FUNCTIONAL INTEGRAL REPRESENTATIONS
OF A HEAT SEMIGROUP IN QED

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Abstract
This note presents functional integral representations for heat semigroups with infinitesimal generators given by self-adjoint Hamiltonians describing an interaction of a non-relativistic charged particle and a quantized radiation field in the Coulomb gauge without the dipole approximation.

1 INTRODUCTION

Until now many authors give functional integral representations for heat semigroups with infinitesimal generators given by self-adjoint Hamiltonians describing quantum systems. Using functional integral representations, they analyze quantum systems. In this note we concern with quantum systems which describe interactions of a non-relativistic charged particle and a quantized radiation field in the Coulomb gauge without the dipole approximation, which is the so called “Pauli-Fierz model” ([1,2,3,4,5,6]).

Mathematically, the set of state vectors $\mathcal{M}_B$ of this quantum system can be described by the tensor product of $L^2(\mathbb{R}^d)$ and the Boson Fock space $\mathcal{F}(\mathcal{W})$ over $\mathcal{W} = \bigoplus_{d-1}^{d} L^2(\mathbb{R}^d)$. Pauli-Fierz Hamiltonians $H_{\rho,B}$ are formally defined as operators acting in $\mathcal{M}$ as follows ((2.2)):

$$H_{\rho,B} = \frac{1}{2} \sum_{\mu=1}^{d} \left( -iD_{\mu} \otimes -A_{\mu}(\rho) \right)^2 + I \otimes H_{0,B},$$

where $D_{\mu}$ is the generalized derivative in the $\mu$-th direction, $H_{0,B}$ the free Hamiltonian in the Boson Fock space $\mathcal{F}(\mathcal{W})$ and $A_{\mu}(\rho)$ the time-zero radiation field in the $\mu$-th direction.
Many researches of the Pauli-Fierz model have been devoted to dealing with Hamiltonians with the dipole approximation ([1,2,3]). Without the dipole approximation, it is not known even essential self-adjointness of \( \mathbf{H}_{\rho,B} \) except for special \( \rho \)'s ([4,5,6]). Then analysis of Hamiltonians without the dipole approximation is crucial.

In this note, constructing a Hilbert space \( L^2(Q_{-1}, d\mu_{-1}) \) (see section 2), we define Hamiltonians acting in \( \mathcal{M} = L^2(\mathbb{R}^d) \otimes L^2(Q_{-1}, d\mu_{-1}) \) by the quadratic form sum of the generators \( \mathbf{H}_{\rho,0} \) of strongly continuous 1-parameter semigroups and a self-adjoint operator \( \mathbf{H}_0 \) in \( L^2(Q_{-1}, d\mu_{-1}) \) \((2.3)\). We shall show that \( \mathbf{H}_{\rho,0} + I \otimes \mathbf{H}_0 \) is unitarily equivalent to \( \mathbf{H}_{f,B} \) on some domains with some \( \rho \)'s and \( f \)'s (Theorem 2.1). Again we define Pauli-Fierz Hamiltonians by \( \mathbf{H}_\rho = \mathbf{H}_{\rho,0} + I \otimes \mathbf{H}_0 \).

The Wiener path integral method is useful to get path integral representations of heat semigroups with generators:

\[
\mathbf{H}_\text{cl} = \frac{1}{2} \sum_{\mu=1}^{d} (-iD_\mu - A_\mu)^2 + V,
\]

where \( A_\mu \) is a vector potential and \( V \) a scalar potential. It is known as the Feynman-Kac-Itô formula ([7]). In connection with construction of quantum field models from markoff fields, E.Nelson introduced a “generalized path space” (functional space). He also introduced a natural embedding of a Boson Fock space in \( d \) space dimension into a constant time subspace in the \( L^2 \) space over the “generalized path space” in \( d + 1 \) dimensions. The natural embedding gives us a functional integral representation of a heat semigroup with the free Hamiltonian in the Boson Fock space as the generator. It is called the “Feynman-Kac-Nelson formula ([8]).

The purpose of this note is to give functional integral representations to the expectation \( \langle F, e^{-t\mathbf{H}_\rho} G \rangle_{\mathcal{M}} \). In order to do so, we shall use the FKI and the FKN formulas simultaneously. And we shall need to define the “time-ordered Hilbert space-valued stochastic integral associated with a family of isometry \( \{\mathbf{J}_t\}_{t \in \mathbb{R}^d} \) from a Hilbert space \( \mathcal{H}_{-1} \) to another one \( \mathcal{H}_{-2} \)” (see Theorem 3.7).

## 2 PAULI-FIERZ HAMILTONIAN

In this section, we define Pauli-Fierz Hamiltonians and give their probabilistic descriptions. For mathematical generality, we consider the situation where a charged particle
moves in $\mathbb{R}^d$ and a quantized radiation field is over $\mathbb{R}^d$. We define polarization vectors $e^r(r = 1, \ldots, d - 1)$ as measurable functions $e^r : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$e^r(k) \cdot e^s(k) = \delta_{rs}, \quad k \cdot e^r(k) = 0, \quad a.e. k \in \mathbb{R}^d.$$ 

Put

$$d_{\mu\nu}(k) \equiv \sum_{r=1}^{d-1} e^r_{\mu}(k)e^r_{\nu}(k) = \delta_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{|k|^2}.$$ 

The Boson Fock space $\mathcal{F}(\mathcal{W})$ over $\mathcal{W} = \bigoplus_{d-1} L^2(\mathbb{R}^d)$ is defined by

$$\mathcal{F}(\mathcal{W}) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{W}), \quad \mathcal{F}_n(\mathcal{W}) = \otimes^n \mathcal{W}, \quad n \geq 1, \quad \mathcal{F}_0 = \mathbb{C},$$

where $\otimes^n$ denotes the $n$-fold symmetric tensor product. Put $\Omega = \{1, 0, 0, \ldots\}$. Let

$$\mathcal{F}^N(\mathcal{W}) = \bigoplus_{n=0}^{N} \mathcal{F}_n(\mathcal{W}) \bigoplus \{0\}, \quad \mathcal{F}^\infty(\mathcal{W}) = \bigcup_{N=0}^{\infty} \mathcal{F}^N(\mathcal{W}).$$

The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ ($f \in \mathcal{W}$) act on $\mathcal{F}^\infty(\mathcal{W})$ and leave it invariant with the canonical commutation relations (CCR): for $f, g \in \mathcal{W}$

$$[a(f), a^\dagger(g)] = \langle f, g \rangle_{\mathcal{W}}, \quad [a^\dagger(f), a^\dagger(g)] = 0,$$

where $[A, B] = AB - BA$, $a^2$ denotes either $a$ or $a^\dagger$. It is well known that

$$\mathcal{F}(\mathcal{W}) = \mathcal{L}\{a^\dagger(f_1)\ldots a^\dagger(f_n)\Omega, \Omega | f_j \in \mathcal{W}, j = 1, \ldots, n, n \geq 1\}.$$ 

We recall here second quantizations of operators. For any contraction operator $T$ on $\mathcal{W}$, the “second quantization of $T$”, $\Gamma_B(T) : \mathcal{F}(\mathcal{W}) \to \mathcal{F}(\mathcal{W})$, is a contraction operator uniquely determined by

$$\Gamma_B(T)\Omega = 0, \quad \Gamma_B(T)a^\dagger(f_1)a^\dagger(f_2)\ldots a^\dagger(f_n)\Omega = a^\dagger(T f_1)a^\dagger(T f_2)\ldots a^\dagger(T f_n)\Omega, \quad n \geq 1.$$ 

For a nonnegative self-adjoint operator $A$ in $\mathcal{W}$, the “second quantization of $A$”, $d\Gamma_B(A)$, is defined by the infinitesimal generator of the strongly continuous 1-parameter semigroup

$$\Gamma_B(e^{-tA}) = e^{-td\Gamma_B(A)}.$$
We define a maximal multiplication operator \( \omega_B \) in \( L^2(\mathbb{R}^d) \) by
\[
(\omega_B f)(k) = h(k)f(k),
\]
where \( h(k) = |k| \). Put \( \hat{\omega}_B = \omega_B \oplus \ldots \oplus \omega_B \). Then \( H_{0,B} = d\Gamma_B(\hat{\omega}_B) \) shall be the free Hamiltonian in \( \mathcal{F}(\mathcal{W}) \). We define the \( \mu \)-th direction time-zero radiation field by
\[
A_\mu(x,f) = \frac{1}{\sqrt{2}} \left\{ a^+ \left( \bigoplus_{r=1}^{d-1} \frac{e^{i\mu \hat{f} \cdot e^{-ik}}}{\sqrt{\hbar}} \right) + a \left( \bigoplus_{r=1}^{d-1} \frac{e^{i\mu \hat{f} \cdot e^{ik}}}{\sqrt{\hbar}} \right) \right\}, \mu = 1, \ldots, d,
\]
where \( \hat{f} \) is the Fourier transformation of \( f \) (the inverse Fourier transformation of \( f \) in what follows) and \( \hat{g}(k) = g(-k) \). A Hilbert space of state vectors in a system of the non-relativistic charged particle interacting with the quantized radiation field is given by \( \mathcal{M}_B = L^2(\mathbb{R}^d) \otimes \mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^{d+1}; \mathcal{F}(\mathcal{W})) \). We shall use this identification without notice. Then interaction Hamiltonians (Pauli-Fierz Hamiltonians) of the non-relativistic charged particle with mass one and the quantized radiation field is “formally” defined as an operator acting in \( \mathcal{M}_B \) by
\[
H_\rho = \frac{1}{2} \sum_{\mu=1}^{d} (-iD_\mu \otimes I - A_\mu(\rho))^2 + I \otimes H_{0,B},
\]
where we take the natural unit \( c = \hbar = 1 \) and
\[
A_\mu(\rho) = \int_{\mathbb{R}^d} A_\mu(x,\rho)dx.
\]
Generally, it is crucial whether Hamiltonians defined on some domains have unique self-adjoint extensions, since the unique extensions lead to the uniqueness of time evolutions of state vectors in quantum systems. Nevertheless it is not known whether the formally defined Hamiltonians \( H_\rho \) restricted to some concrete domains have unique self-adjoint extensions. Then we must construct self-adjoint extensions of \( H_\rho \) in some way.

We have to give probabilistic descriptions to the Hamiltonian \( H_\rho \). First we define two real Hilbert spaces \( \mathcal{H}_{-1} \) and \( \mathcal{H}_{-2} \) by
\[
\mathcal{H}_{-1} \equiv \left\{ f \in S'_d(\mathbb{R}^d) \left| \int_{\mathbb{R}^d} \frac{|\hat{f}(k)|^2}{|k|^2}dk < \infty \right. \right\}, \mathcal{H}_{-2} \equiv \left\{ f \in S'_d(\mathbb{R}^{d+1}) \left| \int_{\mathbb{R}^{d+1}} \frac{|\hat{f}(k)|^2}{|k|^2}dk < \infty \right. \right\},
\]
where \( S'_d(\mathbb{R}^n) \) denotes the set of the real tempered distributions on \( \mathbb{R}^n(n = d, d + 1) \). Put
\[
\mathcal{H}_{-1} = \bigoplus_{d}^{d} \mathcal{H}_{-1}, \quad \mathcal{H}_{-2} = \bigoplus_{d}^{d} \mathcal{H}_{-2}.
\]
We introduce bilinear forms $(\cdot,\cdot)_{-1}$ and $(\cdot,\cdot)_{-2}$ in $\mathcal{H}_{-1}$ and $\mathcal{H}_{-2}$ by

$$(f,g)_{-1} = \frac{d}{dk} \int_{\mathbb{R}^d} \frac{d\mu(k)}{|k|} \bar{f}(k) \bar{\tilde{g}}(k) dk, \quad (f,g)_{-2} = \frac{d}{dk} \int_{\mathbb{R}^{d+1}} \frac{d\mu(k)}{|k|^2} \bar{f}(k) \bar{\tilde{g}}(k) dk.$$  

We denote the associated semi-norms by $| \cdot |_{-1}$ and $| \cdot |_{-2}$, respectively and put

$$N_{-1} = \{f \in \mathcal{H}_{-1} | |f|_{-1} = 0 \}, \quad N_{-2} = \{f \in \mathcal{H}_{-2} | |f|_{-2} = 0 \}.$$  

Then we define pre-Hilbert spaces by the quotient spaces

$$[\mathcal{H}_{-1}] = \mathcal{H}_{-1}/N_{-1}, \quad [\mathcal{H}_{-2}] = \mathcal{H}_{-2}/N_{-2},$$

with inner products $\langle \cdot, \cdot \rangle_{-1}$ and $\langle \cdot, \cdot \rangle_{-2}$ defined by

$$\langle \pi_{-1}(f), \pi_{-1}(g) \rangle_{-1} \equiv (f,g)_{-1}, \quad \langle \pi_{-2}(f), \pi_{-2}(g) \rangle_{-2} \equiv (f,g)_{-2}.$$  

Here $\pi_{-1}(f)$ and $\pi_{-2}(f)$ denote the equivalence classes of $f$ in $\mathcal{H}_{-1}$ and $\mathcal{H}_{-2}$, respectively. We denote the norms associated with the inner products $\langle \cdot, \cdot \rangle_{-1}$ and $\langle \cdot, \cdot \rangle_{-2}$ by $| | \cdot | |_{-1}$ and $| | \cdot | |_{-2}$, respectively. The Hilbert spaces constructed by the completions of $[\mathcal{H}_{-1}]$ and $[\mathcal{H}_{-2}]$ with respect to $| | \cdot | |_{-1}$ and $| | \cdot | |_{-2}$ are denoted by the same symbols.

Let $\{\phi_{-1}(\pi_{-1}(f)) | f \in \mathcal{H}_{-1}\}$ and $\{\phi_{-2}(\pi_{-2}(f)) | f \in \mathcal{H}_{-2}\}$ be the Gaussian random processes indexed by the Hilbert spaces $[\mathcal{H}_{-1}]$ and $[\mathcal{H}_{-2}]$ such that the characteristic functions are given by

$$\int_{Q_j} e^{i\phi_{-j}(\pi_{j}(f))} d\mu_j = e^{-\frac{1}{4} | |\pi_{j}(f)| |^2}, \quad j = -1, -2,$$

where $(Q_j, d\mu_j), j = -1, -2$ denote the underlying measure spaces of these processes. It is well known that $L^2(Q_j, d\mu_j)$ has the orthogonal decomposition

$$L^2(Q_j, d\mu_j) = \bigoplus_{n=0}^{\infty} \Gamma_n([\mathcal{H}_j]), \quad j = -1, -2,$$

with

$$\Gamma_0([\mathcal{H}_j]) = \mathcal{C},$$

$$\Gamma_n([\mathcal{H}_j]) = \mathcal{L}\{ : \phi_j(\pi_j(f_1))\phi_j(\pi_j(f_2))\ldots\phi_j(\pi_j(f_n)) : |f_k | \in \mathcal{H}_j, k = 1, \ldots, n \}^- \quad n \geq 1.$$  

Here $:\ldots:$ means the Wick product and $\mathcal{L}$ the linear span of the vectors in $\{\ldots\}$ over $\mathcal{C}$. We denote the complexifications of $[\mathcal{H}_j]$ by $[\mathcal{H}_j]_\mathbb{C}$. Suppose that $T$ is a contraction operator
from $[\mathcal{H}_i]_c$ to $[\mathcal{H}_j]_c$. Corresponding to each such $T$ we can define a contraction operator $\Gamma(T) : L^2(Q_i; d\mu_i) \rightarrow L^2(Q_j; d\mu_j)$ by

$$\Gamma(T) \Omega_i = 0,$$

$$\Gamma(T) : \phi_i(\pi_i(f_1))...\phi_i(\pi_i(f_n)) : = : \phi_j(T\pi_j(f_1))\phi_j(T\pi_j(f_2))...\phi_j(T\pi_j(f_n)) :,$$

where $\Omega_i$ denotes the constant function 1 in $L^2(Q_i, d\mu_i)$. For a nonnegative self-adjoint operator $A : [\mathcal{H}_i]_c \rightarrow [\mathcal{H}_i]_c$ ($i = -1, -2$) we define $d\Gamma(A)$ by

$$d\Gamma(A)\Omega_i = 0,$$

$$d\Gamma(A) : \phi_i(\pi_i(f_1))...\phi_i(\pi_i(f_n)) :$$

$$= : \phi_i(A\pi_i(f_1))\phi_i(\pi_i(f_2))...\phi_i(\pi_i(f_n)) : + : \phi_i(\pi_i(f_1))\phi_i(\pi_i(f_2))...\phi_i(\pi_i(f_n)) :$$

$$+...+ : \phi_i(\pi_i(f_1))\phi_i(\pi_i(f_2))...\phi_i(A\pi_i(f_n)) :,$$

where $\pi_i(\mu) \in D(A)$, $k = 1, ..., n$.

It is well known that $d\Gamma(A)$ has the unique self-adjoint extension in $L^2(Q_i; d\mu_i)$. We denote it by the same symbol $d\Gamma(A)$. We define an operator $\omega$ in $\mathcal{H}_{-1}$ by

$$\omega f(k) = h(k)\hat{f}(k),$$

and put $\tilde{\omega} = \omega \oplus ... \oplus \omega$. Furthermore, $[\tilde{\omega}] : [\mathcal{H}_{-1}] \rightarrow [\mathcal{H}_{-1}]$ is defined by

$$[\tilde{\omega}]\pi_{-1}(f) = \pi_{-1}(\tilde{\omega}f), \quad D([\tilde{\omega}]) = \{\pi_{-1}(f) \in [\mathcal{H}_{-1}]|\tilde{\omega}f \in \mathcal{H}_{-1}\}.$$ 

Set $d\Gamma([\tilde{\omega}]) = H_0$, $L^2(Q_{-1}, d\mu_{-1}) = \mathcal{F}$, $L^2(Q_{-2}, d\mu_{-2}) = \mathcal{E}$, $\phi_{-1}(\cdot) = \phi_{\mathcal{F}}(\cdot)$ and $\phi_{-2}(\cdot) = \phi_{\mathcal{E}}(\cdot)$. Similarly to the Boson Fock space $\mathcal{F}(\mathcal{W})$, we put

$$\mathcal{F}^N = \bigoplus_{n=0}^N \Gamma_n([\mathcal{H}_{-1}]) \bigoplus_{n>N+1} \{0\}, \quad \mathcal{F}^\infty = \bigcup_{N=0}^\infty \mathcal{F}^N.$$ 

For an $\mathcal{H}_{-1}$-valued function on $\mathbb{R}^d$, $\rho(\cdot) : \mathbb{R}^d \rightarrow \mathcal{H}_{-1}$, we put $\tilde{\rho}_\mu(\cdot) = (0, ..., \rho_{\mu}(\cdot), ..., 0)$.

Then we define an operator in $\mathcal{M} = L^2(\mathbb{R}^d) \otimes \mathcal{F} \cong L^2(\mathbb{R}^d; \mathcal{F})$ by

$$\phi_{\mathcal{F}, \mu} = \int_{\mathbb{R}^d} (\pi_{-1} (\tilde{\rho}_\mu(x))) dx.$$ 

Define operators in $\mathcal{M}$ by

$$H_\rho = \frac{1}{2} \sum_{\mu=1}^d \left(-i D_\mu \otimes I - \phi_{\mathcal{F}, \mu}^* \right)^2 + I \otimes H_0 \equiv H_{\rho, 0} + I \otimes H_0. \quad (2. 3)$$

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Theorem 2.1 ([4, Theorem 3.1]) Set $D_B = C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}^\infty(W) \cap D(I \otimes H_{0,B})$ and $D = C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}^\infty \cap D(I \otimes H_0)$. Let $\rho = (\bar{f}(\cdot)e^{ix})^\vee$, $f \in \mathcal{H}_{-1}$. Then there exists a unitary operator $U$ from $\mathcal{F}(W)$ to $\mathcal{F}$ such that $U$ maps $D_B$ onto $D$ and

$$U^{-1}H_\rho U\big|_D = H_{f,B}\big|_D.$$ 

By Theorem 2.1, we call $H_\rho$ the Pauli-Fierz Hamiltonian again. We can give connection between $\mathcal{F}$ and $\mathcal{E}$. For $t \in \mathbb{R}$ we define an operator $j_t$ by

$$j_t : \mathcal{H}_{-1} \to \mathcal{H}_{-2}, \quad j_t f = \delta_t \otimes f,$$

where $\delta_t$ is the one-dimensional delta function with mass at $\{t\}$. We put $\tilde{j}_t = j_t \oplus \cdots \oplus j_t$ and define

$$\tilde{j}_t \pi_-(f) = \pi_-(\tilde{j}_t f).$$

It can be easily seen that $\tilde{j}_t$ is a linear isometry. Hence the range of $\tilde{j}_t$ is a closed subspace of $[\tilde{\mathcal{H}}_{-2}]$. We denote the projection onto $\text{Ran}(\tilde{j}_t)$ by $[e_t]$. We denote the projection onto $\text{Ran}(\tilde{j}_t)$ by $[e_t]$. Let

$$U_{[a,b]} \equiv L \left\{ \pi_-(f) \in [\tilde{\mathcal{H}}_{-2}] \big| \pi_-(f) \in \text{Ran}(\tilde{j}_t), a \leq t \leq b \right\}.$$ 

We denote the projection onto the closure $\overline{U_{[a,b]}}$ by $[e_{[a,b]}].$

Proposition 2.2 ([9, Propositions III.3 and III.4])

(a) $[\tilde{j}_t]\tilde{j}_t^* = [e_t].$

(b) $[\tilde{j}_t]^*[\tilde{j}_s] = e^{-|t-s|\tilde{\omega}}.$

(c) Let $a \leq b \leq c$. Then $[e_a][e_b][e_c] = [e_a][e_c].$

(d) Let $a \leq b \leq t \leq c \leq d$. Then $[e_{[a,b]}][e_t][e_{[c,d]}] = [e_{[a,b]}][e_{[c,d]}].$

Proof: (a) is straightforwardly seen. Since we have

$$\left\langle [\tilde{j}_t]^*[\tilde{j}_s]\pi_-(f), \pi_-(g) \right\rangle_{-2} = \left\langle \pi_-(\tilde{j}_s f), \pi_-(\tilde{j}_t g) \right\rangle_{-2} = \frac{1}{\pi} \sum_{\mu,\nu=1}^d \int_{\mathbb{R}^{d+1}} \tilde{f}_\mu(\vec{k})\tilde{g}_\nu(\vec{k})d_{\mu\nu}(\vec{k})e^{i(t-s)k_0} d\vec{k}dk_0$$

$$= \frac{d}{\pi} \sum_{\mu,\nu=1}^d \int_{\mathbb{R}^d} \tilde{f}_\mu(\vec{k})\tilde{g}_\nu(\vec{k})d_{\mu\nu}(\vec{k})e^{-|t-s|\vec{k}} d\vec{k},$$
the statement (b) holds. Since
\[ [e_a][e_b][e_c] = [\hat{e}_a][\hat{e}_b][\hat{e}_c]^* = [\hat{e}_a]e^{-(c-a)[\hat{\omega}][\hat{e}_c]^*} = [e_a][e_c], \]
the statement (c) follows. For any \( \pi_2(f) \) and \( \pi_2(g) \), by the definition of \([e_{[a,b]}]\) and \([e_{[c,d]}]\), they can be presented as follows
\[
[e_{[c,d]}]\pi_2(f) = \lim_{n \to \infty} \sum_{\alpha=1}^{N_n} f_{n_\alpha}, \quad f_{n_\alpha} \in \text{Ran}([e_{t_n}]), t_n \in [c,d],
\]
\[
[e_{[a,b]}]\pi_2(g) = \lim_{m \to \infty} \sum_{\beta=1}^{M_m} g_{m_\beta}, \quad g_{m_\beta} \in \text{vRan}([e_{t_m}]), t_m \in [a,b].
\]
Hence by (c) we have
\[
\langle [e_{[a,b]}][e_{t}][e_{[c,d]}]\pi_2(f), \pi_2(g) \rangle = \lim_{n,m \to \infty} \sum_{\alpha,\beta=1}^{N_n,M_m} \langle f_{n_\alpha}, g_{m_\beta} \rangle_{-2}
\]
\[
= \lim_{n,m \to \infty} \sum_{\alpha,\beta=1}^{N_n,M_m} \langle f_{n_\alpha}, g_{m_\beta} \rangle_{-2}
\]
\[
= \langle [e_{[a,b]}][e_{[c,d]}]\pi_2(f), \pi_2(g) \rangle_{-2}.
\]
Then (d) follows. \( \square \)

We introduce notations; \( \Gamma([e_{[a,b]}]) \equiv E_{[a,b]}, \Gamma([\hat{e}_t]) \equiv J_t, \Gamma([e_t]) \equiv E_t. \)

**Proposition 2.3 ([9,Theorem III.5])**

(a) \( J_t \) is a linear isometry from \( \mathcal{F} \) to \( \mathcal{E} \).

(b) \( J_t J_t^* = E_t \).

(c) \( J_t^* J_s = e^{-|t-s|H_0} \).

(d) Let \( \Sigma_{[a,b]} \) denote the \( \sigma \)-algebra generated by \( \mathcal{L} \{ \phi_{\mathcal{E}}(\pi_2(f)) \mid \pi_2(f) \in \overline{U_{[a,b]}} \} \) and the set of \( \Sigma_{[a,b]} \)-measurable functions in \( \mathcal{E} \) by \( \mathcal{E}_{[a,b]} \). Then \( \text{Ran } (E_{[a,b]}) = \mathcal{E}_{[a,b]} \).

(e) (Markoff property) Let \( a \leq b \leq t \leq c \leq d \). Then
\[
E_{[a,b]} E_t E_{[c,d]} = E_{[a,b]} E_{[c,d]}.
\]

**Proof:** Eqs.(a),(b),(c) and (e) follow from Proposition 2.2. We shall show (d). Let \( \{e_n\}_{n \geq 1} \) be a complete orthonormal system in \( \overline{U_{[a,b]}} \). Then any vectors \( \Psi \in \text{Ran}(E_{[a,b]}) \) can be given by the strong limit of finite linear sum of vectors : \( \phi_{\mathcal{E}}(e_1)^{n_1}...\phi_{\mathcal{E}}(e_k)^{n_k} \). Then
we have \( \exp(i\phi_\xi(f)) \in \text{Ran}(E_{[a,b]}) \). Since for \( F \in S(\mathbb{R}^n) \) (the set of rapidly decreasing infinitely differentiable functions),

\[
F(\phi_\xi(f_1), \ldots, \phi_\xi(f_n)) = (2\pi)^{-n/2} \int \hat{F}(t_1, \ldots, t_n) e^{i\phi_\xi\left(\sum_{j=1}^{n} t_j f_j\right)} dt_1 \ldots dt_n,
\]

we see that \( F(\phi_\xi(f_1), \ldots, \phi_\xi(f_n)) \in \text{Ran}(E_{[a,b]}), f_1, \ldots, f_n \in U_{[a,b]} \). By virtue of the fact that the following subset is dense in \( \mathcal{E}_{[a,b]} \) ([9, section I]),

\[
\left\{ F(\phi_\xi(f_1), \ldots, \phi_\xi(f_n)) | f_1, \ldots, f_n \in U_{[a,b]}, F \in S(\mathbb{R}^n), n \geq 1 \right\},
\]

we have \( \mathcal{E}_{[a,b]} \subset \text{Ran}(E_{[a,b]}) \). The proof is complete. \( \square \)

**Proposition 2.4 ([9, Theorem III.6], FKN formula)** Let \( f_1, \ldots, f_n \in \mathcal{H}_{-} \) and \( G_0, \ldots, G_k \) be bounded measurable functions on \( \mathbb{R}^d \). Let \( t_1, \ldots, t_n \geq 0 \) be given. Then

\[
\left\langle \Omega_{\mathcal{F}}, G_0^{s_0} e^{-t_1 H_0} G_1^{s_1} \ldots e^{-t_n H_0} G_n^{s_n} \Omega_\mathcal{F} \right\rangle_{\mathcal{F}} = \left\langle \Omega_\mathcal{E}, G_0^{s_0} \cdots G_n^{s_n} \Omega_\mathcal{E} \right\rangle_\mathcal{E},
\]

where \( \Omega_{\mathcal{F}} \) and \( \Omega_\mathcal{E} \) are the function 1 in \( \mathcal{F} \) and \( \mathcal{E} \) respectively, and \( s_0 \) is arbitrary and

\[
s_j = s_0 + \sum_{i=1}^{j} t_i, \quad j = 1, \ldots, n,
\]

\[
G_j^{s_j} = G_j \left( \phi_{\mathcal{F}}(\pi_{-1}(f_1)), \ldots, \phi_{\mathcal{F}}(\pi_{-1}(f_n)) \right),
\]

\[
G_j^{s_j} = G_j \left( \phi_{\mathcal{E}}(\pi_{-1}(f_1)), \ldots, \phi_{\mathcal{E}}(\pi_{-1}(f_n)) \right).
\]

**Proof:** From Proposition 2.3 it follows that

the l.h.s. of (2.4) = \( \left\langle \Omega_\mathcal{E}, J_{s_0} G_0^{s_0} J_{s_1}^{*} J_{s_1} G_1^{s_1} J_{s_2}^{*} \ldots J_{s_n} G_n^{s_n} J_{s_n}^{*} \Omega_\mathcal{E} \right\rangle \).

It can be easily seen that as operators in \( \mathcal{E} \),

\[
J_{t} e^{i\phi_{\mathcal{F}}(\pi_{-1}(f))} J_{t}^{*} = E_{t} e^{i\phi_{\mathcal{E}}(n_2(\delta_{0} f))} E_{t}^{*}.
\]

Since for \( G \in S(\mathbb{R}^d) \),

\[
G^{s_j} \left( \phi_{\mathcal{F}}(f_1), \ldots, \phi_{\mathcal{F}}(f_d) \right) = (2\pi)^{-n/2} \int \hat{G}(t_1, \ldots, t_d) e^{i\phi_{\mathcal{F}}(\sum_{j=1}^{d} t_j f_j)} dt_1 \ldots dt_d,
\]

we have \( J_{s} G^{s_j} J_{s}^{*} = E_{s} G^{s_j} E_{s} \). Then it follows that

the l.h.s. of (2.4) = \( \left\langle \Omega_\mathcal{E}, E_{s_0} G_0^{s_0} E_{s_1} G_1^{s_1} E_{s_2} \ldots E_{s_n} G_n^{s_n} E_{s_n} \Omega_\mathcal{E} \right\rangle \).

Since \( E_{s_0} \Omega_\mathcal{E} = \Omega_\mathcal{E} \), we have

\[
= \left\langle G_0^{s_0} \Omega_\mathcal{E}, E_{s_0} E_{s_1} G_1^{s_1} E_{s_2} \ldots E_{s_n} G_n^{s_n} E_{s_n} \Omega_\mathcal{E} \right\rangle.
\]
Since $G_0^{n}\Omega_{\xi} \in \text{Ran}(E_{s_0})$ and $G_1^{s_1} E_{s_1} E_{s_2} G_2^{s_2} \ldots E_{s_n} G_n^{s_n} \Omega_{\xi} \in \text{Ran}(E_{s_1})$, we have
\[
\langle G_0^{n}\Omega_{\xi}, G_1^{s_1} E_{s_1} E_{s_2} G_2^{s_2} \ldots E_{s_n} G_n^{s_n} E_{s_n} \Omega_{\xi} \rangle = \langle G_1^{s_1} G_0^{n}\Omega_{\xi}, E_{s_1} E_{s_2} G_2^{s_2} \ldots E_{s_n} G_n^{s_n} E_{s_n} \Omega_{\xi} \rangle.
\]
Since $G_1^{s_1} G_0^{n}\Omega_{\xi} \in \text{Ran}(E_{s_0,s_1})$, and $G_2^{s_2} E_{s_2} \ldots E_{s_n} G_n^{s_n} E_{s_n} \Omega_{\xi} \in \text{Ran}(E_{s_2})$, by Proposition 2.3, we have
\[
\langle G_1^{s_1} G_0^{n}\Omega_{\xi}, G_2^{s_2} E_{s_2} \ldots E_{s_n} G_n^{s_n} E_{s_n} \Omega_{\xi} \rangle = \langle G_2^{s_2} G_1^{s_1} G_0^{n}\Omega_{\xi}, E_{s_2} E_{s_3} G_3^{s_3} \ldots E_{s_n} G_n^{s_n} E_{s_n} \Omega_{\xi} \rangle.
\]
Repeating this procedure we have
\[
= \langle G_n^{s_n} G_{n-1}^{s_{n-1}} \ldots G_0^{n}\Omega_{\xi}, \Omega_{\xi} \rangle.
\]
By a limiting argument, the proof is complete.

However corresponding projection $[e_{[a,b]}]$ can be characterized in such a way. (see [4]).

3 FUNCTIONAL INTEGRALS

For each $x, y \in \mathbb{R}^d$ and an $\mathcal{H}_{-1}$-valued function $\rho$, we can define a unitary operator on $\mathcal{F}$ by
\[
U_\rho(x, y) \equiv \exp \left\{ \frac{1}{2} i \phi_F \left( \sum_{\mu=1}^{d} \pi_{-1} (\tilde{\rho}_\mu(x) + \tilde{\rho}_\mu(y)) (x_\mu - y_\mu) \right) \right\}.
\]
Then we define a family of contractive self-adjoint operators $\{Q_{\rho,s}\}_{s \geq 0}$ on $\mathcal{M}$ by
\[
(Q_{\rho,s} F)(x) = \int_{\mathbb{R}^d} p_s(x-y) U_\rho(x, y) F(y) dy, \quad s > 0,
\]
\[
(Q_{\rho,0} F)(x) = F(x),
\]
where $F(\cdot) \in \mathcal{M}$, the integral is the $\mathcal{F}$-valued Bochner integral and $p_s(x)$ the $d$-dimensional heat kernel. Let
\[
[C_n^0(\mathbb{R}^d; \mathcal{H}_j)] = \left\{ \rho(\cdot) : \mathbb{R}^d \rightarrow \mathcal{H}_j | \pi_j (\tilde{\rho}(\cdot)) \in C_n^0(\mathbb{R}^d; [\mathcal{H}_j]), \mu = 1, \ldots, d \right\}, j = -1, -2,
\]
where $C^n_b(\mathbb{R}^d; K)$ denotes the set of $K$-valued $n$-times strongly continuously differentiable together with bounded functions up to $n$.

**Definition 3.1** For $\rho \in [C^1_b(\mathbb{R}^d; H_{-1})]$, we say that $F \in \mathcal{M}_\rho^\infty$ if and only if the following (i)-(iii) hold

(i) $F(\cdot) \in C^2(\mathbb{R}^d; F)$ such that $||\partial^k F(\cdot)||_F \in L^2(\mathbb{R}^d)$, $|k| \leq 2$.

(ii) For each $y \in \mathbb{R}^d$,

$$F(y) \in \mathcal{F}^\infty, \quad \partial_\mu F(y) \in \mathcal{F}^\infty, \quad \mu = 1, \ldots, d.$$  

(iii) (Integration by parts condition) For all $G \in \mathcal{M}$, $x \in \mathbb{R}^d$,

$$\lim_{y \to \infty} \partial_{y_\mu} p_s(x - y) \cdot \langle F(y), U_\rho(x, y) G(x) \rangle_F = 0,$$

$$\lim_{y \to \infty} p_s(x - y) \cdot \partial_{y_\mu} \langle F(y), U_\rho(x, y) G(x) \rangle_F = 0, \quad \mu = 1, \ldots, d.$$  

Note that $C^\infty_0(\mathbb{R}^d) \otimes \mathcal{F}^\infty \subset \mathcal{M}_\rho^\infty$, where $\otimes$ denotes the algebraic tensor product.

**Lemma 3.2** ([4, Lemmas 4.4 and 4.5]) Let $\rho \in [C^2_b(\mathbb{R}^d; H_{-1})]$, $F \in \mathcal{M}_\rho^\infty$, and $G \in \mathcal{M}$. Then $(Q_{\rho,s} F, G)_{\mathcal{M}}$ is the right side differentiable at $s = 0$ with

$$\frac{d}{ds} (Q_{\rho,s} F, G)_{\mathcal{M}} \big|_{s=0^+} = - \langle H_{\rho,0} F, G \rangle_{\mathcal{M}}. \quad (3.1)$$

Let $(\Omega, Db)$ be a probability space for the $d$-dimensional Brownian motion $b(t) = (b_\mu(t))_{1 \leq \mu \leq d, t \geq 0}$ and $d\mu$ be the Wiener measure on $\mathbb{R}^d \times \Omega$ defined by $d\mu = dx \otimes Db$.

In what follows, for simplicity, we put $n^* = 2^n$.

**Lemma 3.3** Let $\rho \in [C^2_b(\mathbb{R}^d; H_{-1})]$. Then, for all $t \geq 0$, the strong limit

$$s - \lim_{n \to \infty} Q_{\rho,\frac{s}{n^*}}^{n^*} \equiv G_\rho(t)$$

exists. Moreover, $G_\rho(t)$ has the following functional integral representation for $F, H \in \mathcal{M}$

$$\langle F, G_\rho(t) H \rangle_{\mathcal{M}} = \int_{\mathbb{R}^d} d\mu \int_{Q_{-1}} d\mu_1 e^{i\phi_F(\pi_{-1}(t,x))} \mathcal{F}(b(t) + x) H(x) \quad (3.2)$$

$$\pi_{-1}(t, x) = \sum_{\mu=1}^d \left( \int_0^t \pi_{-1}(\hat{\rho}_\mu(b(s) + x)) \, db_\mu + \frac{1}{2} \int_0^t \partial_\mu \pi_{-1}(\hat{\rho}_\mu(b(s) + x)) \, ds \right).$$
Proof: We see that
\[
\left\| Q_{\rho, \frac{1}{m^*}}^{n*} F - Q_{\rho, \frac{1}{m^*}}^{m*} F \right\|_{\mathcal{M}}^2 = \left\langle F, Q_{\rho, \frac{1}{m^*}}^{n*} F \right\rangle_{\mathcal{M}} + \left\langle F, Q_{\rho, \frac{1}{m^*}}^{2m*} F \right\rangle_{\mathcal{M}} - 2\Re \left\langle F, Q_{\rho, \frac{1}{m^*}}^{n*} Q_{\rho, \frac{1}{m^*}}^{m*} F \right\rangle_{\mathcal{M}}.
\]
(3.3)

From the definition of \( Q_{\rho, t} \) it follows that
\[
\left\langle F, Q_{\rho, \frac{1}{m^*}}^{n*} Q_{\rho, \frac{1}{m^*}}^{m*} F \right\rangle_{\mathcal{M}} = \int_{\mathbb{R}^d} dx \left\langle F(b(2t) + x), e^{i\phi x} \left( \sum_{\mu=1}^d \pi_{-1}(\Box_{\mu, m, n}(x)) \right) F(x) \right\rangle_{L^2(\Omega; F)},
\]
where
\[
\Box_{\mu, m, n}(x) = \sum_{k=1}^{m^*} \left\{ b_{\mu} \left( \frac{t}{m^*} k + x \right) + \tilde{\rho}_{\mu} \left( \frac{t}{m^*} (k-1) + x \right) \right\} \times \left\{ b_{\mu} \left( \frac{t}{m^*} - b_{\mu} \left( \frac{t}{m^*} (k-1) \right) \right) \right\}
\]
\[
+ \sum_{k=1}^{n^*} \left\{ b_{\mu} \left( \frac{t}{n^*} k + t + x \right) + \tilde{\rho}_{\mu} \left( \frac{t}{n^*} (k-1) + t + x \right) \right\} \times \left\{ b_{\mu} \left( \frac{t}{n^*} k + t \right) - b_{\mu} \left( \frac{t}{n^*} (k-1) + t \right) \right\}.
\]

We can see that for each \( x \in \mathbb{R}^d \)
\[
s - \lim_{m \to \infty} \lim_{n \to \infty} \sum_{\mu=1}^d \pi_{-1}(\Box_{\mu, m, n}(x)) = \pi_{-1}(2t, x)
\]
in \( L^2(\Omega; [\mathcal{H}_{-1}]) \). By the Lebesgue dominated convergence theorem, we have
\[
\lim_{n \to \infty} \lim_{m \to \infty} \left\langle F, Q_{\rho, \frac{1}{m^*}}^{n*} Q_{\rho, \frac{1}{m^*}}^{m*} F \right\rangle_{\mathcal{M}} = \int_{\mathbb{R}^d} dx \left\langle F(b(2t) + x), e^{i\phi x} \left( \sum_{\mu=1}^d \pi_{-1}(\Box_{\mu, m, n}(x)) \right) F(x) \right\rangle_{L^2(\Omega; F)}. \]
(3.4)

Similarly it can be easily seen that \( \left\langle F, Q_{\rho, \frac{1}{m^*}}^{2m*} F \right\rangle_{\mathcal{M}} \) and \( \left\langle F, Q_{\rho, \frac{1}{m^*}}^{2m*} F \right\rangle_{\mathcal{M}} \) converge to the r.h.s. of (3.4) as \( n, m \to \infty \), respectively. Then it follows that \( \{Q_{\rho, \frac{1}{m^*}}^{n*}\}_{n \geq 0} \) is a Cauchy.

Eq.(3.2) easily follows from (3.4).

Lemma 3.4 Let \( \rho \in [C^2_b(\mathbb{R}^d; \mathcal{H}_{-1})] \). Then the family \( \{G_\rho(t)\}_{t \geq 0} \) is a strongly continuous 1-parameter semigroup on \( \mathcal{M} \).

Proof: The group properties follow from the proof of Lemma 3.3 and the strong continuity in \( t \) a direct calculation using (3.2).
By Lemma 3.4, Hille-Yoshida’s theorem yields that for each \( \rho \in [C^2_b(\mathbb{R}^d; \mathcal{H}_1)] \), there exists a unique nonnegative self-adjoint operator \( \widetilde{H}_{\rho, 0} \) in \( \mathcal{M} \) such that

\[
G_\rho(t) = e^{-t\widetilde{H}_{\rho, 0}}.
\]

**Lemma 3.5** Let \( \rho \in [C^2_b(\mathbb{R}^d; \mathcal{H}_1)] \). Then the self-adjoint operator \( \widetilde{H}_{\rho, 0} \) is a self-adjoint extension of \( H_{\rho, 0} \mid_{\mathcal{M}_\rho} \).

**Proof:** Let \( F \in D(\widetilde{H}_{\rho, 0}) \) and \( G \in \mathcal{M}_\rho \). Then we have

\[
\left\langle \frac{1}{t} \left( e^{-t\widetilde{H}_{\rho, 0}} - I \right) G, F \right\rangle_{\mathcal{M}} = \lim_{n \to \infty} \frac{n^*}{n} \sum_{j=0}^{n-1} \left\langle \frac{n^*}{t} \left( Q_{\rho, \frac{n^*}{n}} - I \right) G, Q_{\rho, \frac{n^*}{n}}^{[n^*]} F \right\rangle_{\mathcal{M}} = \lim_{n \to \infty} \int_0^1 \left\langle \frac{n^*}{t} \left( Q_{\rho, \frac{n^*}{n}} - I \right) G, Q_{\rho, \frac{n^*}{n}}^{[n^*]} F \right\rangle_{\mathcal{M}} ds.
\]

Because of the weak right differentiability of \( Q_{\rho, t} G \) in \( t = 0 \) ((3.1)), and the definition of \( Q_{\rho, t} \) (Lemma 3.3), we have

\[
w - \lim_{n \to 0} \frac{n^*}{t} \left( Q_{\rho, \frac{n^*}{n}} - I \right) G = -H_{\rho, 0} G, \quad s - \lim_{n \to \infty} Q_{\rho, \frac{n^*}{n}}^{[n^*]} = G_\rho(ts).
\]

Hence

\[
\left\langle \frac{1}{t} \left( e^{-t\widetilde{H}_{\rho, 0}} - I \right) G, F \right\rangle_{\mathcal{M}} = \int_0^1 ds \left\langle -H_{\rho, 0} G, e^{-ts\widetilde{H}_{\rho, 0}} F \right\rangle_{\mathcal{M}}. \tag{3.5}
\]

As \( t \to 0 \) on the both sides of (3.5), we get

\[
\left\langle G, \widetilde{H}_{\rho, 0} F \right\rangle_{\mathcal{M}} = \left\langle H_{\rho, 0} G, F \right\rangle_{\mathcal{M}}.
\]

which implies that \( G \in D(\widetilde{H}_{\rho, 0}) \) and \( \widetilde{H}_{\rho, 0} G = H_{\rho, 0} G \). \( \square \)

We denote the extension \( H_{\rho, 0} \) by the same symbol \( H_{\rho, 0} \). For \( \rho \in [C^2_b(\mathbb{R}^d; \mathcal{H}_1)] \), we give a rigorous definition of \( H_\rho \) in terms of the form sum + of \( H_{\rho, 0} \) and \( I \otimes H_0 \);

\[
H_\rho = H_{\rho, 0} + I \otimes H_0.
\]

Note that \( \mathcal{M}_\rho \cap D(I \otimes H_0) \) is dense in \( \mathcal{M} \). We introduce a multiplication operator in \( L^2(\mathbb{R}^d) \otimes \mathcal{E} \cong L^2(\mathbb{R}^d; \mathcal{E}) \) by

\[
\phi_{E, \mu}^{\rho, s} \equiv \int_{\mathbb{R}^d} \phi_{\mathcal{E}}(\pi_{-2} \left( \hat{s} \hat{\rho}_{\mu}(x) \right)) \, dx.
\]
Moreover we formally define an operator acting in $L^2(\mathbb{R}^d; \mathcal{E})$ by

$$H_{\rho,0,s} = \frac{1}{2} \sum_{\mu=1}^{d} (-iD_\mu \otimes I - \phi_{\rho,0,s}^{\mu})^2.$$ 

Since for $\rho \in [C_b^2(\mathbb{R}^d; \mathcal{H}_{-1})]$, we see that $j_\rho \rho \in [C_b^2(\mathbb{R}^d; \mathcal{H}_{-2})]$, one can construct a self-adjoint extension of $H_{\rho,0,s}$ in the same manner as that of $H_{\rho,0}$. We denote it by the same symbol $H_{\rho,0,s}$. Similarly to that of $H_{\rho,0}$, we define $Q_{\rho,t,s}$, contraction operators in $L^2(\mathbb{R}^d; \mathcal{E})$, corresponding to $Q_{\rho,t}$ i.e.,

$$s - \lim_{n \to \infty} Q_{\rho,0,s}^{n} = e^{-tH_{\rho,0,s}}. \quad (3.6)$$

**Lemma 3.6 ([4, Lemma 4.9])** Let $\rho \in [C_b^2(\mathbb{R}^d; \mathcal{H}_{-1})]$. Then the following equation holds on $L^2(\mathbb{R}^d; \mathcal{E})$

$$J_x e^{-tH_{\rho,0,s}} J_x^* = E_x e^{-tH_{\rho,0,s}} E_x.$$ 

Now we are ready to state main theorem in this note.

**Theorem 3.7 ([4, Theorem 4.10])** Let $F, G \in \mathcal{M}$, $V \in C_b(\mathbb{R}^d)$ and $\rho \in [C_b^2(\mathbb{R}^d; \mathcal{H}_{-1})]$ such that

$$(1) \, \sup_{\mu=1, \ldots, d, x \in \mathbb{R}^d} |||\tilde{\omega}_{\mu-1}(\tilde{\rho}_\mu(x))|||_{-1} < \infty, \quad (2) \, \left| \sum_{\mu=1}^{d} \partial_{\mu} \pi_{-1}(\tilde{\rho}_\mu(x)) \right|_{-1} = 0.$$ 

Then the following limit exists in $L^2(\Omega; [\mathcal{H}_{-2}])$ for each $x \in \mathbb{R}^d$:

$$s - \lim_{n \to \infty} \frac{n \mu - 1}{t} \int_{\frac{m}{n}}^{\frac{m+1}{n}} \int_{\frac{m}{n} + \frac{1 - s}{n}}^{\frac{m + 1 - s}{n}} \tilde{\omega}_{\mu-1}(\tilde{\rho}_\mu(b(s) + x)) \, db_\mu \equiv \pi_{-2}^{\mu}(t, x).$$ 

Moreover

$$\left\langle F, e^{-t(H_{\rho,V} + V)} \right\rangle_{\mathcal{M}} = \left\langle F, e^{-t V(b(t))} e^{-t \int_{0}^{t} V(b(s) + x) \, ds} e^{-t \int_{0}^{t} \pi_{-2}^{\mu}(t, x) \, ds} \right\rangle_{\mathcal{M}} = \int_{\mathbb{R}^d} d\mu d\mu_{-2} e^{-t \int_{0}^{t} V(b(s)) \, ds} e^{-t \int_{0}^{t} \pi_{-2}^{\mu}(t, x) \, ds} J_t \bar{F}(b(t) + x) J_0 G(x) \quad (3.7)$$

**Proof:** The existence of the strong limit follows directly from (1) (see [4, section 2]). Let $\frac{k}{n} = s$. By the strong Trotter product formula, Markoff properties of $E_j$ ([4, Proposition 3.3 (e)]), Lemma 3.6 and (3.6), we see that

$$\left\langle F, e^{-t(H_{\rho,V} + V)} \right\rangle_{\mathcal{M}} = \lim_{n \to \infty} \lim_{k \to \infty} S_{n,k,s},$$

$S_{n,k,s} = \left\langle F, J_t^* \left( Q_{\rho,0,s}^{k} \right) e^{-sV} \left( Q_{\rho,\frac{k}{n},t-s}^{k} \right) e^{-sV} \cdots \left( Q_{\rho,\frac{k}{n},t-s}^{k} \right) e^{-sV} J_0 G \right\rangle_{\mathcal{M}}$. 

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Then the definition of $Q_{\rho,t',t'}$ yields that

$$S_{n*,k*} = \int_{\mathbb{R}^d} dx \left\langle F(b(t) + x), \right. \left. J_t^* \exp \left( i \phi \left( \sum_{\mu=1}^{d} \sum_{j=0}^{n*-1} [\tilde{J}_{j\mu}]_{\pi_{-1}} (\Box_{\mu,j,k}(x)) \right) - s \sum_{j=1}^{n*} V(b(js) + x) \right) J_0 G(x) \right\rangle_{L^2(\Omega; \mathcal{F})},$$

where

$$\Box_{\mu,j,k}(x) = \sum_{m=1}^{k*} \left\{ \tilde{\rho}_\mu \left( b(\frac{m}{k*} s + j s) + x \right) + \tilde{\rho}_\mu \left( b(\frac{m-1}{k*} s + j s) + x \right) \right\} \times \left\{ b_\mu (\frac{m}{k*} s + j s) - b_\mu (\frac{m-1}{k*} s + j s) \right\}, \quad j = 0, \ldots, n* - 1.$$

By the Coulomb gauge condition (2), it can be easily seen that for $x \in \mathbb{R}^d$

$$s - \lim_{k \to \infty} \Box_{\mu,j,k}(x) = \int_{js}^{(j+1)s} [\tilde{J}_{j\mu}]_{\pi_{-1}} (\tilde{\rho}_\mu (b(s') + x)) db_\mu \equiv \Box_{\mu,j}(x)$$

in $L^2(\Omega; [\tilde{\mathcal{H}}_{-2}])$. By the Lebesgue dominated convergence theorem, we have

$$\lim_{k \to \infty} S_{n*,k*} = \int_{\mathbb{R}^d} dx \left\langle J_0 F(b(t) + x), \right. \left. \exp \left( i \phi \left( \sum_{\mu=1}^{d} \sum_{j=0}^{n*-1} \Box_{\mu,j}(x) \right) - s \sum_{j=1}^{n*} V(b(js) + x) \right) J_0 G(x) \right\rangle_{L^2(\Omega; \mathcal{F})}.$$\]

Hence by the first statement of the Theorem and again by the Lebesgue dominated convergence theorem, we get

$$\lim_{n \to \infty} \lim_{k \to \infty} S_{n*,k*} = (3.7).$$

We call $\pi_{-2}^c(t, x)$ “time-ordered $[\tilde{\mathcal{H}}_{-2}]$-valued stochastic integral associated with the family of isometries $[\tilde{J}_t]$ from $[\tilde{\mathcal{H}}_{-1}]$ to $[\tilde{\mathcal{H}}_{-2}]$”.

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