}_0}$.

(d) Let $\Sigma_{[a,b]}$ denote the σ-algebra generated by $\mathcal{L}\left\{\phi_{\mathcal{E}}(\pi_{-2}(f)) \,\middle|\, \pi_{-2}(f) \in \overline{U_{[a,b]}}\right\}$ and the set of $\Sigma_{[a,b]}$-measurable functions in $\mathcal{E}$ by $\mathcal{E}_{[a,b]}$. Then $Ran\left(E_{[a,b]}\right) = \mathcal{E}_{[a,b]}$.

(e) (Markoff property) Let $a \le b \le t \le c \le d$. Then

$$E_{[a,b]}E_t E_{[c,d]} = E_{[a,b]}E_{[c,d]}.$$

Proof: Eqs.(a),(b),(c) and (e) follow from Proposition 2.2. We shall show (d). Let $\{e_n\}_{n\ge 1}$ be a complete orthonormal system in $\overline{U_{[a,b]}}$. Then any vectors $\Psi \in Ran(E_{[a,b]})$ can be given by the strong limit of finite linear sum of vectors : $\phi_{\mathcal{E}}(e_1)^{n_1}...\phi_{\mathcal{E}}(e_k)^{n_k}$:. Then

$Ran(E_{[a,b]}) \subset \mathcal{E}_{[a,b]}$. On the other hand, for $f \in \overline{U_{[a,b]}}$, we have $\exp(i\phi_{\mathcal{E}}(f)) \in Ran(E_{[a,b]})$. Since for $F \in S(\mathbb{R}^n)$ (the set of rapidly decreasing infinitely differentiable functions),

$$F(\phi_{\mathcal{E}}(f_1), ..., \phi_{\mathcal{E}}(f_n)) = (2\pi)^{-n/2} \int \widehat{F}(t_1, ..., t_n) e^{i\phi_{\mathcal{E}}\left(\sum_{j=1}^{n} t_j f_j\right)} dt_1 ... dt_n,$$

we see that $F(\phi_{\mathcal{E}}(f_1), ..., \phi_{\mathcal{E}}(f_n)) \in Ran(E_{[a,b]}), f_1, ..., f_n \in \overline{U_{[a,b]}}$. By virtue of the fact that the following subset is dense in $\mathcal{E}_{[a,b]}$ ([9.section I]);

$$\left\{F(\phi_{\mathcal{E}}(f_1), ..., \phi_{\mathcal{E}}(f_n)) | f_1, ..., f_n \in \overline{U_{[a,b]}}, F \in S(\mathbb{R}^n), n \geq 1\right\},$$

we have $\mathcal{E}_{[a,b]} \subset Ran(E_{[a,b]})$. The proof is complete. □

Proposition 2.4 ([9,Theorem III.6], FKN formula) *Let $f_1, ..., f_n \in \widetilde{\mathcal{H}}_{-1}$ and $G_0, ..., G_k$ be bounded measurable functions on $\mathbb{R}^d$. Let $t_1, ..., t_n \geq 0$ be given. Then*

$$\left\langle \Omega_{\mathcal{F}}, G_0^{\mathcal{F}} e^{-t_1 \mathbf{H}_0} G_1^{\mathcal{F}} ... e^{-t_n \mathbf{H}_0} G_n^{\mathcal{F}} \Omega_{\mathcal{F}} \right\rangle_{\mathcal{F}} = \langle \Omega_{\mathcal{E}}, G_0^{s_0} ... G_n^{s_n} \Omega_{\mathcal{E}} \rangle_{\mathcal{E}}, \tag{2. 4}$$

where $\Omega_{\mathcal{F}}$ and $\Omega_{\mathcal{E}}$ are the function 1 in $\mathcal{F}$ and $\mathcal{E}$ respectively, and s_0 is arbitrary and

$$\begin{aligned} s_j &= s_0 + \sum_{i=1}^{j} t_i, \quad j = 1, ..., n, \\ G_j^{\mathcal{F}} &= G_j\left(\phi_{\mathcal{F}}(\pi_{-1}(f_1)), ..., \phi_{\mathcal{F}}(\pi_{-1}(f_n))\right), \\ G_j^{s_j} &= G_j\left(\phi_{\mathcal{E}}(\pi_{-2}(\widetilde{j}_{s_j} f_1)), ..., \phi_{\mathcal{E}}(\pi_{-2}(\widetilde{j}_{s_j} f_n))\right). \end{aligned}$$

Proof: From Proposition 2.3 it follows that

$$the\ l.h.s.of\ (2.4) = \left\langle \Omega_{\mathcal{E}}, J_{s_0} G_0^{\mathcal{F}} J_{s_0}^* J_{s_1} G_1^{\mathcal{F}} J_{s_1}^* ... J_{s_n} G_n^{\mathcal{F}} J_{s_n}^* \Omega_{\mathcal{E}} \right\rangle.$$

It can be easily seen that as operators in $\mathcal{E}$ $J_t e^{i\phi_{\mathcal{F}}(\pi_{-1}(f))} J_t^* = E_t e^{i\phi_{\mathcal{E}}(\pi_{-2}(\delta_t \otimes f))} E_t^*$. Since for $G \in S(\mathbb{R}^d)$,

$$G^{\mathcal{F}}(\phi_{\mathcal{E}}(f_1), ..., \phi_{\mathcal{E}}(f_d)) = (2\pi)^{-n/2} \int \widehat{G}(t_1, ..., t_d) e^{i\phi_{\mathcal{E}}\left(\pi_{-2}(\sum_{j=1}^{d} t_j f_j)\right)} dt_1 ... dt_d,$$

we have $J_s G^{\mathcal{F}} J_s^* = E_s G^{s_j} E_s$. Then it follows that

$$the\ l.h.s.of\ (2.4) = \langle \Omega_{\mathcal{E}}, E_{s_0} G_0^{s_0} E_{s_0} E_{s_1} G_1^{s_1} E_{s_1} ... E_{s_n} G_n^{s_n} E_{s_n} \Omega_{\mathcal{E}} \rangle.$$

Since $E_{s_0} \Omega_{\mathcal{E}} = \Omega_{\mathcal{E}}$, we have

$$= \left\langle \overline{G_0^{s_0}} \Omega_{\mathcal{E}}, E_{s_0} E_{s_1} G_1^{s_1} E_{s_1} ... E_{s_n} G_n^{s_n} E_{s_n} \Omega_{\mathcal{E}} \right\rangle.$$

Since $\overline{G_0^{s_0}}\Omega_{\mathcal{E}} \in Ran(E_{s_0})$ and $G_1^{s_1}E_{s_1}E_{s_2}G_2^{s_2}...E_{s_n}G_n^{s_n}\Omega_{\mathcal{E}} \in Ran(E_{s_1})$, we have

$$\begin{aligned} &= \left\langle \overline{G_0^{s_0}}\Omega_{\mathcal{E}}, G_1^{s_1}E_{s_1}E_{s_2}G_2^{s_2}E_{s_2}...E_{s_n}G_n^{s_n}E_{s_n}\Omega_{\mathcal{E}} \right\rangle, \\ &= \left\langle \overline{G_1^{s_1}G_0^{s_0}}\Omega_{\mathcal{E}}, E_{s_1}E_{s_2}G_2^{s_2}E_{s_2}...E_{s_n}G_n^{s_n}E_{s_n}\Omega_{\mathcal{E}} \right\rangle. \end{aligned}$$

Since $\overline{G_1^{s_1}G_0^{s_0}}\Omega_{\mathcal{E}} \in Ran(E_{[s_0,s_1]})$, and $G_2^{s_2}E_{s_2}...E_{s_n}G_n^{s_n}E_{s_n}\Omega_{\mathcal{E}} \in Ran(E_{s_2})$, by Proposition 2.3, we have

$$\begin{aligned} &= \left\langle \overline{G_1^{s_1}G_0^{s_0}}\Omega_{\mathcal{E}}, G_2^{s_2}E_{s_2}...E_{s_n}G_n^{s_n}E_{s_n}\Omega_{\mathcal{E}} \right\rangle, \\ &= \left\langle \overline{G_2^{s_2}G_1^{s_1}G_0^{s_0}}\Omega_{\mathcal{E}}, E_{s_2}E_{s_3}G_3^{s_3}E_{s_3}...E_{s_n}G_n^{s_n}E_{s_n}\Omega_{\mathcal{E}} \right\rangle. \end{aligned}$$

Repeating this procedure we have

$$= \left\langle \overline{G_n^{s_n}G_{n-1}^{s_{n-1}}...G_0^{s_0}}\Omega_{\mathcal{E}}, \Omega_{\mathcal{E}} \right\rangle.$$

By a limiting argument, the proof is complete. □

In scalar field theory [9], the range of the projection $e_{[a,b]}$ (notations follow [9]) can be characterized by some support properties ,i.e.

$$Ran(e_{[a,b]}) = \{f \in N | \text{supp} f \subset (a,b) \times \mathbb{R}^d\}^-.$$

However corresponding projection $[e_{[a,b]}]$ can be characterized in such a way. (see [4]).

3 FUNCTIONAL INTEGRALS

For each $x, y \in \mathbb{R}^d$ and an $\mathcal{H}_{-1}$-valued function ρ, we can define a unitary operator on $\mathcal{F}$ by

$$U_\rho(x,y) \equiv \exp\left\{\frac{1}{2}i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^d \pi_{-1}\left(\tilde{\rho}_\mu(x)+\tilde{\rho}_\mu(y)\right)(x_\mu - y_\mu)\right)\right\}.$$

Then we define a family of contractive self-adjoint operators $\{Q_{\rho,s}\}_{s\geq 0}$ on $\mathcal{M}$ by

$$\begin{aligned} \left(Q_{\rho,s}F\right)(x) &= \int_{\mathbb{R}^d} p_s(x-y)U_\rho(x,y)F(y)dy, \quad s>0, \\ \left(Q_{\rho,0}F\right)(x) &= F(x), \end{aligned}$$

where $F(\cdot) \in \mathcal{M}$, the integral is the $\mathcal{F}$-valued Bochner integral and $p_s(x)$ the d-dimensional heat kernel. Let

$$[C_b^n(\mathbb{R}^d;\mathcal{H}_j)] = \left\{\rho(\cdot) : \mathbb{R}^d \to \mathcal{H}_j \middle| \pi_j\left(\tilde{\rho}_\mu(\cdot)\right) \in C_b^n(\mathbb{R}^d;[\widetilde{\mathcal{H}}_j]), \mu = 1,...,d\right\}, j=-1,-2,$$

where $C_b^n(\mathbb{R}^d;\mathcal{K})$ denotes the set of $\mathcal{K}$-valued n-times strongly continuously differentiable together with bounded functions up to n .

Definition 3.1 *For $\rho \in [C_b^1(\mathbb{R}^d;\mathcal{H}_{-1})]$, we say that $F \in \mathcal{M}_\rho^\infty$ if and only if the following (i)-(iii) hold*
(i) $F(\cdot) \in C^2(\mathbb{R}^d;\mathcal{F})$ such that $||\partial^k F(\cdot)||_{\mathcal{F}} \in L^2(\mathbb{R}^d), \quad |k| \leq 2.$
(ii) For each $y \in \mathbb{R}^d$,

$$F(y) \in \mathcal{F}^\infty, \quad \partial_\mu F(y) \in \mathcal{F}^\infty, \quad \mu = 1, ..., d.$$

(iii) (Integration by parts condition) For all $G \in \mathcal{M}$, $x \in \mathbb{R}^d$,

$$\lim_{y\to\infty} \partial_{y_\mu} p_s(x-y) \cdot \langle F(y), U_\rho(x,y)G(x)\rangle_{\mathcal{F}} = 0,$$
$$\lim_{y\to\infty} p_s(x-y) \cdot \partial_{y_\mu} \langle F(y), U_\rho(x,y)G(x)\rangle_{\mathcal{F}} = 0, \qquad \mu = 1, ..., d.$$

Note that $C_0^\infty(\mathbb{R}^d)\widehat{\otimes}\mathcal{F}^\infty \subset \mathcal{M}_\rho^\infty$, where $\widehat{\otimes}$ denotes the algebraic tensor product.

Lemma 3.2 ([4,Lemmas 4.4 and 4.5]) *Let $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$, $F \in \mathcal{M}_\rho^\infty$, and $G \in \mathcal{M}$. Then $\langle Q_{\rho,s}F, G\rangle_{\mathcal{M}}$ is the right side differentiable at $s = 0$ with*

$$\frac{d}{ds}\langle Q_{\rho,s}F, G\rangle_{\mathcal{M}} |_{s=0+} = -\langle \mathbf{H}_{\rho,0}F, G\rangle_{\mathcal{M}}. \tag{3. 1}$$

Let (Ω, Db) be a probability space for the d-dimensional Brownian motion $b(t) = (b_\mu(t))_{1\leq\mu\leq d, t\geq 0}$ and $d\mu$ be the Wiener measure on $\mathbb{R}^d \times \Omega$ defined by $d\mu = dx \otimes Db$. In what follows, for simplicity, we put $n* = 2^n$.

Lemma 3.3 *Let $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$. Then, for all $t \geq 0$, the strong limit*

$$s - \lim_{n\to\infty} Q^{n*}_{\rho,\frac{t}{n*}} \equiv G_\rho(t)$$

exists. Moreover, $G_\rho(t)$ has the following functional integral representation for $F, H \in \mathcal{M}$

$$\langle F, G_\rho(t)H\rangle_{\mathcal{M}} = \int_{\mathbb{R}^d\times\Omega} d\mu \int_{Q_{-1}} d\mu_{-1} e^{i\phi_{\mathcal{F}}\left(\pi_{-1}^\rho(t,x)\right)} \overline{F}(b(t)+x)H(x) \tag{3. 2}$$

$$\pi_{-1}^\rho(t,x) = \sum_{\mu=1}^d \left(\int_0^t \pi_{-1}\left(\tilde{\rho}_\mu(b(s)+x)\right) db_\mu + \frac{1}{2}\int_0^t \partial_\mu \pi_{-1}\left(\tilde{\rho}_\mu(b(s)+x)\right) ds \right).$$

Proof: We see that

$$\left\|Q^{n*}_{\rho,\frac{t}{n*}}F - Q^{m*}_{\rho,\frac{t}{m*}}F\right\|^2_{\mathcal{M}} = \left\langle F, Q^{2n*}_{\rho,\frac{t}{n*}}F\right\rangle_{\mathcal{M}} + \left\langle F, Q^{2m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}} - 2\Re\left\langle F, Q^{n*}_{\rho,\frac{t}{n*}}Q^{m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}}. \tag{3. 3}$$

From the definition of $Q_{\rho,t}$ it follows that

$$\left\langle F, Q^{n*}_{\rho,\frac{t}{n*}}Q^{m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}} = \int_{\mathbb{R}^d} dx \left\langle F(b(2t)+x), e^{i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^d \pi_{-1}(\square_{\mu,m,n}(x))\right)}F(x)\right\rangle_{L^2(\Omega;\mathcal{F})},$$

where

$$\begin{aligned}\square_{\mu,m,n}(x) &= \sum_{k=1}^{m*}\left\{\widetilde{\rho}_\mu\left(b\left(\frac{t}{m*}k\right)+x\right)+\widetilde{\rho}_\mu\left(b\left(\frac{t}{m*}(k-1)\right)+x\right)\right\} \\ &\qquad\times\left\{b_\mu\left(\frac{t}{m*}k\right)-b_\mu\left(\frac{t}{m*}(k-1)\right)\right\} \\ &+ \sum_{k=1}^{n*}\left\{\widetilde{\rho}_\mu\left(b\left(\frac{t}{n*}k+t\right)+x\right)+\widetilde{\rho}_\mu\left(b\left(\frac{t}{n*}(k-1)+t\right)+x\right)\right\} \\ &\qquad\times\left\{b_\mu\left(\frac{t}{n*}k+t\right)-b_\mu\left(\frac{t}{n*}(k-1)+t\right)\right\}.\end{aligned}$$

We can see that for each $x \in \mathbb{R}^d$

$$s-\lim_{m\to\infty}\lim_{n\to\infty}\sum_{\mu=1}^d \pi_{-1}(\square_{\mu,m,n}(x)) = \pi^\rho_{-1}(2t,x)$$

in $L^2(\Omega;[\widetilde{\mathcal{H}}_{-1}])$. By the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\left\langle F, Q^{n*}_{\rho,\frac{t}{n*}}Q^{m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}} = \int_{\mathbb{R}^d} dx\left\langle F(b(2t)+x),\, e^{i\phi_{\mathcal{F}}\left(\pi^\rho_{-1}(2t,x)\right)}F(x)\right\rangle_{L^2(\Omega;\mathcal{F})}. \tag{3. 4}$$

Similarly it can be easily seen that $\left\langle F, Q^{2n*}_{\rho,\frac{t}{n*}}F\right\rangle_{\mathcal{M}}$ and $\left\langle F, Q^{2m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}}$ converge to the r.h.s. of (3.4) as $n, m \to \infty$, respectively. Then it follows that $\{Q^{n*}_{\rho,\frac{t}{n*}}\}_{n\geq 0}$ is a Cauchy. Eq.(3.2) easily follows from (3.4). □

Lemma 3.4 *Let $\rho \in [C^2_b(\mathbb{R}^d;\mathcal{H}_{-1})]$. Then the family $\{G_\rho(t)\}_{t\geq 0}$ is a strongly continuous 1-parameter semigroup on $\mathcal{M}$.*

Proof: The group properties follow from the proof of Lemma 3.3 and the strong continuity in t a direct calculation using (3.2). □

By Lemma 3.4, Hille-Yoshida's theorem yields that for each $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$, there exists a unique nonnegative self-adjoint operator $\widetilde{\mathbf{H}}_{\rho,0}$ in $\mathcal{M}$ such that

$$G_\rho(t) = e^{-t\widetilde{\mathbf{H}}_{\rho,0}}.$$

Lemma 3.5 *Let $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$. Then the self-adjoint operator $\widetilde{\mathbf{H}}_{\rho,0}$ is a self-adjoint extension of $\mathbf{H}_{\rho,0}|_{\mathcal{M}_\rho^\infty}$.*

Proof: Let $F \in D(\widetilde{\mathbf{H}}_{\rho,0})$ and $G \in \mathcal{M}_\rho^\infty$. Then we have

$$\begin{aligned}\left\langle \frac{1}{t}\left(e^{-t\widetilde{\mathbf{H}}_{\rho,0}} - I\right)G, F\right\rangle_{\mathcal{M}} &= \lim_{n\to\infty}\sum_{j=0}^{n*-1}\frac{1}{n*}\left\langle \frac{n*}{t}\left(Q_{\rho,\frac{t}{n*}} - I\right)G, Q_{\rho,\frac{t}{n*}}^{n*\frac{j}{n*}}F\right\rangle_{\mathcal{M}} \\ &= \lim_{n\to\infty}\int_0^1 \left\langle \frac{n*}{t}\left(Q_{\rho,\frac{t}{n*}} - I\right)G, Q_{\rho,\frac{t}{n*}}^{[n*s]}F\right\rangle_{\mathcal{M}} ds.\end{aligned}$$

Because of the weak right differentiability of $Q_{\rho,t}G$ in $t = 0$ ((3.1)), and the definition of $Q_{\rho,t}$ (Lemma 3.3), we have

$$w - \lim_{n\to 0}\frac{n*}{t}\left(Q_{\frac{t}{n*}} - I\right)G = -\mathbf{H}_{\rho,0}G, \quad s - \lim_{n\to\infty} Q_{\rho,\frac{t*}{n}}^{[n*s]} = G_\rho(ts).$$

Hence

$$\left\langle \frac{1}{t}\left(e^{-t\widetilde{\mathbf{H}}_{\rho,0}} - I\right)G, F\right\rangle_{\mathcal{M}} = \int_0^1 ds\left\langle -\mathbf{H}_{\rho,0}G, e^{-ts\widetilde{\mathbf{H}}_{\rho,0}}F\right\rangle_{\mathcal{M}}. \tag{3. 5}$$

As $t \to 0$ on the both sides of (3.5), we get

$$\left\langle G, \widetilde{\mathbf{H}}_{\rho,0}F\right\rangle_{\mathcal{M}} = \langle \mathbf{H}_{\rho,0}G, F\rangle_{\mathcal{M}},$$

which implies that $G \in D(\widetilde{\mathbf{H}}_{\rho,0})$ and $\widetilde{\mathbf{H}}_{\rho,0}G = \mathbf{H}_{\rho,0}G$. □

We denote the extension $\widetilde{\mathbf{H}}_{\rho,0}$ by the same symbol $\mathbf{H}_{\rho,0}$. For $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$, we give a rigorous definition of $\mathbf{H}_\rho$ in terms of the form sum $\dot{+}$ of $\mathbf{H}_{\rho,0}$ and $I \otimes \mathbf{H}_0$;

$$\mathbf{H}_\rho \equiv \mathbf{H}_{\rho,0} \dot{+} I \otimes \mathbf{H}_0.$$

Note that $\mathcal{M}_\rho^\infty \cap D(I \otimes \mathbf{H}_0)$ is dense in $\mathcal{M}$. We introduce a multiplication operator in $L^2(\mathbb{R}^d) \otimes \mathcal{E} \cong L^2(\mathbb{R}^d;\mathcal{E})$ by

$$\phi_{\mathcal{E},\mu}^{\rho,s} \equiv \int_{\mathbb{R}^d}^{\oplus} \phi_{\mathcal{E}}\left(\pi_{-2}\left(\widetilde{j}_s\widetilde{\rho}_\mu(x)\right)\right) dx.$$

Moreover we formally define an operator acting in $L^2(\mathbb{R}^d;\mathcal{E})$ by

$$\mathbf{H}_{\rho,0,s} = \frac{1}{2}\sum_{\mu=1}^{d}\left(-iD_\mu \otimes I - \phi_{\mathcal{E},\mu}^{\rho,s}\right)^2.$$

Since for $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$, we see that $j_s\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-2})]$, one can construct a self-adjoint extension of $\mathbf{H}_{\rho,0,s}$ in the same manner as that of $\mathbf{H}_{\rho,0}$. We denote it by the same symbol $\mathbf{H}_{\rho,0,s}$. Similarly to that of $\mathbf{H}_{\rho,0}$, we define $Q_{\rho,t,s}$, contraction operators in $L^2(\mathbb{R}^d;\mathcal{E})$, corresponding to $Q_{\rho,t}$ i.e.,

$$s - \lim_{n\to\infty} Q^{n*}_{\rho,\frac{t}{n*},s} = e^{-t\mathbf{H}_{\rho,0,s}}. \qquad (3.\ 6)$$

Lemma 3.6 ([4,Lemma 4.9]) *Let $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$. Then the following equation holds on $L^2(\mathbb{R}^d;\mathcal{E})$*

$$J_s e^{-t\mathbf{H}_{\rho,0,}} J_s^* = E_s e^{-t\mathbf{H}_{\rho,0,s}} E_s.$$

Now we are ready to state main theorem in this note.

Theorem 3.7 ([4,Theorem 4.10]) *Let $F, G \in \mathcal{M}$, $V \in C_b(\mathbb{R}^d)$ and $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$ such that*

$$(1) \sup_{\mu=1,\ldots,d,x\in\mathbb{R}^d} ||[\widetilde{\omega}]\pi_{-1}(\widetilde{\rho}_\mu(x))||_{-1} < \infty, \quad (2) \left|\left|\sum_{\mu=1}^{d}\partial_\mu \pi_{-1}(\widetilde{\rho}_\mu(x))\right|\right|_{-1} = 0.$$

Then the following limit exists in $L^2(\Omega;[\widetilde{\mathcal{H}}_{-2}])$ for each $x \in \mathbb{R}^d$:

$$s - \lim_{n\to\infty}\sum_{j=0}^{n*-1}\int_{\frac{jt}{n*}}^{\frac{j+1}{n*}t} [\widetilde{j}_{\frac{jt}{n*}}]\pi_{-1}\left(\widetilde{\rho}_\mu(b(s)+x)\right) db_\mu \equiv \pi_{-2}^{\rho}(t,x).$$

Moreover

$$\left\langle F, e^{-t(\mathbf{H}_\rho+V)}G\right\rangle_{\mathcal{M}} = \int_{\mathbb{R}^d\times\Omega\times Q_{-2}} d\mu d\mu_{-2} e^{-\int_0^t V(b(s)+x)ds} e^{i\phi_{\mathcal{E}}\left(\pi_{-2}^{\rho}(t,x)\right)} J_t\overline{F}(b(t)+x)J_0G(x) \qquad (3.\ 7)$$

Proof: The existence of the strong limit follows directly from (1) (see [4,section 2]). Let $\frac{t}{n*} = s$. By the strong Trotter product formula, Markoff properties of E_j ([4,Proposition 3.3 (e)]), Lemma 3.6 and (3.6), we see that

$$\begin{aligned}
\left\langle F, e^{-t(\mathbf{H}_\rho+V)}G\right\rangle_{\mathcal{M}} &= \lim_{n\to\infty}\lim_{k\to\infty} S_{n*,k*}, \\
S_{n*,k*} &= \left\langle F, J_t^*\left(Q^{k*}_{\rho,\frac{s}{k*},t}\right)e^{-sV}\left(Q^{k*}_{\rho,\frac{s}{k*},t-s}\right)e^{-sV}\ldots\left(Q^{k*}_{\rho,\frac{s}{k*},s}\right)e^{-sV}J_0G\right\rangle_{\mathcal{M}}.
\end{aligned}$$

Then the definition of $Q_{\rho,t,t'}$ yields that

$$S_{n*,k*} = \int_{\mathbb{R}^d} dx \Big\langle \; F(b(t)+x),$$
$$J_t^* \exp\left(i\phi_{\mathcal{E}} \left(\sum_{\mu=1}^{d} \sum_{j=0}^{n*-1} [\widetilde{j}_{js}]\pi_{-1}\left(\square_{\mu,j,k}(x)\right) \right) - s\sum_{j=1}^{n*} V\left(b(js)+x\right) \right) J_0 G(x) \Big\rangle_{L^2(\Omega;\mathcal{F})},$$

where

$$\begin{aligned}\square_{\mu,j,k}(x) \quad &= \quad \sum_{m=1}^{k*} \left\{ \widetilde{\rho}_\mu \left(b\left(\frac{m}{k*}s + js\right) + x \right) + \widetilde{\rho}_\mu \left(b\left(\frac{m-1}{k*}s + js\right) + x \right) \right\} \\ &\times \left\{ b_\mu\left(\frac{m}{k*}s+js\right) - b_\mu\left(\frac{m-1}{k*}s+js\right) \right\}, \quad j = 0, ..., n*-1.\end{aligned}$$

By the Coulomb gauge condition (2), it can be easily seen that for $x \in \mathbb{R}^d$

$$s - \lim_{k\to\infty} \square_{\mu,j,k}(x) = \int_{js}^{(j+1)s} [\widetilde{j}_{js}]\pi_{-1}\left(\widetilde{\rho}_\mu(b(s')+x)\right) db_\mu \equiv \square_{\mu,j}(x)$$

in $L^2(\Omega; [\widetilde{\mathcal{H}}_{-2}])$. By the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty} S_{n*,k*} = \int_{\mathbb{R}^d} dx \Big\langle \; J_t F(b(t)+x),$$
$$\exp\left(i\phi_{\mathcal{E}} \left(\sum_{\mu=1}^{d} \sum_{j=0}^{n*-1} \square_{\mu,j}(x) \right) - s\sum_{j=1}^{n*} V\left(b(js)+x\right) \right) J_0 G(x) \Big\rangle_{L^2(\Omega;\mathcal{E})}.$$

Hence by the first statement of the Theorem and again by the Lebesgue dominated convergence theorem , we get

$$\lim_{n\to\infty} \lim_{k\to\infty} S_{n*,k*} = (3.7).$$

We call $\pi_{-2}^{\rho}(t,x)$ "time-ordered $[\widetilde{\mathcal{H}}_{-2}]$-valued stochastic integral associated with the family of isometries $[\widetilde{j}_t]$ from $[\widetilde{\mathcal{H}}_{-1}]$ to $[\widetilde{\mathcal{H}}_{-2}]$".

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