The weak limit of renormalized Rabi Hamiltonian

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Harmonic oscillators

$$a = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\omega}}\frac{d}{dx} + \sqrt{\omega}x\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{\omega}}\frac{d}{dx} + \sqrt{\omega}x\right)$$

are the annihilation op. and the creation op. in $L^2(\mathbb{R})$. They satisfy CCR

$$[a, a^\dagger] = 1\!\!1.$$

Let

$$\omega a^\dagger a = \frac{1}{2}\left(-\frac{d^2}{dx^2} + \omega^2 x^2 - \omega\right).$$

Then $\varphi_{\mathrm{g}}(x) = \left(\frac{\omega}{\pi}\right)^{1/4} e^{-\omega x^2/2}$ is the eigenvector associated with the lowest eigenvalue 0 of $\omega a^\dagger a$. The nth excited state of $\omega a^\dagger a$ is defined by

$$\varphi_n = \frac{1}{\sqrt{n!}}\prod^n a^\dagger \varphi_{\mathrm{g}}, \quad n = 0, 1, 2, \cdots$$

with $\varphi_0 = \varphi_{\mathrm{g}}$. It follows that $\omega a^\dagger a \varphi_n = \omega n \varphi_n$ and $\mathrm{Spec}(\omega a^\dagger a) = \{\omega n\}_{n=0}^\infty$.

$$L^2(\mathbb{R}) = \oplus_{n=0}^\infty [\mathbb{C}\varphi_n].$$

Spin 1/2

Let $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ be elements of $SU(2)$.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Let

$$\Delta\sigma_z, \quad \Delta > 0.$$

Then $\mathrm{Spec}(\Delta\sigma_z) = \{\pm\Delta\}$.

The ground state of $\Delta\sigma_z$ is $s_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the excited state is $s_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$\mathbb{C}^2 = \mathbb{C}s_+ \oplus \mathbb{C}s_-.$$

Rabi model

The total Hilbert space is $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ and the free Rabi Hamiltonian is given by

$$\Delta\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \omega a^\dagger a$$

Introducing the interaction $\sigma_x \otimes \left(a^\dagger + a\right)$, the Rabi Hamiltonian H_{Rabi} is given by

$$H_{\text{Rabi}} = \Delta\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \omega a^\dagger a + g\sigma_x \otimes \left(a^\dagger + a\right), \quad g \in \mathbb{R}.$$

It has the parity symmetry:

$$[\sigma_z \otimes (-1)^{N/\omega}, H_{\text{Rabi}}] = 0.$$

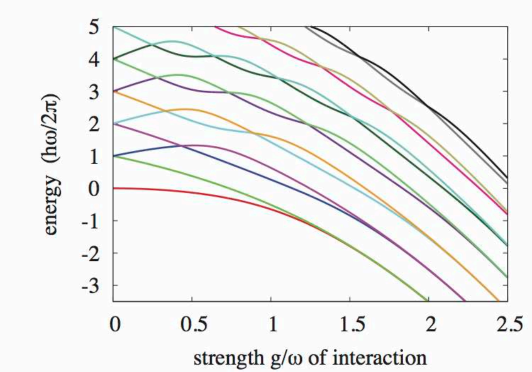

Figure: Eigenvalue curves

1) The lowest curve is cancave.
2) Asymptotic behavior

Rotation U of spin

The rotation group in $\mathbb{R}^3$ has an adjoint representation on $SU(2)$. Let $n \in \mathbb{R}^3$ be a unit vector and $\theta \in [0, 2\pi)$. Thus $e^{(i/2)\theta n\cdot\sigma}$ satisfies that

$$e^{(i/2)\theta n\cdot\sigma}\sigma_\mu e^{-(i/2)\theta n\cdot\sigma} = (R\sigma)_\mu, \quad \mu = x, y, z$$

where $R = R(n, \theta)$ denotes 3×3 matrix representing the rotation around n with angle θ. In particular $U = e^{(i\pi/4)\sigma_y}$ yields that

$$U\sigma_x U^{-1} = \sigma_z,$$
$$U\sigma_z U^{-1} = -\sigma_x.$$

Then

$$UH_{\mathrm{Rabi}}U^{-1} = -\Delta\sigma_x + \omega a^\dagger a + g\sigma_z(a^\dagger + a).$$

Ground state transformation U_g

Since φ_{g} is strictly positive, we can define the unitary operator

$$U_g : L^2(\mathbb{R}) \to L^2(\mathbb{R}, \varphi_{\mathrm{g}}^2 \mathrm{d}x), \quad U_g f = \varphi_{\mathrm{g}}^{-1} f.$$

We set the probability measure on $\mathbb{R}$ by

$$\varphi_{\mathrm{g}}^2(x)\mathrm{d}x = \mathrm{d}\mu(x).$$

Thus

$$\begin{aligned} U_g U H_{\mathrm{Rabi}} U^{-1} U_g^{-1} &= -\Delta\sigma_x - \frac{1}{2}\frac{d^2}{dx^2} + \omega x \frac{d}{dx} + g\sigma_z\sqrt{2\omega}x \\ &= \begin{pmatrix} -\frac{1}{2}\frac{d^2}{dx^2} + \omega x \frac{d}{dx} + g\sqrt{2\omega}x & -\Delta \\ -\Delta & -\frac{1}{2}\frac{d^2}{dx^2} + \omega x \frac{d}{dx} - g\sqrt{2\omega}x \end{pmatrix} \end{aligned}$$

Scalar transform

Let $\mathbb{Z}_2 = \{-1, 1\}$. We identify $\mathbb{C}^2 \otimes L^2(\mathbb{R}, \mathrm{d}\mu)$ with

$$\mathscr{H} = L^2(\mathbb{R} \times \mathbb{Z}_2, \mathrm{d}\mu) = \left\{ f = f(x,\alpha) \,\middle|\, \sum_{\alpha \in \mathbb{Z}_2} \int |f(x,\alpha)|^2 \mathrm{d}\mu(x) < \infty \right\}$$

by the map $\mathbb{C}^2 \otimes L^2(\mathbb{R}, \mathrm{d}\mu) \ni \begin{bmatrix} f_+(x) \\ f_-(x) \end{bmatrix} \mapsto f(x,\alpha) \in \mathscr{H}$. Here $f(x,+1) = f_+(x)$ and $f(x,-1) = f_-(x)$. $H : \mathscr{H} \to \mathscr{H}$ is defined by

$$Hf(x,\alpha) = \left\{ -\frac{1}{2}\frac{d^2}{dx^2} + \omega x \frac{d}{dx} + g\sqrt{2\omega}\alpha x \right\} f(x,\alpha) - \Delta f(x,-\alpha), \quad \alpha \in \mathbb{Z}_2$$

Lemma

H_{Rabi} in $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ is unitarily equivalent to H in $L^2(\mathbb{R} \times \mathbb{Z}_2, \mathrm{d}\mu)$.

In what follows we deal with H instead of H_{Rabi}.

OU process

$(X_t)^x_{t\in\mathbb{R}}$ is an Ornstein-Uhrenbeck process on $(\mathscr{X},\mathscr{B},P)$ st

$$\mathbb{E}_P[X^x_t] = xe^{-\omega t}, \quad \mathbb{E}_P[X^x_t X^x_s] = e^{-\omega(t+s)}\left(x^2+\frac{e^{2\omega(t\wedge s)}-1}{2\omega}\right)$$

Here $\mathbb{E}_Q[\cdots]$ denotes the expectation wrt Q. We have

$$\int \mathbb{E}_P[X^x_t]\,\mathrm{d}\mu(x)=0, \quad \int \mathbb{E}_P[X^x_t X^x_s]\,\mathrm{d}\mu(x)=\frac{e^{-|t-s|\omega}}{2\omega}.$$

Let N be redefined by

$$\mathrm{N}=-\frac{1}{2}\frac{d^2}{dx^2}+\omega x\frac{d}{dx} \quad \text{(the number op. in mind)}$$

The generator of X^x_t is given by $-\mathrm{N}$:

$$(f,e^{-t\mathrm{N}}g)_{L^2(\mathbb{R},\mathrm{d}\mu)}=\int \mathbb{E}_P\left[\overline{f(X^x_0)}g(X^x_t)\right]\mathrm{d}\mu(x).$$

Spin process

Let $(N_t)_{t\in\mathbb{R}}$ be a Poisson process on $(\mathscr{X}', \mathscr{B}', \nu)$ with unit intensity, i.e.,

$$\mathbb{E}_\nu\left[\mathbb{1}_{N_t=n}\right] = \frac{t^n}{n!}e^{-t}, \quad n \geq 0.$$

We define the spin process by

$$\sigma_t^\alpha = (-1)^{N_t}\alpha, \quad \alpha \in \mathbb{Z}_2.$$

Let σ_F be the fermionic harmonic oscillator:

$$\sigma_F = \frac{1}{2}(\sigma_z + i\sigma_y)(\sigma_z - i\sigma_y) = -\sigma_1 + \mathbb{1}.$$

The generator of σ_t^α is given by $-\sigma_F$:

$$(f, e^{-t\sigma_F} g)_{L^2(\mathbb{Z}_2)} = \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\nu\left[\overline{f(\sigma_0^\alpha)}g(\sigma_t^\alpha)\right].$$

Feynman-Kac Formula

Theorem (FH+Hirokawa, (14))

Set $\sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_P \mathbb{E}_\nu [\cdots] \mathrm{d}\mu(x) = \mathbb{E}[\cdots]$. *Then*

$$(f, e^{-tH} g)_{\mathscr{H}} = e^t \mathbb{E}\left[\overline{f(X_0^x, \sigma_0^\alpha)} g(X_t^x, \sigma_t^\alpha) e^{-g\sqrt{2\omega}\int_0^t \sigma_s^\alpha X_s^x \mathrm{d}s} \Delta^{N_t} \right].$$

Corollary (FH+Hirokawa (14))

e^{-tH} *is positivity improving for* $\forall t > 0$. *In particular* $e^{-tH} \mathbb{1} > 0$.

Vacuum expectation

Let $\mathbb{1} \in L^2(\mathbb{R} \times \mathbb{Z}_2, \mathrm{d}\mu)$ be the constant function. Note that

$$\mathbb{1} \cong \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \varphi_{\mathrm{g}} \in \mathbb{C}^2 \otimes L^2(\mathbb{R}).$$

We set

$$\mathbb{1}_t = e^{-tH}\,\mathbb{1}, \quad t \geq 0.$$

Let $\Phi_{\mathrm{g}} (> 0)$ be the ground state of H. Since

$$\Phi_{\mathrm{g}} = \lim_{t\to\infty} \mathbb{1}_t / \|\,\mathbb{1}_t\|,$$

we have

$$(\Phi_{\mathrm{g}}, A\Phi_{\mathrm{g}}) = \lim_{t\to\infty} \frac{(\mathbb{1}_t, A\,\mathbb{1}_t)}{(\mathbb{1}_t, \mathbb{1}_t)}$$

Here $A = e^{+\beta \mathrm{N}}, (-1)^{\mathrm{N}}, \sigma_x(-1)^{\mathrm{N}}$ etc.

$(B_t)_{t\geq 0}$ is Brownian motion. $\mathbb{E}[B_s B_t] = s \wedge t$.

$$X_s^x = e^{-s\omega}\left(x + \frac{1}{\sqrt{2\omega}} B_{e^{2s\omega}-1}\right) \quad s \geq 0$$

$$(\mathbb{1}, \mathbb{1}_t) = e^t \sum_{\alpha\in\mathbb{Z}_2} \int \mathbb{E}_\nu \mathbb{E}_P \left[e^{-g\sqrt{2\omega}\int_0^t \sigma_s X_s^x \mathrm{d}s} \Delta^{N_t} \right] \mathrm{d}\mu(x)$$

$$= e^t \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_t} \mathbb{E}_P \left[e^{-g\int_0^t \sigma_s^\alpha e^{-s\omega} B_{e^{2s\omega}-1} \mathrm{d}s} \right] \int \sqrt{\frac{\omega}{\pi}} e^{-\omega x^2 - \sqrt{2\omega} g \int_0^t \sigma_s^\alpha e^{-s\omega} x} \mathrm{d}x \right]$$

$$1) \mathbb{E}_P \left[e^{-g\int_0^t \sigma_s^\alpha e^{-s\omega} B_{e^{2s\omega}-1} \mathrm{d}s} \right] = \exp\left(\frac{g^2}{2} \left\| \int_0^t \sigma_s^\alpha e^{-s\omega} \mathbb{1}_{[0, e^{2s\omega}-1]}(\cdot) \mathrm{d}s \right\|_{L^2(\mathbb{R})}^2 \right)$$

$$= \exp\left(\frac{g^2}{2} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}r \sigma_s^\alpha \sigma_r^\alpha e^{-(s+r)\omega} (e^{2(s\wedge r)\omega} - 1) \right),$$

$$2) \sqrt{\frac{\omega}{\pi}} \int e^{-\omega x^2 - \sqrt{2\omega} g \int_0^t \sigma_s^\alpha e^{-s\omega} x} \mathrm{d}x = \exp\left(\frac{g^2}{2} \left(\int_0^t \sigma_s^\alpha e^{-s\omega} \mathrm{d}s \right)^2 \right)$$

Concave and parity

$$(\mathbb{1}, \mathbb{1}_t) = e^t \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_t} \exp\left(\frac{g^2}{2} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}r \sigma_s^\alpha \sigma_r^\alpha e^{-\omega|s-r|} \right) \right]$$

Let $E_0(g) = \inf \mathrm{Spec}(H_{\mathrm{Rabi}})$.

Theorem (Concave)

$g \mapsto E_0(g)$ *is concave and differentiable a.e. in* g.

Proof: This follows from $E_0(g) = \lim_{t\to\infty} -\frac{1}{t} \log(\mathbb{1}, \mathbb{1}_t)$. □

Theorem (Parity)

Let $P = \sigma_x \otimes (-1)^N$. *Then* $P\Phi_{\mathrm{g}} = -\Phi_{\mathrm{g}}$.

Proof: Since $P\mathbb{1} = -\mathbb{1}$ and $\Phi_{\mathrm{g}} = \lim_{t\to\infty} \mathbb{1}_t / \|\mathbb{1}_t\|$, the theorem follows. □

$(\mathbb{1}_t, \mathbb{1}_t)$ and $(\mathbb{1}_t, e^{-\beta N}\mathbb{1}_t)$

$$
\begin{aligned}
(\mathbb{1}_t, \mathbb{1}_t) &= e^{2t} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_{2t}} \exp\left(\frac{g^2}{2} \int_0^{2t} \mathrm{d}s \int_0^{2t} \mathrm{d}r \sigma_s^\alpha \sigma_r^\alpha e^{-\omega|s-r|} \right) \right] \\
&= e^{2t} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_t+N_{-t}} \exp\left(\frac{g^2}{2} \int_{-t}^{t} \mathrm{d}s \int_{-t}^{t} \mathrm{d}r \sigma_s^\alpha \sigma_r^\alpha e^{-\omega|s-r|} \right) \right]
\end{aligned}
$$

Here we can extend $(N_t)_{t\geq 0}$ to $(N_t)_{t\in\mathbb{R}}$, where N_t is independent of N_{-s} for any $s,t > 0$, and we used the shift invariance of σ_t.

Since

$$(f, e^{-\beta \mathrm{N}} g) = \sum_{\alpha \in \mathbb{Z}_2} \int \mathbb{E}_P[\bar{f}(X_0^x, \alpha) g(X_\beta^x, \alpha)] \mathrm{d}\mu(x),$$

we see that

$$(\mathbb{1}_t, e^{-\beta \mathrm{N}} \mathbb{1}_t) = \sum_{\alpha \in \mathbb{Z}_2} \int \mathbb{E}_P[\mathbb{1}_t(X_0^x, \alpha) \mathbb{1}_t(X_\beta^x, \alpha)] \mathrm{d}\mu(x).$$

$$\begin{aligned}
\mathbb{1}_t(X_0^x, \alpha) &= e^t \mathbb{E}_\nu \mathbb{E}_P \left[\Delta^{N_t} e^{-\sqrt{2\omega} g \int_0^t \sigma_s^\alpha X_s^x \mathrm{d}s} \right] \\
&= e^t \mathbb{E}_\nu \mathbb{E}_P \left[\Delta^{N_t} e^{-\sqrt{2\omega} g \int_0^t \sigma_s^\alpha e^{-s\omega} \left(x + \frac{1}{\sqrt{2\omega}} B_{e^{2s\omega} - 1} \right) \mathrm{d}s} \right] \\
&= e^t \mathbb{E}_\nu \left[\Delta^{N_t} e^{-\sqrt{2\omega} g \int_0^t \sigma_s^\alpha e^{-s\omega} \mathrm{d}s x} \mathbb{E}_P \left[e^{-g \int_0^t \sigma_s^\alpha e^{-s\omega} B_{e^{2s\omega} - 1} \mathrm{d}s} \right] \right] \\
&= e^t \mathbb{E}_\nu \left[\Delta^{N_t} e^{-\sqrt{2\omega} g \int_0^t \sigma_s^\alpha e^{-s\omega} \mathrm{d}s x} e^{\frac{g^2}{2} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}r \sigma_s^\alpha \sigma_r^\alpha e^{-(s+r)\omega} (e^{2(s \wedge r)\omega} - 1)} \right]
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}_P\left[\mathbb{1}_t(X_\beta^x,\alpha)\right]\\
&= e^t\,\mathbb{E}_P\left[\mathbb{E}_\nu\left[\Delta^{N_t} e^{-\sqrt{2\omega}g\int_0^t \sigma_s^\alpha e^{-s\omega}\mathrm{d}s X_\beta^x} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \sigma_s^\alpha \sigma_r^\alpha e^{-(s+r)\omega}(e^{2(s\wedge r)\omega}-1)}\right]\right]\\
&= e^t\,\mathbb{E}_P\left[\mathbb{E}_\nu\left[\Delta^{N_t} e^{-\sqrt{2\omega}g\int_0^t \sigma_s^\alpha e^{-s\omega}\mathrm{d}s e^{-\beta\omega}\left(x+\frac{1}{\sqrt{2}\omega}B_{e^{2\beta\omega}-1}\right)}\right.\right.\\
&\qquad\qquad\qquad\qquad \left.\left.\times\; e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \sigma_s^\alpha \sigma_r^\alpha e^{-(s+r)\omega}(e^{2(s\wedge r)\omega}-1)}\right]\right]\\
&= e^t\,\mathbb{E}_\nu\left[\Delta^{N_t} e^{-\sqrt{2\omega}g\int_0^t \sigma_s^\alpha e^{-s\omega}\mathrm{d}s e^{-\beta\omega}x}\right.\\
&\qquad \left.\times\; e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \sigma_s^\alpha \sigma_r^\alpha e^{-(s+r)\omega}(1-e^{-2\beta\omega})} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \sigma_s^\alpha \sigma_r^\alpha e^{-(s+r)\omega}(e^{2(s\wedge r)\omega}-1)}\right]
\end{aligned}
$$

Let $W(s,r) = \sigma_s^\alpha \sigma_r^\alpha e^{-\omega|s-r|}$. Finally we have

$$
\begin{aligned}
&(\mathbb{1}_t, e^{-\beta \mathrm{N}} \mathbb{1}_t) \\
&= e^{2t} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_{-t}+N_t} \exp\left(\frac{g^2}{2} \int_{-t}^{t} \int_{-t}^{t} W(s,r) \mathrm{d}s\mathrm{d}r \right) \right. \\
&\qquad \left. \times \exp\left(-g^2 (1 - e^{-\beta\omega}) \int_{-t}^{0} \int_{0}^{t} W(s,r) \mathrm{d}s\mathrm{d}r \right) \right]
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sigma_s^\alpha \sigma_r^\alpha &= (-1)^{N_s} (-1)^{N_r} \alpha^2 \\
&= (-1)^{N_s} (-1)^{N_r} = (-1)^{N_s} (-1)^{-N_r} = (-1)^{N_s - N_r} = (-1)^{N_{|s-r|}},
\end{aligned}
$$

which implies that $\sigma_s^\alpha \sigma_r^\alpha$ is independent of $\alpha \in \mathbb{Z}_2$. Thus

$$
e^{i\pi N_{|s-r|}} = \sigma_s^\alpha \sigma_r^\alpha
$$

Let $W(s-r) = e^{i\pi N_{|s-r|}} e^{-\omega|s-r|}$.

$$(\Phi_{\rm g}, e^{-\beta \rm N}\Phi_{\rm g}) = \lim_{t\to\infty} \frac{(\mathbb{1}_t, e^{-\beta \rm N}\mathbb{1}_t)}{(\mathbb{1}_t, \mathbb{1}_t)}$$

$$= \lim_{t\to\infty} \frac{\sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_t+N_{-t}} e^{\frac{g^2}{2}\int_{-t}^t\int_{-t}^t W(s-r){\rm d}s{\rm d}r - g^2(1-e^{-\beta\omega})\int_{-t}^0\int_0^t W(s-r){\rm d}s{\rm d}r}\right]}{\sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_t+N_{-t}} e^{\frac{g^2}{2}\int_{-t}^t\int_{-t}^t W(s-r){\rm d}s{\rm d}r}\right]}$$

$$= \lim_{t\to\infty} \mathbb{E}_{\nu_t}\left[e^{-g^2(1-e^{-\beta\omega})\int_{-t}^0\int_0^t W(s-r){\rm d}s{\rm d}r}\right]$$

$$\nu_t(\mathscr{O}) = \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_t+N_{-t}} e^{\frac{g^2}{2}\int_{-t}^t\int_{-t}^t W(s-r){\rm d}s{\rm d}r}\mathscr{O}\right] \Big/ \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_t+N_{-t}} e^{\frac{g^2}{2}\int_{-t}^t\int_{-t}^t W(s-r){\rm d}s{\rm d}r}\right]$$

Theorem (FH Adv. Math. (14))

Let $W(s-r) = e^{i\pi N_{|s-r|}} e^{-\omega|s-r|}$. There exists a prob. measure ν_∞ st

$$(\Phi_{\rm g}, e^{-\beta \rm N}\Phi_{\rm g}) = \mathbb{E}_{\nu_\infty}\left[e^{-g^2(1-e^{-\beta\omega})\int_{-\infty}^0\int_0^\infty W(s-r){\rm d}s{\rm d}r}\right] \quad \beta > 0$$

Localization

$\mathbb{C}^2 \otimes L^2(\mathbb{R}) = \bigoplus_{n=0}^{\infty}[\mathbb{C}^2 \otimes [\mathbb{C}\varphi_n]]$. Thus $\Phi_{\rm g} = \bigoplus_{n=0}^{\infty} \Phi_{\rm g}^{(n)}$ and

$$1 = \|\Phi_{\rm g}\|^2 = \sum_{n=0}^{\infty} \|\Phi_{\rm g}^{(n)}\|^2_{\mathbb{C}^2 \otimes [\mathbb{C}\varphi_n]}$$

The question is $\|e^{\beta \mathrm{N}}\Phi_{\rm g}\|^2 = \sum_{n=0}^{\infty} e^{2\beta n}\|\Phi_{\rm g}^{(n)}\|^2_{[\mathbb{C}^2 \otimes \mathbb{C}\varphi_n]} < \infty$?

Corollary (Localization)

Let $W(s-r) = e^{i\pi N_{|s-r|}} e^{-\omega|s-r|}$. By the analytic continuation on β we see that

$$(\Phi_{\rm g}, e^{\beta \mathrm{N}}\Phi_{\rm g}) = \mathbb{E}_{\nu_\infty}\left[e^{-g^2(1-e^{\beta\omega})\int_{-\infty}^0 \int_0^\infty W(s-r)\mathrm{d}s\mathrm{d}r}\right] \quad \beta \in \mathbb{C},$$

$$(\Phi_{\rm g}, \mathrm{N}\Phi_{\rm g}) = \omega g^2 \mathbb{E}_{\nu_\infty}\left[\int_{-\infty}^0 \int_0^\infty W(s-r)\mathrm{d}s\mathrm{d}r\right]$$

In particular

$$\|e^{\beta \mathrm{N}}\Phi_{\rm g}\|^2 < \infty, \quad \forall \beta > 0.$$

$\sum_{n=0}^{\infty} \|\Phi_{\mathrm{g}}^{(n)}\|^2 = 1$, but

$$(\Phi_{\mathrm{g}}, (-1)^{\mathrm{N}/\omega}\Phi_{\mathrm{g}}) = \sum_{n=0}^{\infty} (-1)^n \|\Phi_{\mathrm{g}}^{(n)}\|^2 \lesseqgtr 0?$$

Theorem

$(\Phi_{\mathrm{g}}, (-1)^{\mathrm{N}/\omega}\Phi_{\mathrm{g}}) > 0$

Proof: Let $\beta = i\pi/\omega$. Then

$$(\Phi_{\mathrm{g}}, (-1)^{\mathrm{N}/\omega}\Phi_{\mathrm{g}}) = \mathbb{E}_{\nu_\infty}\left[e^{2g^2 \int_{-\infty}^{0} \int_{0}^{\infty} e^{i\pi N_{|s-r|}} e^{-\omega|s-r|} \mathrm{d}s \mathrm{d}r} \right] > 0$$

□

Spectral zetafunction

Let $\omega = 1$ and $p = -id/dx$.

$$H_{\mathrm{Rabi}} + g^2 \cong H_{\mathrm{ren}} = \begin{pmatrix} a^\dagger a & 0 \\ 0 & a^\dagger a \end{pmatrix} - \Delta \begin{pmatrix} 0 & e^{-i2\sqrt{2}gp} \\ e^{+i2\sqrt{2}gp} & 0 \end{pmatrix}$$

Let $\mathrm{Spec}(H_{\mathrm{Rabi}}) = \{E_n(g)\}_{n=0}^{\infty}$.
If $\Delta < 1/4$, then $E_{2n}, E_{2n+1} \in (n - 1/4, n + 1/4)$. Let

$$\zeta_g(s) = \sum_{n=0}^{\infty} \frac{1}{(E_n(g) + g^2)^s}.$$

Let $H = \begin{pmatrix} a^\dagger a & 0 \\ 0 & a^\dagger a \end{pmatrix}$.

Lemma

Let $\tau > 0$ and $s > 0$. Then $s - \lim_{g \to \infty} (H_{\mathrm{ren}} + \tau)^{-s} = (H + \tau)^{-s}$.

Theorem

$$\lim_{g\to\infty}\sum_{n=0}^{\infty}\frac{1}{(E_n(g)+g^2+\tau)^s}=2\sum_{n=0}^{\infty}\frac{1}{(n+\tau)^s}$$

Proof: Let $e_{2n}=\begin{bmatrix}\varphi_n\\ 0\end{bmatrix}$ and $e_{2n+1}=\begin{bmatrix}0\\ \varphi_n\end{bmatrix}$.

$$\lim_{g\to\infty}\zeta_g(s)=\lim_{g\to\infty}Tr(H_{\text{ren}}+\tau)^{-s}=\lim_{g\to\infty}\sum_{n=0}^{\infty}(e_n,(H_{\text{ren}}+\tau)^{-s}e_n)$$
$$=\sum_{n=0}^{\infty}(e_n,(H+\tau)^{-s}e_n)$$

□

The same result is obtained in
Cid Reyes-Bustos and Masato Wakayama, arXiv:2304.08943, 2023.

Summary

- $(f, e^{-tH} g) =$
  $$e^t \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_P \mathbb{E}_\nu \left[\overline{f(X_0^x, \sigma_0^\alpha)} g(X_t^x, \sigma_t^\alpha) e^{-g\sqrt{2\omega} \int_0^t \sigma_s^\alpha X_s^x \mathrm{d}s} \Delta^{N_t} \right] \mathrm{d}\mu(x).$$
- $(1\!\!1, e^{-tH} 1\!\!1) = e^t \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_\nu \left[\Delta^{N_t} \exp\left(\frac{g^2}{2} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}r e^{i\pi N_{|s-r|}} e^{-\omega|s-r|} \right) \right]$
- Prob. meas. $\exists \nu_\infty$ st
  $$(\Phi_g, e^{\beta N} \Phi_g) = \mathbb{E}_{\nu_\infty} \left[e^{-g^2(1-e^{\beta\omega}) \int_{-\infty}^0 \int_0^\infty e^{i\pi N_{|s-r|}} e^{-\omega|s-r|} \mathrm{d}s\mathrm{d}r} \right] \quad \forall \beta \in \mathbb{C}.$$
- $g \mapsto E_0(g)$ is concave and differentiable a.e. in g.
- $\sigma_x (-1)^{N/\omega} \Phi_g = -\Phi_g$.
- $\|e^{\beta N} \Phi_g\|^2 < \infty, \quad \forall \beta > 0.$
- $$\lim_{g \to \infty} \sum_{n=0}^\infty \frac{1}{(E_n(g) + g^2 + \tau)^s} = 2 \sum_{n=0}^\infty \frac{1}{(n+\tau)^s}$$

Thank you.