

微分積分

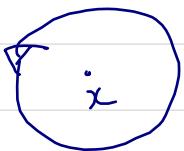
4/10 No.1

§1 記号の説明

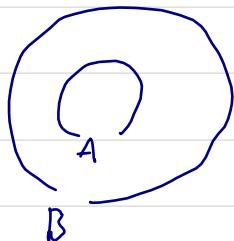
$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Zahl quotient Real complex

$$x \in A$$



$$A \subset B$$



$$A = B \Leftrightarrow A \subset B \text{ かつ } A \supset B$$

$$\{ \quad ; \quad \text{条件} \} \quad \{ x ; |x| < \epsilon \text{ かつ } |x| \neq 0 \}$$

$$[a, b] \subset [a, b] \quad [a, b] \rightarrow \{ x \in \mathbb{R} ; |x| < 1 \}$$

有界閉区間

有界閉区間

, \mathbb{R} .

$$[a, +\infty) \quad (-\infty, b] \quad (a, +\infty) \quad (-\infty, b) \quad (-\infty, \infty)$$

全称記号 \forall と存在記号 \exists

例) 全ての実数 x は $x^2 + x + 1 > 0$.

$$\forall x \in \mathbb{R} \quad x^2 + x + 1 > 0$$

複素数

$$x^2 + x^2 + 2 = 0 \quad \exists \text{ たとえやく } x \text{ が 存在する}$$

\Downarrow

$$\exists x \in \mathbb{C} \quad \text{s.t.} \quad x^2 + x + 1 = 0.$$

$\overline{\text{such that}}$

There exists $x \in \mathbb{C}$ such that $x^2 + x + 1 = 0$

例題 1.7 $\cdot \forall x \in \mathbb{R} \exists y \in \mathbb{R} \text{ s.t. } y > x^2 - 0$

$\cdot \exists y \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R} \text{ s.t. } y > x^2. \times$

三向不等式

Theorem 1. 10 $a, b \in \mathbb{R}$

$$(1) |a+b| \leq |a| + |b|$$

$$(2) | |a| - |b| | \leq |a-b|$$

$$\textcircled{(1)} (1) \bar{a}^2 + 2ab + b^2 \leq a^2 + 2|a||b| + b^2$$

$$(2) |a| \leq |a-b| + |b| \therefore |a| - |b| \leq |a-b|$$

$$|b| \leq |b-a| + |a| \therefore |b| - |a| \leq |a-b|$$

$$| |a| - |b| | \leq |a-b|$$

= 互展開

$$(a+b)^n = \sum_{j=0}^n \binom{n}{j} a^{n-j} b^j$$

$$\textcircled{(1)} n=1 \quad a+b = a+b \quad \sum_{j=0}^1 \binom{1}{j} a^{1-j} b^j = |a+b|$$

$$n=k \text{ 时 } a+b = \sum_{j=0}^k \binom{k}{j} a^{k-j} b^j = (a+b)^k$$

$$\begin{aligned} n=k+1 \quad & a+b = \\ & (a+b)^k (a+b) = \sum_{j=0}^k \binom{k}{j} a^{k-j} b^j (a+b) \\ & = \sum_{j=0}^k \binom{k}{j} \left(a^{k+1-j} b^j + a^{k-j} b^{j+1} \right) \\ & \quad a^{k+1-j} b^j \quad a^{k+1-(j+1)} b^{j+1} \end{aligned}$$

$$\begin{aligned} j+1=i \\ & \sum_{i=1}^{k+1} \binom{k}{i-1} a^{k+1-i} b^i + \sum_{i=0}^k \binom{k}{i} a^{k+1-i} b^i \end{aligned}$$

$$= a^{k+1} + \sum_{i=1}^k \left[\binom{k}{i-1} + \binom{k}{i} \right] a^{k+1-i} b^i + b^{k+1}$$

$$\begin{aligned}
 & a^{k+1} + \sum_{i=1}^k \left[\binom{k}{i-1} + \binom{k}{i} \right] a^{k+1-i} b^i + b^{k+1} \\
 & = a^{k+1} + \sum_{i=1}^k \binom{k+1}{i} a^{k+1-i} b^i + b^{k+1} \\
 & = \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i \quad //
 \end{aligned}$$

§2 数列と関数の不正確

Def 2.3 $\alpha \in \mathbb{R}$ $\{a_n\} \subset \mathbb{R}$ 数列

$\{a_n\}$ は α に収束する

$\iff \forall \varepsilon > 0 \exists N \text{ s.t. } \forall n \geq N \quad |a_n - \alpha| < \varepsilon$

$\varepsilon - N$ の定義を用いて Weierstrass

Example 2.6 $\alpha \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} a_n = \alpha \text{ を示す。}$$

$$S_n = \frac{\sum_{j=1}^n a_j}{n} \quad n \rightarrow \lim_{n \rightarrow \infty} S_n = \alpha$$

$$\because |a_n - \alpha| < \varepsilon \quad (\forall n > N_1)$$

$$S_n - \alpha = \frac{(a_1 - \alpha) + \dots + (a_n - \alpha)}{n}$$

$$= \frac{(a_1 - \alpha) + \dots + (a_{N_1} - \alpha)}{n} + \frac{(a_{N_1+1} - \alpha) + \dots + (a_n - \alpha)}{n}$$

$$\leq \left| \dots \right| + \dots \leq \frac{|a_1 - \alpha| + \dots + |a_{N_1} - \alpha|}{n} + \left(\frac{n - N_1}{n} \right) \varepsilon$$

$$\frac{|a_1 - \alpha| + \dots + |a_{N_1} - \alpha|}{n} < \varepsilon \quad \forall n > N_2 (> N_1)$$

$$\therefore |S_n - \alpha| \leq \varepsilon + \varepsilon = 2\varepsilon \quad \forall n > N_2,$$

Def 2.8

$$\lim_{n \rightarrow \infty} a_n = +\infty \Leftrightarrow \forall L > 0, \exists N \text{ s.t. } a_n > L \quad (\forall n > N)$$

Example 2.6

$$\lim_{n \rightarrow \infty} a_n = +\infty$$

$$S_n = \frac{\sum_{i=1}^n a_i}{n} \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

$$\begin{aligned} & \forall L > 0, \exists N \text{ s.t. } a_n > N \quad \frac{a_1 + \dots + a_{N_1}}{n} + \frac{a_{N_1+1} + \dots + a_n}{n} \\ & \geq \frac{a_1 + \dots + a_{N_1}}{n} + \left(\frac{n - N_1}{n} \right) L \geq -\frac{L}{4} + \frac{L}{2} = \frac{L}{4}, \end{aligned}$$

$$\left| \frac{a_1 + \dots + a_{N_1}}{n} \right| < \frac{L}{4} \quad \frac{n - N_1}{n} > \frac{1}{2} \quad n \geq N_2 > N_1$$

• 4 • 17
No 2

Def 2.9 $\{a_n\}$ 加有界 $\exists M > 0 \text{ s.t. } |a_n| < M \forall n$

Prop 2.11

(1) 亂又車子り は 有界

(2) $a_n \rightarrow \alpha$ 加 $\alpha > \beta \Leftrightarrow \exists N > 0 \text{ s.t. } a_n > \beta \forall n > N$

$\therefore (1) \{a_n\} \lim_{n \rightarrow \infty} a_n = \alpha \text{ は } \therefore$

$|a_n - \alpha| < \varepsilon \quad \forall n > N,$

$\therefore |a_n| \leq |a_n - \alpha| + |\alpha| < \varepsilon + |\alpha| \quad \forall n > N,$

$\max \{a_1, \dots, a_{N+1}\} = m \therefore |a_n| \leq \max \left\{ \frac{\varepsilon + |\alpha|}{m} \right\}$

(2) $\alpha - \beta = \varepsilon > 0$ は \exists .

$\exists N > 0 \quad \forall n > N \quad |a_n - \alpha| < \varepsilon$

$\therefore -\varepsilon < a_n - \alpha < \varepsilon \quad \therefore a_n > \alpha - \varepsilon = \beta$,

問題 2.13

$$\textcircled{1} \quad a_n \rightarrow \alpha \Rightarrow |a_n| \rightarrow |\alpha|$$

$$\textcircled{2} \quad |a_n| \rightarrow |\alpha| \Rightarrow a_n \rightarrow \alpha \text{ の根拠}$$

$$\textcircled{3} \quad (1) \quad \left| |a_n| - |\alpha| \right| \leq |a_n - \alpha| \quad \text{証明}$$

$\forall \varepsilon > 0, \exists N, n > N, |a_n - \alpha| < \varepsilon \Rightarrow |a_n| \rightarrow |\alpha|$

$$| |a_n| - |\alpha| | < \varepsilon \Rightarrow |a_n| \rightarrow |\alpha|$$

$$(2) \quad \left| -1 + \frac{1}{2^n} \right| \rightarrow 1 \quad (\text{証明}) \quad -1 + \frac{1}{2^n} \rightarrow -1.$$

Prop 2.14 $\lim a_n = \alpha \quad \lim b_n = \beta$

$$\textcircled{1} \quad \lim (a_n + b_n) = \alpha + \beta$$

$$\textcircled{2} \quad \lim a_n b_n = \alpha \beta$$

$$\textcircled{3} \quad \beta \neq 0 \quad a_n \neq -\frac{b_n}{\beta} \quad \lim \frac{a_n}{b_n} = \frac{\alpha}{\beta}$$

$\therefore \textcircled{1}$ 各自

$$\textcircled{2} \quad |a_n - \alpha| < \varepsilon \quad \forall n > N_1$$

$$|b_n - \beta| < \varepsilon \quad \forall n > N_2$$

$$|a_n b_n - \alpha \beta| \leq |a_n b_n - \alpha b_n| + |\alpha b_n - \alpha \beta|$$

$$\leq |a_n - \alpha| \cdot \underbrace{|b_n|}_{\leq M(>0)} + |\alpha| |b_n - \beta|$$

$$\leq |a_n - \alpha| M + |\alpha| |b_n - \beta| \leq \varepsilon M + |\alpha| \varepsilon$$

$$= \varepsilon (M + |\alpha|)$$

改めて

$$\exists N_1 \text{ s.t. } \forall n \geq N_1, |a_n - \alpha| < \frac{\epsilon}{M+|\alpha|}$$

$$\exists N_2 \text{ s.t. } \forall n \geq N_2, |b_n - \beta| < \frac{\epsilon}{M+|\beta|}$$

$$\therefore |a_n b_n - \alpha \beta| \leq |a_n - \alpha| M + |\alpha| |b_n - \beta|$$

$$< \frac{M+|\alpha|}{M+|\alpha|} \epsilon = \epsilon \quad \forall n > \max\{N_1, N_2\},$$

(3) Prop 2.11 より $b_n \rightarrow \beta > 0 \therefore \underline{b_n > 0}$

\downarrow 定理 3. $\exists M > 1 \forall n > M$

$$\therefore \left| \frac{a_n}{b_n} - \frac{\alpha}{\beta} \right| = \left| \frac{a_n \beta - \alpha b_n}{b_n \beta} \right|$$

$$\leq \frac{|a_n \beta - \alpha \beta| + |\alpha \beta - \alpha b_n|}{|b_n \beta|} \leq \frac{|a_n - \alpha| |\beta| + |\alpha| |b_n - \beta|}{|b_n \beta|}$$

$b_n > \beta > \frac{\beta}{2} > 0$ すな

$$\leq \frac{|a_n - \alpha| |\beta|}{|\beta|^2 / 2} + \frac{|b_n - \beta| |\alpha|}{|\beta|^2 / 2}$$

$$\therefore |a_n - \alpha| < \frac{2|\beta|}{|\beta|^2} \epsilon, \quad |b_n - \beta| < \frac{2|\alpha|}{|\beta|^2} \epsilon$$

$$\therefore \left| \frac{a_n}{b_n} - \frac{\alpha}{\beta} \right| < \frac{2(|\beta| + |\alpha|)}{|\beta|^2} \epsilon = C \epsilon \text{ とす}$$

$$\exists N'_1 \quad \forall n > N'_1 \text{ s.t. } |a_n - \alpha| < \frac{\varepsilon}{c}$$

$$\exists N'_2 \quad \forall n > N'_2 \text{ s.t. } |b_n - \beta| < \frac{\varepsilon}{c}$$

$\therefore \forall n > \max\{N'_1, N'_2\}$ に a_n, b_n は α, β の近似

$$\left| \frac{b_n}{a_n} - \frac{\beta}{\alpha} \right| < \varepsilon \quad //$$

Prop 2.16 (上界と下界の原理.) $\lim a_n = \alpha$
 $\lim b_n = \beta$

$$(1) \quad a_n \leq b_n \Rightarrow \alpha \leq \beta$$

$$(2) \quad a_n \leq c_n \leq b_n \Rightarrow \alpha = \beta \Rightarrow \lim c_n = \alpha.$$

$$\therefore (1) \quad \beta - \alpha = \underline{\beta - b_n} - \underline{(\alpha - a_n)} + \underline{b_n - a_n}$$

$$(\text{教科書は階差法}) \quad \beta - b_n \geq -\varepsilon \quad \alpha - a_n \leq \varepsilon$$

$$\therefore \beta - \alpha \geq -2\varepsilon + b_n - a_n \geq -2\varepsilon$$

したがって $\beta - \alpha \geq 0$.

$$(2) \quad a_n \leq c_n \leq b_n$$

$$a_n - \alpha > -\varepsilon \quad \therefore a_n > \alpha - \varepsilon$$

$$b_n - \beta < \varepsilon \quad \therefore b_n < \beta + \varepsilon$$

$$\therefore \alpha - \varepsilon < c_n < \beta + \varepsilon \quad \alpha = \beta \text{ 且}$$

$$\alpha - \varepsilon < c_n < \alpha + \varepsilon$$

$$\therefore -\varepsilon < c_n - \alpha < \varepsilon \quad \therefore |c_n - \alpha| < \varepsilon$$

$$\text{Ex 2.19 } 0 < a \leq b \leq c$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a^n + b^n + c^n} = C \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{a}{c}\right)^n + \left(\frac{b}{c}\right)^n + 1} = c$$

$$1 \leq \sqrt[n]{\left(\frac{a}{c}\right)^n + \left(\frac{b}{c}\right)^n + 1} \leq \left(\frac{a}{c}\right)^n + \left(\frac{b}{c}\right)^n + 1 \rightarrow 1$$

• 4 • 24

No. 3

Def 2.8 ① $\forall L > 0 \exists N$ st. $a_n > L \forall n > N$
 つまり $\lim_{n \rightarrow \infty} a_n = +\infty$ と表す。

② $\forall L < 0 \exists N$ st. $a_n < L \forall n > N$
 つまり $\lim_{n \rightarrow \infty} a_n = -\infty$

(注) 発散する = 収束しない (振動も入る)

Prop 2.20 $a_n \leq b_n$

(1) $\lim_{n \rightarrow \infty} a_n = +\infty \Rightarrow \lim_{n \rightarrow \infty} b_n = +\infty$

(2) $\lim_{n \rightarrow \infty} b_n = -\infty \Rightarrow \lim_{n \rightarrow \infty} a_n = -\infty$

(\because) $\forall L \exists N$ st. $a_n > L \forall n > N$ かつ $L' = L$

$\forall L' \exists N'$ st. $b_n > L' \forall n > N'$ すなはち $L' = L$

したがって

$\forall L \exists N$ st. $b_n \geq a_n > L \forall n > N$
 $\therefore b_n > L \forall n > N$ //

(2) 同じ

$$\frac{1+2+\dots+n}{n+h-1+\dots+1} \rightarrow \frac{n(n+1)}{2}$$

Ex $\lim_{n \rightarrow \infty} \sqrt[n]{n!}$

$$(n!)^2 = \prod_{k=1}^n k(n-k+1) = 1 \cdot n \cdot 2(n-1) \cdots n \cdot 1 = (n!)^2$$

$$k(n-k+1)-n = kn-n-k^2+k = (k-1)n - k(k-1) = (k-1)(n-k) \geq 0$$

$$\therefore k(n-k+1) \geq n$$

$$(n!)^2 \geq n^n \therefore n! \geq \sqrt{n^n} \therefore \underline{\sqrt{n!}} \geq \sqrt{n} \rightarrow \infty$$

$$\text{Ex } \lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0 \quad (a > 0)$$

$\because \frac{a}{\sqrt[n]{n!}} \rightarrow 0 \quad (n \rightarrow \infty)$

正確~~解~~ = 1154

$$\frac{a}{\sqrt[n]{n!}} \leq \frac{a}{\sqrt{n}} \quad (\text{なぜか}) \quad \forall \varepsilon, \exists N \text{ s.t. } \frac{a}{\sqrt[n]{n!}} < \varepsilon \quad (\forall n > N)$$

$$\text{実際 } N \in \mathbb{Z} \quad \left(\frac{a}{\sqrt{n}}\right)^2 < \varepsilon^2 \quad \text{と} \quad n > N \quad \therefore \frac{a^n}{n!} < \varepsilon^n < \varepsilon \quad \forall n > N$$

$$n_1 < n_2 < n_3 < \dots$$

$\{a_{n_k}\}$ $\{a_n\}$ の部分列のことを.

$$\lim_{n \rightarrow \infty} a_n = \alpha \quad n \in \mathbb{Z} \quad \lim_{k \rightarrow \infty} a_{n_k} = \alpha$$

$\because \forall \varepsilon \exists N \text{ s.t. } |a_n - \alpha| < \varepsilon \quad \forall n > N \quad (\text{なぜか})$

$$\forall \varepsilon \exists N' \text{ s.t. } |a_{n_k} - \alpha| < \varepsilon \quad \forall k > N' \quad (\text{なぜか})$$

実際 $N' \in \mathbb{Z}$ $n_{N'} \geq N' + 1 \geq N + 1 \geq n$.

§ 2.2 陰爻の本質

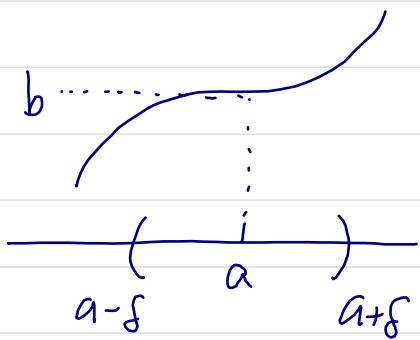
$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$\text{Def 2.25} \quad \lim_{x \rightarrow a} f(x) = b \quad \text{if } \forall \epsilon > 0 \exists$$

$$\forall \varepsilon \exists \delta > 0 \text{ such that } \forall x (0 < |x - a| < \delta) \Rightarrow |f(x) - b| < \varepsilon.$$

注 a_n は $\frac{1}{n}$ の形の数である。

$f(x)$ x は連続 $f(x)$



Def 2.27

$$(1) \lim_{n \rightarrow \infty} f(n) = +\infty$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (|x - a| < \delta) f(x) > L$$

$$(2) \lim_{x \rightarrow a} f(x) = -\infty \quad \text{且 同上.}$$

$$(3) \lim_{x \rightarrow +\infty} f(x) = b \quad 4/12$$

$$\forall \varepsilon > 0 \exists R > 0 \text{ such that } \forall x (x > R) \text{ implies } |f(x) - b| < \varepsilon$$

$$(4) \lim_{x \rightarrow -\infty} f(x) = b \text{ 且 } r$$

$\forall \varepsilon > 0 \exists R > 0 \text{ such that } \forall x (|x| > R) \Rightarrow |f(x) - b| < \varepsilon$

$$(5) \lim_{x \rightarrow +\infty} f(x) = +\infty \text{ 且}$$

$$\forall L > 0 \exists R > 0 \forall x (|x| > R) \Rightarrow f_n(x) > L.$$

以下同様.

Def 2.28 $\lim_{x \rightarrow a+0} f(x) = b$ は 

$\forall \varepsilon > 0 \exists \delta > 0 \forall n (\alpha < x < \alpha + \delta) \mid f(x_n) - b \mid < \varepsilon$

$\lim_{x \rightarrow a-0} f(x) = b$ は

$\forall \varepsilon > 0 \exists \delta > 0 \forall n (\alpha - \delta < x < \alpha) \mid f(x_n) - b \mid < \varepsilon$.

$\lim_{x \rightarrow a+0} f(x) = +\infty$ は

$\forall L \exists \delta > 0 \forall n (\alpha < x < \alpha + \delta) \mid f(x_n) \mid > L$.

L \times F \bar{A} U.

$\varepsilon - \delta$ 論法

間数の極限

$\varepsilon - N$ 論法

数列の

はさみうちの原理 $g(x) \leq f(x) \leq h(x)$

$\lim_{x \rightarrow a} g(x) = a = \lim_{x \rightarrow a} h(x) \quad \text{or } \lim_{x \rightarrow a} f(x) = a$

$\therefore \forall \varepsilon_1 \exists \delta_1 > 0 \forall x (\mid x - a \mid < \delta_1) \mid g(x) - a \mid < \varepsilon_1$

$\exists \delta_2 > 0 \forall x (\mid x - a \mid < \delta_2) \mid h(x) - a \mid < \varepsilon_2$

$$(2) a_n \leq c_n \leq b_n$$

$$a_n - \alpha > -\varepsilon \quad \therefore a_n > \alpha - \varepsilon$$

$$b_n - \beta < \varepsilon \quad \therefore b_n < \beta + \varepsilon$$

$$\therefore \alpha - \varepsilon < c_n < \beta + \varepsilon \quad \alpha = \beta \text{ 且}$$

$$\alpha - \varepsilon < c_n < \alpha + \varepsilon$$

$$\therefore -\varepsilon < c_n - \alpha < \varepsilon \quad \therefore |c_n - \alpha| < \varepsilon \quad //$$

上證証明：

$$g(x) - \alpha > -\varepsilon \quad \therefore g(x) > \alpha - \varepsilon \quad \forall x \quad |x - a| < \delta_1$$

$$h(x) - \alpha < \varepsilon \quad h(x) < \alpha + \varepsilon \quad \forall x \quad |x - a| < \delta_2$$

$$\alpha - \varepsilon < f(x) < \alpha + \varepsilon \quad \forall x \quad |x - a| < \delta_3$$

$$\therefore -\varepsilon < f(x) - \alpha < \varepsilon \quad \text{where } \delta_3 = \min\{\delta_1, \delta_2\}$$

$$\therefore \exists \delta_3 \in \mathbb{R}^+ \quad \forall x \quad (|x - a| < \delta_3) \quad |f(x) - \alpha| < \varepsilon.$$

]

$$\lim_{n \rightarrow \infty} n \sin \frac{1}{n} \quad 0 \leq |n \sin \frac{1}{n}| \leq |x|$$

$$\therefore \lim_{n \rightarrow \infty} n \sin \frac{1}{n} = 0.$$

$$\text{Prop 2.31} \quad \lim_{x \rightarrow a} f(x) = b$$

$$\Leftrightarrow \lim_{x \rightarrow a+0} f(x) = b \Leftrightarrow \lim_{x \rightarrow a-0} f(x) = b$$

$\therefore (\Rightarrow)$ 由の用

(\Leftarrow)

$$\{a \in \mathbb{R} : \exists \delta_1, \delta_2 \}$$

$$\forall \epsilon \exists \delta_1 \quad \forall x (a < x < a + \delta_1) \quad |f(x) - b| < \epsilon$$

$$\delta = \min \{\delta_1, \delta_2\} \quad \text{を定義}$$

$$\forall x (|x - a| < \delta) \quad \therefore |f(x) - b| < \epsilon \quad //$$

Def 2.32 \Rightarrow [左]極限の定義

(1)

$a \in \mathbb{R}$. f が a の左側で連続

$$\lim_{x \rightarrow a^-} f(x) = f(a^-) \quad T_1, T_2 \in \mathbb{R} \quad a \in T_1 \quad \text{左端点}$$

左極限
左端点

$a \in T_2$

$a \in T_2 \quad a \in T_1 \quad \text{左極限}$

(2) $\forall a \in \mathbb{R}$ 連続 $\exists \epsilon \in \mathbb{R} \quad \forall x \in \mathbb{R} \quad |x - a| < \epsilon \Rightarrow |f(x) - f(a)| < \epsilon$.

Prop 2.33 f が a の cont \downarrow 考察する
 $\Leftrightarrow a = \text{数列の極限} \forall_{\epsilon > 0} \exists N \in \mathbb{N} \text{ 使得} \lim_n f(a_n) = f(a)$

$\therefore (\Rightarrow) \forall \epsilon \exists \delta \text{ 使得 } \forall x (|x - a| < \delta) |f(x) - f(a)| < \epsilon$

左辺 $= \exists \delta \exists N \text{ 使得 } |a - a_n| < \delta \forall n > N.$

右辺 $\forall \epsilon \exists N \text{ 使得 } |f(a_n) - f(a)| < \epsilon \quad (\forall n > N),$

(\Leftarrow) 背理法 f が a の cont でないことを假定する

つまり cont は否定する:

$\exists \epsilon \text{ 使得 } \forall \delta \exists x (|x - a| < \delta) |f(x) - f(a)| \geq \epsilon$

$\delta = \frac{1}{n} \text{ 使得 } \exists x_n (|x_n - a| < \frac{1}{n}) \text{ 使得 } |f(x_n) - f(a)| \geq \epsilon$

$x_n \in \{x_n\} \subset \{x_n\} = \{x_n\}$

$\therefore \lim_n x_n = a$

$\therefore \lim_n |f(x_n) - a| = 0$ しかしこれは矛盾 \therefore cont (a).

第3章 実数の連続性

有理数 \mathbb{Q} の存在を仮定して実数を定義する

Dedekind Schnitt (=cut)

切断 — 実数

(1) 有理数

Def 3.1 $\mathcal{N} \supset A, B$ ($\neq \emptyset$) が
 \mathcal{N} の切断とは 次の 2 条件を満たす

(1) $A \cap B = \emptyset, A \cup B = \mathcal{N}$

(2) $a \in A, b \in B \Rightarrow a < b$



ここで \mathcal{N} は 川原序の二つた数の集合

漸くは $\mathcal{N} = \mathbb{Q}$ といふ \mathbb{Q} の接続性を示す

実数の公理

1) 四則演算

$$\alpha + \beta$$

① 可換 ② 結合 ③ ④ 逆

$$\alpha \times \beta$$

① “ ② “ ③ ④ $\frac{\exists 1}{\exists 4}$

$$⑤ (\alpha + \beta) \times \gamma = \alpha \gamma + \beta \gamma \quad (\neq 0)$$

2 順序

$$\left| \begin{array}{l} \alpha \leq \alpha \quad \alpha \leq \beta \Leftrightarrow \beta \geq \alpha \\ \alpha \leq \beta \quad \beta \leq \gamma \rightarrow \alpha \leq \gamma \end{array} \right.$$

$$① \alpha < \beta, \alpha = \beta, \alpha > \beta \text{ 不定}$$

$$② \alpha \leq \beta \rightarrow \alpha + \gamma \leq \beta + \gamma$$

$$\alpha \leq \beta, \gamma > 0 \rightarrow \alpha \gamma \leq \beta \gamma$$

3 連続性

接続形には 4>の可換でないもの

$$(1) A := \max_{x \in \mathbb{Q}} \alpha, B := \min_{x \in \mathbb{Q}} \beta$$

$$(2) A := \max_{x \in \mathbb{Q}} \beta, B := \min_{x \in \mathbb{Q}} \alpha$$

$$(3) A := \max_{x \in \mathbb{Q}} \beta, B := \min_{x \in \mathbb{Q}} \alpha$$

$$(4) A = \max_{x \in \mathbb{Q}} \alpha, B = \min_{x \in \mathbb{Q}} \beta$$

$$(4) \text{ は } \max = \alpha, \min = \beta \text{ と 3つで } \frac{\alpha + \beta}{2}$$

$$\alpha < \frac{\alpha + \beta}{2} < \beta \quad \therefore \frac{\alpha + \beta}{2} \in A \cap B \leftarrow \text{プロト}$$

例 (1) $A = \{x \in \mathbb{Q}; x < 0\}, B = \{x \in \mathbb{Q}; x \geq 0\}$

$$(2) \text{ は } 1$$

$$(3) A = \{x \in \mathbb{Q}; x^2 > 2 \Rightarrow x < 0\}$$

$$B = \mathbb{Q} \setminus A$$

(1) と (2) の 場合 \max or $\min (\in \mathbb{Q})$ が存在する

例 1 例 (1) の は $A \cap \max r \in \mathbb{Q}$

$$A = \{x \in \mathbb{Q}; x \leq r\} \quad B = \{x \in \mathbb{Q}; x > r\}$$

$$A' = \{x \in \mathbb{Q}; x < r\} \quad B' = \{x \in \mathbb{Q}; x \leq r\}$$

(A, B) と (A', B') は 同じ もの で 可

$$r \in \mathbb{Q} \leftrightarrow \text{接続形 } (1) (A, B) \text{ or } (2) (A', B')$$

接続形の全体 \mathcal{D} と表す。

$$\{(A, A') \in \mathcal{D}; (A, A') \text{ は (1) or (2)}\} = \mathcal{D}$$

Dedekind の順序は 大小と演算を導入する:

$$A = (A, A') \quad B = (B, B') \quad (= \neq, \subset)$$

$$A \subset B \iff A' \subset B$$
$$A \subset B \text{ かつ } A \neq B \implies A \subset B$$

大小関係の公理

$$A \leq A$$

$$A \leq B, B \leq A \Rightarrow A = B$$

$$A \subset B, B \subset C \rightarrow A \subset C$$

$$A \leq B \text{ または } A \geq B \text{ 成立}$$

① $C' = \{a' + b' \mid a' \in A', b' \in B'\}$

$$C = \mathbb{Q} \setminus C$$

(C, C') は順序: \subset , \leq

加群 $\left\{ \begin{array}{l} a+b = b+a \\ (a+b)+c = a+(b+c) \\ A+0 = A \\ A+B = 0 \end{array} \right.$

② $A + B = 0 \iff B = -A \iff c$

$$\iff A - B = A + (-B) \iff c$$

$$A \leq B \rightarrow A + C \leq B + C$$

$$A \leq 0 \Rightarrow A \in \text{負の整数}$$

$$A \geq 0 \Rightarrow A \in \text{正の整数}$$

$$\textcircled{X} \quad C' = \{ a'b' \mid a' \in A, b' \in B \} \quad (A \geq 0, B \geq 0)$$

$$C = (C'^c, C') = \forall z \ A \times B = C$$

と定めよ:

$$A \times B = -((\neg A) \times B) \quad A \leq 0, B \geq 0$$

$$A \times B = - (A \times (\neg B)) \quad A \geq 0, B \leq 0$$

$$A \times B = +A \vee (\neg B) \quad A \leq 0, B \leq 0$$

$$A \times B = B > A$$

$$A \times (B \times C) = (A \times B) \times C$$

$$A \times \overline{1} = A$$

$$A \times \overline{B} = 1 \quad A \neq 0 \quad B = \frac{1}{A} \text{ とする。}$$

$$\textcircled{\text{÷}} \quad A \div B = A \times \frac{1}{B} \quad \text{と定めよ。}$$

+ × の関係: $A \times (B+C) = A \times B + A \times C$
 $0 \neq 1$

\mathbb{Q} には 大小関係，四則演算が定められた。
一一が 積互

- 大小。
- 四則演算の公理

$$\mathbb{Q} \supset \mathbb{Q} \quad Q \hookrightarrow Q \quad 1:1 \text{ onto}$$
$$\therefore S : Q \rightarrow Q$$

$$P(A+B) = P(A) + P(B)$$

$$P(A-B) = P(A) - P(B)$$

$$P(AB) = P(A)P(B)$$

$$P\left(\frac{A}{B}\right) = P(A)/P(B)$$

$\mathbb{Q} \setminus Q$ は無理数と呼ばれる。

\mathbb{A} ： 実数

定理 \mathbb{A} の切片全体 $\tilde{\mathbb{A}}$

$\tilde{\mathbb{A}} \ni (A|B)$ と呼ぶ

(1) A の max と B の min

(2) A の max と B の min

これが実数の連続性である

$$\mathbb{Q} = \mathbb{Q}$$

? と思ふ

$$\mathbb{Q} \subset \mathbb{D}$$

\mathbb{D} には 大小、四則演算が定義される

$$\text{例) } (A, B) \quad (A', B')$$

$$\rightarrow (A, B) + (A', B') = (A + A', \frac{B}{A + A'})$$

↑
七則運算による

$\mathbb{D} \setminus \mathbb{Q}$ を無理数といい、 \mathbb{D} の実数部分。

\mathbb{D} の接続性は (1) と (2) しか走行しない。

(Dedekind の公理)

Dedekind の接続性により具体的に実数の存在を証明した。これは実数の連続性を示す。

上界と下界

Def 3.3 $S \subset \mathbb{R}$ / $\exists x \in \mathbb{R}$ st $s \leq x \forall s \in S$ $\Rightarrow S$ は上界

Def 3.3a $a \in \mathbb{R}$ 上界 an upper bound など

Thm 3.4 上界をもつ $S (\neq \emptyset)$ の上界は
最小のものである

$\therefore B = \{x \in \mathbb{R} ; x \text{ は } S \text{ の上界}\}$

$B^c = A$

$\mathbb{R} = A \cup B, A \cap B = \emptyset, A \neq \emptyset, B \neq \emptyset$

(A, B) は切妻形

$\therefore a \in A, b \in B$

$s \leq b \forall s \in S$

a は上界 $\exists s_0 \in S$ st $a \leq s_0$

$\therefore a < s_0 \leq b \therefore a < b //$

Redekind の公理 $\exists \max A \text{ or } \exists \min B$

$\exists \max A = a \in \mathbb{Z} \quad a \in B (= A)^c \text{ で } S$

$\exists s_1 \in S$ st $a < s_1$

$a < \frac{a+s_1}{2} < s_1 \therefore \frac{a+s_1}{2} \notin B$



$\therefore a < \frac{a+s_1}{2} \in B^c = \text{上界矛盾}$

$\therefore \exists \min B.$

Def 3.5

上に有界な集合 S の 上界の最小値を

S の 上限 $\sup S : \frac{\text{表記}}{\text{英語}}$.

$\sup S \in S$
or
 $\sup S \notin S$

同様に 下に有界な集合 S ($\neq \emptyset$)

$\Leftrightarrow \exists a \in \mathbb{R} \text{ s.t. } a \leq s \quad \forall s \in S$

a 全体を 下界 lower bound とする

下界の最大値 存在し $\inf S : \frac{\text{表記}}{\text{英語}}$

(注) \mathbb{R} の性質. \mathbb{R} の連続性から導かれる

(2) \mathbb{R} の連続性は \Leftrightarrow 定理 3.4

Th 3.8 ① の 示密・下界.

$\mathbb{R} \ni x, y \rightarrow x < y . \exists r \in \mathbb{Q}$

s.t. $x < r < y$

$\exists m \in \mathbb{Z}$

$\therefore n(y-x) > 2 \quad \forall n \in \mathbb{N} \quad nn < m < ny$

$\therefore x < \frac{m}{n} < y \quad ,$

Th 3.8' 無理の示密・上界

$\mathbb{R} \ni x, y \rightarrow x < y \quad = \exists z \quad \exists r \in \mathbb{R} \setminus \mathbb{Q}$

s.t. $x < r < y \quad n(y-x) > 2 > \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

$\therefore x < \frac{\sqrt{2}}{n} < y \quad = z \quad \frac{\sqrt{2}}{n} \in \mathbb{R} \setminus \mathbb{Q}$

$\therefore \frac{\sqrt{2}}{n} = \frac{q}{p} \quad \therefore \sqrt{2} = \frac{nq}{p} \quad \checkmark$

3.3 有界單調數列の収束

$\{a_n\}$

Def 3.12 $a_1 \leq a_2 \leq \dots$

単調増加

$a_1 < a_2 < \dots$

単調減少

$a_1 > a_2 > \dots$

單調減

$a_1 > a_2 > \dots$

反單調減

The 3.13 有界単調数列の収束定理

$\therefore \{a_n\} \uparrow$ は単調増加
 $\{a_1, a_2, \dots\} = A, \alpha = \sup A = (\sup_n a_n : \mathbb{R})$

$\underline{\underline{a_n \leq \alpha}} \quad \forall n \quad \text{そして } \lim_{n \rightarrow \infty} a_n = \alpha$

$\therefore \exists \varepsilon$ $\alpha - \varepsilon$ は A の上界である (当然)

$\alpha - \varepsilon < \sup^{\exists} a_N \leq a_n \leq \alpha < \alpha + \varepsilon$
 \uparrow
 単調増加

$-\varepsilon < a_n - \alpha < \varepsilon \quad \forall n > N$

$\therefore |a_n - \alpha| < \varepsilon \quad \forall n > N$

$\{a_n\} \downarrow$ は単調減少

$\{a_n\} = B \quad \beta = \inf B = \inf a_n$

$\text{そして } \lim_{n \rightarrow \infty} a_n = \beta$

$\therefore \forall \varepsilon \quad \beta + \varepsilon$ は B の上界である

$\beta + \varepsilon > a_N \geq a_n \geq \beta > \beta - \varepsilon \quad \forall n \geq N$

$\therefore |a_n - \beta| < \varepsilon \quad \forall n \geq N$

Prop 3.14 $A \subset \mathbb{R}$ ($A \neq \emptyset$)

(1) A 有上界 $\exists a_n \in A$ s.t. $\lim_{n \rightarrow \infty} a_n = \alpha$
 $\alpha = \sup A$

(2) A 有上界 $\exists a_n \in A$ s.t. $\lim_{n \rightarrow \infty} a_n = \infty$

$\because \alpha \in A$ 且 $a_n = \alpha$ 时

(1) $\alpha \notin A$ 时

$$a_1 < \alpha$$

ok

$$\frac{1}{2}(a_1 + \alpha) < \alpha \quad \therefore \frac{1}{2}(a_1 + \alpha) < a_2 < \alpha$$

$$\text{由 } a_2 \text{ 为 } \sup A \leq \frac{1}{2}(a_1 + \alpha)$$

$a_1 < a_2 < \dots$

\vdots

$$a_1 \leq a_2 \leq \dots \quad \lim_{n \rightarrow \infty} a_n = \beta$$

$$\frac{1}{2}(a_n + \alpha) < a_{n+1} < \alpha$$

\Downarrow

$$\frac{1}{2}(\beta + \alpha) < \beta < \alpha$$

$$\textcircled{1} \quad \beta \leq \alpha \quad \textcircled{2} \quad \frac{1}{2}\alpha \leq \frac{1}{2}\beta \quad \therefore \alpha \leq \beta \quad //$$

(2) $a_1 \in A$ ($a_1 > 0$)

$$2a_1 < a_2, \quad (\text{'cause})$$

$$2a_n < a_{n+1} \quad \therefore a_n \geq 2^{n-1}a_1 \rightarrow \infty //$$

Thm 3.31

Bolzano-Weierstrass の 定理

有限な数の間に必ず収束する部分がある

prop 3.14 (1) で すべてに証明

∴ $S = \{ n \in \mathbb{N} ; \forall p > n \text{ で } a_n < a_p \}$

(1) $\# S = \infty$ n_1, n_2, \dots で $a_{n_1} < a_{n_2} < \dots$ である
∴ はるかに増加

(2) $\# S < \infty$ $\max S = n_0$

$n_1 = n_0 + 1 \notin S \quad \therefore \exists n_2 \text{ で } a_{n_2} \leq a_{n_1}, \quad n_2 > n_1,$
 $n_2 \notin S \quad \exists n_3 \text{ で } a_{n_3} \leq a_{n_2}, \quad n_3 > n_2$

この操作を繰り返す

Def: $\{a_n\}$ が Cauchy 異なる
 $\Leftrightarrow \forall \varepsilon > 0 \exists N \text{ で } |a_n - a_m| < \varepsilon \quad \forall n, m > N$

① Cauchy 異なるは有限集合OK

② はるかに増加する

$\therefore \forall \varepsilon > 0 \exists N \text{ で } |a_n - \alpha| < \varepsilon \quad (\forall n > N)$

ここで $\forall \varepsilon > 0 \exists N' \text{ で } |a_n - \alpha| < \frac{\varepsilon}{2} \quad (\forall n > N')$

$\therefore |a_n - a_m| \leq |a_n - \alpha| + |\alpha - a_m| < \varepsilon \quad (\forall n, m > N)$

∴ Cauchy 異なる

逆もいえ Cauchy 3n が α に収束する

∴ 4次束する部分列 $\{a_{n_k}\}$ で $\lim_{k \rightarrow \infty} a_{n_k} = \alpha$

$\forall \varepsilon > 0 \exists N \ni |a_n - \alpha| < \varepsilon (\forall n > N) \text{ で } \forall$

$\forall \varepsilon > 0 \exists N \ni |a_n - a_m| < \frac{\varepsilon}{2}$

部分列 $\{a_{n_k}\}$ で $\forall n, m > N$ で $|a_{n_k} - \alpha| < \frac{\varepsilon}{2}$

$(n_k > m)$

$|a_n - \alpha| \leq |a_n - a_{n_k}| + |a_{n_k} - \alpha| < \varepsilon$

$\forall n > N, \forall n_k > \max\{N, M\}$

つまり $\forall \varepsilon > 0 \quad |a_n - \alpha| < \varepsilon \quad (\forall n > N) ,$

13) $a_n = \left(1 + \frac{1}{n}\right)^n$ は 単調, 有界

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.71828\ldots$$

$$a_n < e < b_n$$

$$\prod_{j=1}^n a_j < e^n < \prod_{j=1}^n b_j \quad \text{ok}$$

$$a_1 \dots a_n = \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \dots \left(\frac{n+1}{n}\right)^n$$

$$= \frac{(n+1)^n}{n!} = \frac{n^n}{n!} \left(1 + \frac{1}{n}\right)^n$$

$$b_1 \dots b_n = \left(\frac{2}{1}\right)^2 \left(\frac{3}{2}\right)^3 \dots \left(\frac{n+1}{n}\right)^{n+1}$$

$$= \frac{(n+1)^{n+1}}{n!} = \frac{n^{n+1}}{n!} \left(1 + \frac{1}{n}\right)^{n+1}$$

$$\frac{n^n}{n!} \left(1 + \frac{1}{n}\right)^n < e^n < \frac{n^{n+1}}{n!} \left(1 + \frac{1}{n}\right)^{n+1}$$

$$\frac{n}{\sqrt[n]{n!}} \left(1 + \frac{1}{n}\right) < e \quad \text{ok}$$

$$e < \frac{n}{\sqrt[n]{n!}} \sqrt[n]{n} \left(1 + \frac{1}{n}\right)^{1+\frac{1}{n}} \quad \sqrt[n]{n} \rightarrow 1$$

$$\therefore \frac{n}{\sqrt[n]{n!}} < \frac{e}{1 + \frac{1}{n}} \rightarrow e$$

$$\therefore \frac{e}{\sqrt[n]{n} \left(1 + \frac{1}{n}\right)^{1+\frac{1}{n}}} < \frac{n}{\sqrt[n]{n!}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e$$

2018 · 5 · 23

No. 6.

§ 3, 4 連続関数の基本的性質

復習 \mathbb{R} の割り算, 大小, 連続

① $\mathbb{R} = A \cup B, A \cap B \neq \emptyset, \forall a \in A, \forall b \in B$
 $\rightarrow a < b$
 $\Leftrightarrow (A, B)$ 割り算

② (A, B) 割り算 $\Rightarrow \exists \max A$ または $\exists \min B$
のどちらか一方が成り立つ

③ $A \subset \mathbb{R}$ 上界有界 $\Rightarrow \exists \sup A$ (上界の min)
下界 $\Rightarrow \exists \inf A$ (下界の max)

④ A 上界有界 $\Rightarrow \exists \{a_n\}$ 且 $a_n \rightarrow \sup A$
下界 $\Rightarrow \exists \{b_n\}$ 且 $b_n \rightarrow \inf A$

Thm 3.24 中間値の定理

$I = [a, b]$ b'dd, closed

f is I " cont.

$f(a) < f(b)$ ($f(a) > f(b)$)

$\exists \gamma \in I$ $\forall \gamma (f(a) < \gamma < f(b))$ とされる

$\exists c \in (a, b)$ し $f(c) = \gamma$.

$\therefore A = \{x \in I ; f(x) < \gamma\}$ 上に有界

$\therefore \sup A = c \quad f(a) < \gamma \therefore a \in A$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ し $\forall x (|x-a| < \delta) \rightarrow f(x) < \gamma$

$|f(x) - f(a)| < \varepsilon$. つまり

$- \varepsilon < f(x) - f(a) < \varepsilon$

$f(a) - \varepsilon < f(x) < f(a) + \varepsilon \quad \forall x (|x-a| < \delta)$

$\exists \delta' = \varepsilon + \delta > 0 \quad \forall x (f(x) < \gamma \quad \forall x (|x-a| < \delta'))$

$\therefore a \in A$

$\exists b \quad \gamma < f(b) \quad \therefore b \in A^c$

f cont

$f(x) > \gamma \quad \forall x (|x-b| < \delta)$

つまり

$c < b$

$a < c < b$ ①

prop 3.14 ジ) $\exists x_n \in A$ し $x_n \rightarrow c$

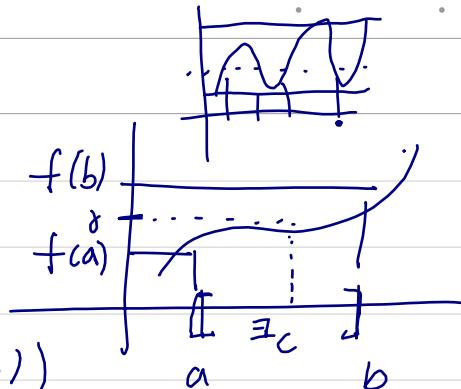
$\therefore f(c) = \liminf f(x_n) \leq \gamma$

EL $f(c) < \gamma$ なぜか f の cont だから

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x (|x-c| < \delta) \quad |f(x) - f(c)| < \varepsilon$

$\therefore -\varepsilon + f(c) < f(x) < f(c) + \varepsilon < \gamma \quad \forall x (|x-c| < \delta)$

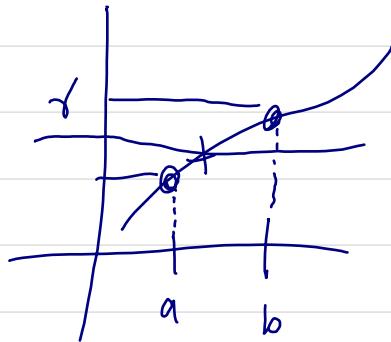
$\therefore -\delta + c < x < c + \delta$ は A に含まれる \therefore ②が成り立



$$(-\delta + c, c + \delta) \subset A \quad \text{よって} \quad \sup A = c$$

$\therefore f(c) = \gamma$

//



Thm 3.35 (関数の連続性と介値定理)

$I = [a, b]$ finite, closed f は cont

\Rightarrow $\exists z \in I$ で $f(z) = \max_{x \in I} f(x)$

∴ $R = f(I)$

(i) R は上に有界 $\Leftrightarrow R$ が上に有界である

$\exists y_n \in R$ かつ $y_n \rightarrow \infty$. $y_n = f(x_n)$, $x_n \in I$

$x_n \in I$ は上に有界. $\exists x_{n_k} \rightarrow x \in I$

$\therefore \lim_{k \rightarrow \infty} f(x_{n_k}) = f(x) < \infty$ \Rightarrow $f(x) = \sup_{x \in I} f(x)$ ($a \leq x_{n_k} \leq b$)
 $(= y_{n_k} \rightarrow \infty)$

(ii) R は下に有界,,

(iii) $\sup R = M$, $\inf R = m$ とする

$\exists y_n \in R$ かつ $y_n \rightarrow M$ $\Rightarrow \exists x_n \in I$ で
 $y_n = f(x_n)$, 且つ x_n の部分列 $x_{n_k} \rightarrow x \in I$

$\therefore \lim_{k \rightarrow \infty} y_{n_k} = M$ $\Rightarrow \lim_{k \rightarrow \infty} f(x_{n_k}) = f(x) \therefore f(x) = M$.

$f(\beta) = m$ で β が示せ.

§4 1変数関数の微分

$$\exists \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

二の極限
 $\underset{a \in \text{接線とn}}{\sqrt{f'(a) + \frac{1}{h}}}$

a は おけい f の 微分係数 とす。

$$\begin{aligned} \lim_{h \rightarrow 0} &\Rightarrow \lim_{h \rightarrow 0^+} \quad \text{右微分可能} \quad f'_+(a) \\ &\Rightarrow \lim_{h \rightarrow 0^-} \quad \text{左微分可能} \quad f'_-(a) \end{aligned}$$

Prop 4.1 f が a で 微分可能 $\Rightarrow f$ は a で 連続

$\therefore \forall \varepsilon > 0$ が given とす。 $\exists \delta > 0$ おいて $\forall x (|x-a| < \delta)$

$$\left| \frac{f(x) - f(a)}{x-a} - f'(a) \right| < \varepsilon$$

$$\Rightarrow |f(x) - f(a)| \leq |f(x) - f(a) - f'(a)(x-a)|$$

$$\leq |x-a| (\varepsilon + |f'(a)|) + |(x-a)f'(a)|$$

$\forall x (|x-a| < \delta) \Rightarrow \text{成立}$

$$\therefore \forall \varepsilon' > 0, \quad \delta = \frac{\varepsilon'}{\varepsilon + |f'(a)|} \quad \text{とすれば}$$

$$\forall x (|x-a| < \delta) \Rightarrow |f(x) - f(a)| \leq \varepsilon'$$

$$|f(x) - f(a)| \leq \varepsilon' \quad //$$

No. 6. 2018. 5. 29

J : open set at \bar{x} $f(x) \rightarrow \frac{1}{\bar{x}}$ same
 $f \in C^1(\bar{x})$ のとき $\exists f'(x)$ $\forall x \in J$

$x \mapsto f'(x)$ は C^1 関数 な.

$f'(x), \frac{df}{dx}(x), \frac{d}{dx}f(x)$. ここで $\exists \frac{1}{x}$ を C^1

prop 4.3 (1) $(f+g)' = f' + g'$
(2) $(\alpha f)' = \alpha f'$ $\alpha \in \mathbb{C}$) 線形性

(3) $(fg)' = f'g + fg'$

(4) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$\therefore (3)$ の証 \vdots

$$fg' = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left\{ \frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right\}$$

$$\downarrow \quad \downarrow$$

$$f'(x) \quad g'(x)$$

$$\downarrow \quad \downarrow$$

$$g'(x) \quad //$$

$$\text{Ex 1) } x^2 \sin \frac{1}{x} = f(x), \quad 0 = f(0)$$

$$\frac{f(0+h) - f(0)}{h} = \frac{h^2 \sin \frac{1}{h}}{h} = h \sin \frac{1}{h} \rightarrow 0$$

连续函数的导数是可积的

Prop 4.5 $f(x)$ 在 $x=a$ 处连续 \Leftrightarrow
 $\exists \epsilon > 0$ 使得 $|x-a| < \delta$ 时 $|f(x) - f(a)| < \epsilon$

$$= \text{def } f'(a) = \varphi(a)$$

$$\exists \delta > 0 \text{ 使得 } |x-a| < \delta \Rightarrow |\varphi(x) - \varphi(a)| < \epsilon$$

$$\therefore (\Rightarrow) \quad \varphi(x) = \begin{cases} \frac{f(x) - f(a)}{x-a} & x \neq a \\ f'(a) & x = a \end{cases} \quad \text{if } a \text{ is cont}$$

$$\text{2. } f(x) = f(a) + (x-a) \varphi(x)$$

$$(\Leftarrow) \quad \frac{f(x) - f(a)}{x-a} = \varphi(x) \quad x \rightarrow a \text{ 时 } \varphi(x) \leftarrow$$

$$\text{即 } \varphi(x) \text{ 在 } x=a \text{ 处连续, } \varphi(a) = f'(a).$$

$$\text{Thm 4.4} \quad (g \circ f)'(x) = g'(f(x)) f'(x)$$

$$\therefore g(f(x)) \quad b=f(a) \text{ 在 } x=a \text{ 处连续}$$

$$\therefore g(y) = g(b) + \varphi(y)(y-b), \quad g'(b) = \varphi(b)$$

$$y = f(a+h)$$

$$\begin{aligned} g(f(a+h)) &= g(f(a) + \varphi(f(a+h)) \underbrace{(f(a+h) - f(a))}_{h}) \\ &= g \circ f(a+h) = g \circ f(a) + \underbrace{\varphi(f(a+h)) \varphi(h)}_{\varphi(a+h) h} \end{aligned}$$

$$= g \circ f |_{(a+h)} = g \circ f(a) + \underbrace{\underbrace{g(f(a+h)) - g(f(a))}_{(g \circ f)(a+h) \cdot f'(a+h)h} h}_{\{(a+h) \cdot h\}}$$

$\{$ je a z. won't

$$\Rightarrow \{ (f \circ g)'(a) = \{ (u) = (g \circ f)(a) \cdot f'(a)$$

$$(g \circ f)(a) = g'(f(a)), \quad f'(a) = f'(u) \quad \therefore g'(f(u)) f'(u), //$$

§§ 4.2 逆関数

f が 単調増加 $\Leftrightarrow x_1 < x_2 \rightarrow f(x_1) \leq f(x_2)$
 積極 $\therefore \Leftrightarrow x_1 < x_2 \rightarrow f(x_1) < f(x_2)$
 減少 $\Leftrightarrow x_1 < x_2 \rightarrow f(x_1) \geq f(x_2)$
 積極 $\therefore x_1 < x_2 \rightarrow f(x_1) > f(x_2)$

Thm 4.8 $I = [a, b]$ で $\det 2$ の $|x|$ の間隔
 積極 単調.

$\because f(x) - f(y) \quad (x > y)$ の $\frac{\partial}{\partial t}$ は

$$F(x, y) = f(x) - f(y)$$

$$S = \{(x, y) \in [a, b] \times [a, b] \mid x > y\}$$

$\circ F \neq 0,$

$\circ F \text{ 連続}$

$$G(t) = F((1-t)x + tx', (1-t)y + ty')$$

$$\begin{aligned} G(0) &> 0 \rightarrow G(1) > 0 \\ G(1) &< 0 \rightarrow G(1) < 0 \end{aligned} \quad \text{由 P. 2 の定理} \quad \therefore \text{同値} //$$

$$S = \{(x, y) \in [a, b] \times [a, b] \mid x \neq y\}$$

$$F(x, y) : f(x) - f(y) \quad (x, y) \in S$$

$$F(x, y) \neq 0 \quad \text{on } S \quad (1:1 \text{ 連続})$$

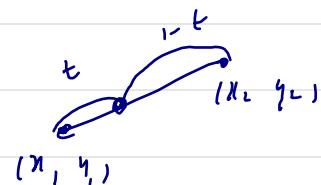
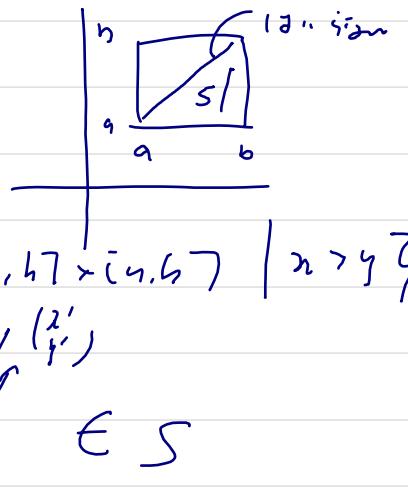
$$G(t) = F((1-t)x_1 + tx_2, (1-t)y_1 + ty_2) \quad \text{由 P. 2 の定理} //$$

$$(G(0) > 0 \rightarrow G(1) > 0 \quad \therefore \text{由 P. 2 の定理} \quad \text{Thm 4.8})$$

$$(G(1) < 0 \rightarrow G(0) < 0) \quad G(\frac{1}{2}) = 0 \quad \therefore \text{由 P. 2 の定理} //$$

$$\therefore F(x_1, y_1) \neq F(x_2, y_2) \rightarrow |x| \neq |y|. \quad \therefore F \neq 0 \text{ on } S$$

$$\therefore f^{-1} = \text{牛込の定理} //$$



Th 4.9 $f: I = [a, b] \rightarrow J \subset \mathbb{R}$ は 1:1, 連続関数

(1) $f(I) = J$ は 開区間

(2) $f^{-1}: J \rightarrow I$ は cont

∴ (1) f 狹ぎ↑ は 連続関数

$$\alpha = f(a) < \beta = f(b) \quad J \subset [\alpha, \beta]$$

$\forall \gamma \in (\alpha, \beta)$ $\exists c \in (a, b)$ s.t. $f(c) = \gamma \in J$

$$\alpha = a < \beta \text{ と } f(a) = \alpha, f(b) = \beta \in J$$

$$\therefore J \supset [\alpha, \beta] \quad \therefore J = [\alpha, \beta]$$

(2) $f: I \rightarrow J$ 1:1 = injective

onto = surjective

$f^{-1}: J \rightarrow I$ bijection

④ $J \ni y_0$ が f^{-1} が cont で示す。 $y_0 \neq \alpha, \beta$

$$y_0 = f(x_0), [x_0 - \varepsilon, x_0 + \varepsilon] = I_\varepsilon \subset I \text{ とす}$$

$$f(I_\varepsilon) = J_\varepsilon \subset J \quad (y_0 \text{ は 端点でない})$$

$$[y_0 - \delta, y_0 + \delta] \subset J_\varepsilon \quad \text{左端点}$$

$$\text{i.e. } \forall y \in [y_0 - \delta, y_0 + \delta] \text{ あり } |f(y) - f(y_0)|$$

$$= |f'(y) - x_0| < \varepsilon \quad \text{証明}$$

$$④ y_0 = f(a), [a, a + \varepsilon] = I_\varepsilon \subset I$$

$$f(I_\varepsilon) = J_\varepsilon \subset J \quad [\alpha, \alpha + \delta] \subset J_\varepsilon \text{ とす}$$

$$\text{i.e. } \forall y \in [a, a + \delta] \text{ あり } |f^{-1}(y) - f^{-1}(a)| = |f'(y) - a| < \varepsilon,$$

$$④ y_0 = \beta = f(b) \neq \text{左端点}$$

Thm 4.10 $f \uparrow (a, b)$ で diff. $f'(x) \neq 0 \forall x \in (a, b)$

$\Rightarrow f(I^\circ)$ 上で f' は diff かつ

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

$\boxed{\begin{array}{l} \exists f^{-1} : f|_I : 1:1 \\ \exists f^{-1} \text{ かつ } f \text{ cont} \\ \Rightarrow f \text{ は 単射} \end{array}}$

∴ $f(I^\circ) = J$ と 互いに
多対一する $y = f(x)$

$$\frac{f^{-1}(y+h) - f^{-1}(y)}{h} = \frac{k}{f(x+k) - f(x)}$$

$\therefore f^{-1}(y+h) - f^{-1}(y) = k$ とする

$$f^{-1}(y+h) = x+k \quad \therefore y+h = f(x+k)$$

$$\therefore h = f(x+k) - y = f(x+k) - f(x)$$

$h \rightarrow 0$ のとき f^{-1} は cont 且つ $k \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{f^{-1}(y+h) - f^{-1}(y)}{h} = \lim_{k \rightarrow 0} \frac{1}{\frac{f(x+k) - f(x)}{k}} = \frac{1}{f'(x)}$$

(主) $f^{-1}(f(x)) = x$

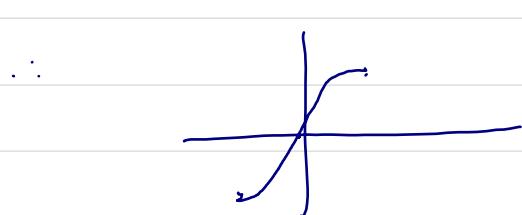
$$\Rightarrow (f^{-1})'(f(x)) f'(x) = 1$$

$$\therefore (f^{-1})'(y) = \frac{1}{f'(x)}$$

\because で $y = f(x)$

微分可で、 x が

例) $\sin x$ $x \in I = [-\frac{\pi}{2}, \frac{\pi}{2}]$



\exists
1:1, cont \Rightarrow 収束定理

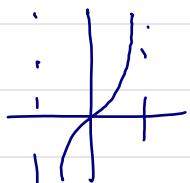
$J = [-1, 1]$ 逆函数 $\text{Arc sin } x$ 主值域

$\text{Arc sin } x = g(x)$ とす

$$g(\sin t) = t \quad \therefore g'(\sin t) \cos t = 1 \quad \therefore g'(\sin t) = \frac{1}{\cos t}$$

$$\therefore g'(x) = \frac{1}{\sqrt{1-x^2}}$$

例) $\tan x$ $x \in I (-\frac{\pi}{2}, \frac{\pi}{2})$



$J = (-\infty, \infty) = \mathbb{R}$ 逆函数 $\text{Arctan } x$

$\text{Arctan } x = g(x)$

$$g(\tan t) = t \quad \therefore g'(\tan t) \cdot \frac{1}{\cos^2 t} = 1$$

$$\therefore g'(x) = \cos t = \frac{1}{1+x^2}$$

例) $\cos x$

$$\text{Arc cos } x, (\text{Arc cos } x)' = \frac{-1}{\sqrt{1-x^2}}$$

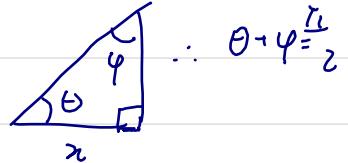
$$\underline{\text{Arc cos}^{\prime\prime}\theta} + \underline{\text{Arc sin}^{\prime\prime}\varphi} = \frac{\pi}{2} \quad \text{OK}$$

$$0 \leq \theta \leq \pi \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \quad \therefore -\frac{\pi}{2} \leq \theta + \varphi \leq \frac{3}{2}\pi$$

$$\sin(\theta + \varphi) = \sin \theta \cos \varphi + \cos \theta \sin \varphi$$

$$0 \leq \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - x^2} \quad \therefore \quad \sin(\theta + \varphi) = 1$$

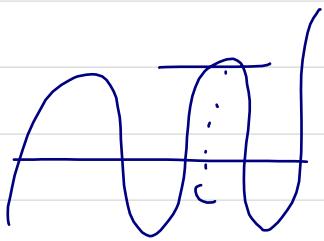
$$0 \leq \cos \varphi = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2} \quad \therefore \quad \theta + \varphi = \frac{\pi}{2}$$



$$\therefore \theta + \varphi = \frac{\pi}{2}$$

§§ 4.4 積分の応用

Thm 4. 21 (Rolle の定理)



$f: I = [a, b] \rightarrow \mathbb{C}$ cont

$f: I = (a, b) \rightarrow \mathbb{C}$ diff

$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$

① $f(x) = \text{定数} \quad a \leq x \leq b$ の場合

$f(x) \neq \text{定数} \Rightarrow \max \neq \min$

$f(c) \neq f(a) \neq f(b)$. ($\exists c \in (a, b) \text{ s.t. } f(c) \neq f(a) \neq f(b)$)

$\therefore c \in (a, b)$

$$\frac{f(x) - f(c)}{x - c} \leq 0 \quad (x > c)$$

$$\frac{f(x) - f(c)}{x - c} \geq 0 \quad (x < c) \Rightarrow f'(c) = 0.$$

② $\max \geq \min \Leftrightarrow f'(c) = 0$

Thm 4.22 (平均値の定理)

f は $I = [a, b]$ で cont

f は $I = (a, b)$ で diff

$$\Leftrightarrow \exists c \in (a, b) \text{ s.t. } \frac{f(b) - f(a)}{b - a} = f'(c)$$

①

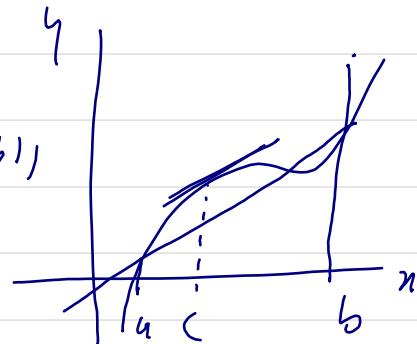
$$y = \frac{f(b) - f(a)}{b - a} (x - a) + f(a)$$

$$y = f(x) \quad \text{かつ } (a + t_1)(b + t_2)$$

左側に注記

$$f(x) - \frac{f(b) - f(a)}{b - a} (x - a) - f(a) = F$$

左側に注記
教科書と違う。



\therefore Rolle の定理 $\exists c \in (a, b)$ s.t.

$$F'(c) = 0$$

$$F' = f'(x) - \frac{f(b) - f(a)}{b - a} \quad T = 0$$

$$\therefore \frac{f(b) - f(a)}{b - a} = f'(c).$$

$$[4.23] \quad f(z) = (\sqrt{z})^c \quad [(\sqrt{z})^c]' = (\sqrt{z})^c \log \sqrt{z}$$

$$\text{証明} \quad \frac{f(a_{n+1}) - f(a_n)}{a_{n+1} - a_n} = f'(z) = (\sqrt{z})^c \log \sqrt{z}$$

$$\frac{f(a_n) - f(z)}{a_n - z} = \frac{a_{n+1} - z}{a_n - z} = (\sqrt{z})^c \log \sqrt{z} \leq z \log \sqrt{z}.$$

$$\underline{2 < c < a_n} \quad z \in a_n \text{ の } \pi_2 \dots ?$$

[4.24] 4.24 Cauchy 平均値の定理

$$f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)) - f(a) = F(x)$$

$$F(a) = 0, \quad F(b) = 0 \quad \therefore \text{Rolle} \quad F'(c) = 0 \quad \exists c \in (a, b)$$

$$F' = f' - \frac{f(b) - f(a)}{g(b) - g(a)} g' \quad \text{では} \quad \text{どう}.$$

Prop 4.25 If f is cont., $\exists \varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \setminus \{a\}$ is diff.

$$\exists h \lim_{x \rightarrow a} f'(x) = l \quad (a \in \mathbb{R}, a \notin \text{discrete})$$

\Rightarrow if $x = a$ is diff s.t. $f'(a) = l$

i.e. f' at $x = a$ cont.

$$\therefore a > a \text{ or } a < a \quad \frac{f(x) - f(a)}{x - a} = f'(c) \quad \exists c \in (a, x)$$

$$a < a \text{ or } a < a \quad \frac{f(x) - f(a)}{x - a} = f'(c) \quad \exists c \in (x, a)$$

$$\therefore \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = \lim_{c \rightarrow a^+} f'(c) = l$$

$$\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} = \lim_{c \rightarrow a^-} f'(c) = l$$

$$\therefore \exists f'(a) \Rightarrow f'(a) = l$$

$$\therefore \lim_{x \rightarrow a} f'(x) = f'(a) \text{ by 3.8.}$$

Th 4.26 If I is con., I° is diff.

$$(1) \quad \forall x \in I^\circ \text{ if } f'(x_0) > 0 \rightarrow I^\circ \text{ 増加} \uparrow$$

$$(2) \quad \text{..} \rightarrow I^\circ \text{ 増加} \downarrow$$

$$(3) \quad \text{..} \rightarrow I^\circ \uparrow$$

$$(4) \quad \text{..} \rightarrow I^\circ \downarrow$$

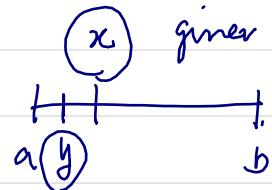
$$(5) \quad \text{..} \leftrightarrow I^\circ \text{ 定数関数}$$

$$\because x_1 <^{\exists} x_2 \text{ at } f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

$$\therefore \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0$$

$\exists z$ s.t. $a < z < x_2$

$$\forall x > a \exists z \text{ s.t. } y(a < y < x)$$



$$f(y) < f(z) \text{ and } |f(y) - f(z)| = \varepsilon \text{ (由題意)}$$

$$\exists s \text{ s.t. } \forall z (z-a < s) \quad |f(a) - f(z)| < \frac{\varepsilon}{2} \quad (\text{To cont.})$$

$$\therefore -\frac{\varepsilon}{2} < f(a) - f(z) < \frac{\varepsilon}{2}$$

$$-\frac{\varepsilon}{2} + f(z) < f(a) < \frac{\varepsilon}{2} + f(z)$$

$$\therefore f(a) + \frac{\varepsilon}{2} < \varepsilon + f(z) = f(x) - f(y) + f(z)$$

$$z \in (x, y) \Rightarrow f(z) - f(y) < 0$$

$$\therefore f(z) < \varepsilon + f(z) < f(x)$$

2018 · 6 · 12

第8回目

Def 4.31 $f(x)$ が $x=a$ の極大 $\Leftrightarrow \exists \delta > 0$ 且 $\forall x (0 < |x-a| < \delta) \Rightarrow f(a) > f(x)$

$f(a)$ が $f(x)$ の極大

(2) $f(a) \geq f(x)$
左の極大

○ 微分との関係

Thm 4.32 f は $a \in I$ で 微分可能かつ $f'(a)$ が
左の極大値 $\Rightarrow f'(a) = 0$

$$\because \frac{f(x) - f(a)}{x - a} \geq 0 \quad (x > a)$$

$$\frac{f(x) - f(a)}{x - a} \leq 0 \quad (x < a)$$

$$\therefore \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = 0 \quad //$$

4.33 $f: I = (c, d) \rightarrow \mathbb{R}$ diff
 $x=a$ で $\vec{\text{下}}\vec{\text{上}}\vec{\text{下}}$ とか $\max_{I \cap [a, b]}$
 $=\vec{\text{下}}\vec{\text{上}}$ f は I で $\vec{\text{下}}\vec{\text{上}}$ の極小値をとる。

$\because \exists \delta > 0 \quad f(a) > f(a+x)$

$$|x| < \delta$$

$\Rightarrow \exists b > a \quad f(b) > f(a)$

つまり $a+x < b$ と仮定すると

$$f(a+x) < f(a) < f(b)$$

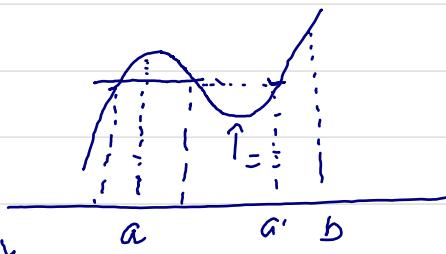
$f(c)$ は $\min_{I \cap [a, b]}$ (か) $c = a+x$ の可能性あり

② $\vec{\text{下}}\vec{\text{上}}$ 中間値定理 $\exists a' \in (a+x, b) \quad \text{s.t. } f(a) = f(a')$

$[a, a'] \in f$ は $\min_{I \cap [a, a']}$ $\exists c \in [a, a']$

(1) $a \neq c$ ($a' \neq c$) $a \leq c$ 極小アリ

(2) $a=c$ ($a'=c$) $a \leq c$ 定義缺、 $\vec{\text{下}}\vec{\text{上}}$ 極小アリ



Σ ④) たとえ $f'(x) \geq 0$ もしくは $f'(x) > 0$ が成り立つ.

$f'(a) > 0 \Leftrightarrow$ 何れかの2つが成り立つ.

例) 基本 $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

\Rightarrow $\frac{f(h) - f(0)}{h} = 0$.

$$\begin{aligned} \text{では } f'(x) &= 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) \\ &= 2x \sin \frac{1}{x} - \cos \frac{1}{x} \end{aligned}$$

$$x_n = \frac{1}{2\pi n} \quad 2^n - \cos(2\pi n) = -1 < 0$$

$$y_n = \frac{1}{\pi(2n+1)} \quad 2^n - \cos(\pi(2n+1)) = +1 > 0$$

① $g(x) = \frac{1}{4}x + f(x)$

$$g'(0) = \frac{1}{4} > 0 \quad g'(x) = \frac{1}{4} + f'(x) = -\frac{3}{4}$$

$$x = \frac{1}{2\pi n}$$

つまり $g'(0) > 0$ で $x \rightarrow 0$ のとき $g'(x) \rightarrow -\infty$

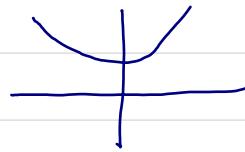
② $\begin{cases} g(x) = 2x^2 + f(x) \geq 0 \\ g(0) = 0 \text{ は } \text{下限} \end{cases}$

$$g'(x) = 4x + f'(x) = \begin{cases} \frac{2}{\pi n} + (-1) \leq 0 \\ \frac{\pm 4}{\pi(2n+1)} + 1 > 0 \end{cases}$$

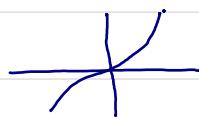
つまり $g'(0) \geq 0$ で $x \rightarrow 0$ のとき $g'(x) \rightarrow -\infty$

§4.5 双曲系関数

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

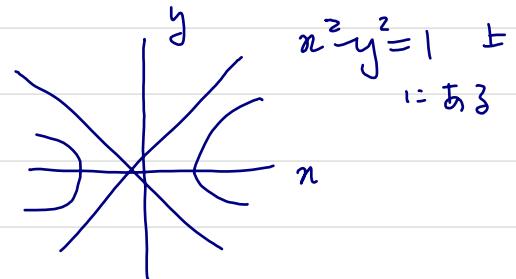


$$\sinh x = \frac{e^x - e^{-x}}{2}$$



$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$



つまに $\sinh x$ は 単増。

$\sinh x$ の 過関数を 理め。

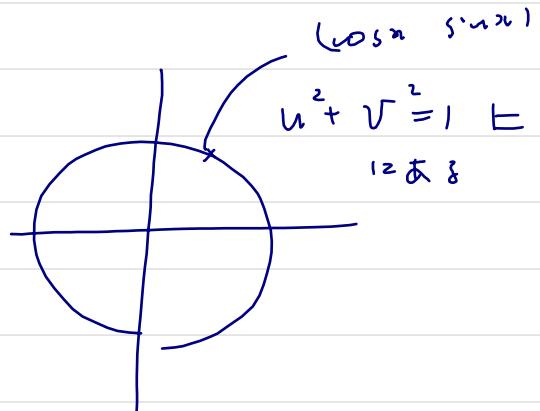
$$y = \frac{e^x - e^{-x}}{2}$$

$$2y = e^x - e^{-x} \therefore 2y e^x = e^{2x} - 1 \therefore e^{2x} - 2ye^x - 1 = 0.$$

$$\therefore e^x = \frac{zy \pm \sqrt{4y^2 + 4}}{2} = y + \sqrt{y^2 + 1}$$

$$\therefore x = \log(y + \sqrt{y^2 + 1})$$

$$\therefore \sinh^{-1} x = \log(x + \sqrt{x^2 + 1}) \quad \text{ただし } (\sinh^{-1} x)' = \frac{1}{\sqrt{x^2 + 1}}$$



§ 4.6 高階導數與 Taylor 定理

$\frac{d^n f}{dx^n}$, etc
高階導數

$$\text{15' } \tilde{s}^n = s(x + \frac{n\pi}{2})$$

$$\text{16' } (e^x)^m = e^x$$

$$\text{17' } e^x \sin x = e^{x+i\pi} \rightarrow e^{(1+i)x} \rightarrow (1+i)^n e^{(1+i)x}$$

$$(1+i)^n = \left(\sqrt{2} e^{\frac{\pi}{4}i}\right)^n = (\sqrt{2})^n e^{\frac{n\pi}{4}i}$$

$$= (\sqrt{2})^n e^{x+i(\frac{n\pi}{4}+x)} = (\sqrt{2})^n e^x \sin(x + \frac{n\pi}{4})$$

imaginary part.

Thm 4.40

$$(fg)^n = \sum_{j=0}^n \binom{n}{j} f^{n-j} g^j$$

$n \in \mathbb{N}$ 且 $0 \leq j \leq n$

$$(fg)^k = \sum_{j=0}^k \binom{k}{j} f^{k-j} g^j$$

$$(fg)^{k+1} = \sum_{j=0}^k \binom{k}{j} \left(f^{k-j+1} g^j + f^{k-j} g^{j+1} \right)$$

= 各自 check 一下

$$13) \quad 4.4, \quad f(x) = x^2 e^{-x}$$

$$f^{(n)} = \sum_{j=0}^n \binom{n}{j} (x^2)^j (e^{-x})^{(n-j)}.$$

Def 4.4) $f \in C^k \Leftrightarrow \overset{\exists}{f^{(k)}} \text{ is const}$
 $f \in C^\infty \Leftrightarrow \overset{\exists}{f^{(k)}} \forall k \in \mathbb{N}$

$$\begin{aligned} 13) \quad \left(x^2 \sin \frac{1}{x} \right)' &= 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \left(-\frac{1}{x^2} \right) \\ &= 2x \sin \frac{1}{x} - \cos \frac{1}{x} \underset{x \rightarrow 0}{\not\rightarrow} 0 \quad (x \rightarrow 0) \end{aligned}$$

$$\textcircled{c} \quad x^k \sin \frac{1}{x} \quad : \quad k \text{ is diff } \text{ i.e. } \notin C^k.$$

9回目

Taylor の定理 $f \in C^{n+1}$

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$\therefore f^{(n+1)} = T_n(x) + \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!} (x-a)^{n+1}$$

$$\exists \theta (0 < \theta < 1) \text{ s.t.}$$

(θ は $x = 1$ のとき, 2.1.3 の $\theta = \theta(x)$ と区別する)

$$\therefore g(t) = f(t) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (t-a)^k - \frac{A}{(n+1)!} (x-t)^{n+1}$$

 $g^{(n+1)} = 0, g(a) = 0 \Rightarrow g(t) \text{ は } A \text{ の定義}$ $\exists c (a < c \text{ or } c < a) \text{ s.t. } g'(c) = 0$

$$g'(t) = - \frac{f^{(n+1)}(t)}{n!} (x-t) + \frac{A}{n!} (x-t)$$

$$A = \frac{(n+1)!}{(x-a)^{n+1}} \left\{ f(t) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k \right\}$$

$$= f^{(n+1)}(c) \quad \text{Euler の定理.}$$

$$c = a + \theta(x-a) \quad \text{OK}$$



11

T_n : n=2 Taylor の定理

n :

(n+1) :

$$\text{13n} \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{x0}}{(n+1)!} x^{n+1}$$

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots$$

$$\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots$$

$$\begin{aligned} [\log(x+1)]' &= \frac{1}{x+1} = (x+1)^{-1} \rightarrow (-1)(x+1)^{-2} \rightarrow (-1)(-2)(x+1)^{-3} \\ &\quad \rightarrow \cdots \rightarrow (-1) \cdots (-n+1)(x+1)^{-n} \\ (\log(x+1))^{(n)} &= (-1)^{n-1} (n-1)! (x+1)^{-n} \rightarrow (-1)^{n-1} (n-1)! \quad n \geq 1 \\ \therefore \log(x+1) &= \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \end{aligned}$$

Prop 4.50 $f \in C^2 \quad f'(a) = 0$

① $f''(a) > 0 \Rightarrow$ 下限小

② $f''(a) < 0 \Rightarrow$ 下限大

$$\begin{aligned} \textcircled{1} \quad f(x) &= f(a) + f'(a)(x-a) + \frac{f''(3)}{2!} (x-a)^2 \\ &= f(a) + \frac{f''(3)}{2!} (x-a)^2 \\ \therefore f(x) - f(a) &= \frac{f''(3)}{2!} (x-a)^2 > 0 \\ &\quad (x=a \text{ or } x < a) \end{aligned}$$

② \Rightarrow ①

§§ 4.7 無限大・無限小の比較

Def 4.5.2 $\lim_{n \rightarrow \infty} f_m = +\infty$, $\lim g_m = +\infty$

$\Rightarrow \lim_{n \rightarrow \infty} \frac{f_m}{g_m} = +\infty \Rightarrow f(x) \gg g(x)$
高位項 \dots 項

Thm 4.53 $\alpha > 0$.

① $\log x \ll x^\alpha \ll e^x$ ($x \rightarrow +\infty$)

② $\alpha < \beta \Rightarrow x^\alpha \ll x^\beta$

∴ $e^x > \frac{x^n}{n!} > \frac{x^{\alpha+1}}{n!}$ ($n > \alpha+1$ と仮定)
 $n > \alpha+1$ と仮定

$$\therefore \frac{e^x}{x^\alpha} > \frac{x}{n!} \rightarrow \infty$$

$$\frac{x^\alpha}{\log x} = \frac{e^{\alpha \ln x}}{u} \rightarrow \infty$$

③ は 明顯 (と思)

Cor 4.54 $\lim_n \sqrt[n]{n} = 1$

$$\therefore \sqrt[n]{n} = e^{\frac{1}{n} \log n} = e^{\frac{\log n}{n}} = 1$$

$$\text{Ex 11} \lim_{n \rightarrow +\infty} x^{\log n} = 0$$

$$x = \frac{1}{y} \quad \lim_{y \rightarrow +\infty} \frac{-\log y}{y^x} = 0$$

$$\lim_{n \rightarrow +\infty} x^n \quad \lim_{n \rightarrow +\infty} n \log n = 0 \quad \therefore \lim_{n \rightarrow +\infty} x^n = 1$$

$$\lim_{x \rightarrow 0} x^{x^x} \quad \lim_{x \rightarrow 0} x^{x^x} \log x = -\infty \quad \therefore \lim_{x \rightarrow 0} x^x = 0$$

$$\text{Def 4.38} \quad \lim_{n \rightarrow a} f_{(n)} = 0, \quad \lim_{n \rightarrow a} g_{(n)} = 0$$

$$\lim_{n \rightarrow a} \frac{f_{(n)}}{g_{(n)}} = 0 \quad f_{(n)} = o(g_{(n)}) \quad (n \rightarrow a) \text{ とき } <$$

$$\text{証明: } f_{(n)} \rightarrow 0 \quad \text{かつ} \quad f_{(n)} = o(g_{(n)}) \quad \text{とき} \quad \frac{o(g_{(n)})}{g_{(n)}} = 0$$

$f = o(g(x)) \leftarrow f \text{ は } o(g(x)) \text{ の極限 } \frac{1}{n} \rightarrow 0$

Thm 4.59

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + o((x-a)^n) \quad (x \rightarrow a)$$

$$\left[\frac{f(x) - T_n(x)}{(x-a)^n} \rightarrow 0 \cdot (x \rightarrow a) \right] \Leftarrow \exists z_1, z_2 \in \mathbb{C}$$

∴ 残余項は

$$c = a + \theta(x-a)$$

$$R_{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

$$\begin{aligned} \left| \frac{R_{n+1}}{(x-a)^n} \right| &= \left| \frac{f^{(n+1)}(c)}{(n+1)!} \right| |x-a| \\ &\leq \frac{M}{(n+1)!} |x-a| \rightarrow 0 \quad (x \rightarrow a) \end{aligned}$$

: たとえ $f \in C^{n+1}$ かつ M あるから

Lem 4.60 $m, n, p \in \mathbb{Z}$ $m \leq n$

$$\textcircled{1} \quad S(x) = o(x^n) \rightarrow S(x) = o(x^m)$$

$$S(x^k) = o(x^{nk})$$

$$\textcircled{2} \quad o(x^m) = o(x^n) = o(x^m)$$

↑
정의
↑
정의

$$\textcircled{3} \quad o(x^n) o(x^m) = o(x^{n+m})$$

$$\textcircled{4} \quad \frac{o(x^m)}{x^m} = o(x^{m-m})$$

$$\textcircled{5} \quad \textcircled{1} \quad S(x) = o(x^m) \therefore \frac{S(x)}{x^m} = \frac{o(x^m)}{x^m} : \frac{o(x^n)}{x^n} x^{n-m} \rightarrow 0$$

$$\frac{S(x^n)}{x^{nk}} = \frac{S(y)}{y^n} \rightarrow 0$$

$$\textcircled{2} \quad \text{正確に } f(x) = o(x^n) \quad g(x) = o(x^n)$$

$$\begin{aligned} &= o(x^n) \\ f(x) \pm g(x) &= o(x^n) \\ \therefore \frac{f(x) \pm g(x)}{x^m} &= \frac{o(x^n)}{x^m} + \frac{g(x)}{x^n} \frac{x^n}{x^m} \rightarrow 0 \end{aligned}$$

全くの

Lem 4.60 は 誤り \rightarrow 演習.

2018 7 3

10回目 原式化と代入法

復習+確認

Thm 4.63

(1) $e^x = \sum_{k=0}^n \frac{x^k}{k!} + o(x^n)$

(2) $\log(1+x) = \sum_{k=1}^n \frac{(-1)^{k-1}}{k} x^k + o(x^n) \quad (\log n \neq n \text{ is not diff})$

(3) $\cos x = \sum_{m=0}^n \frac{(-1)^m}{(2m)!} x^{2m} + o(x^{2n+1}) \quad \text{even}$

(4) $\sin x = \sum_{m=0}^n \frac{(-1)^m}{(2m+1)!} x^{2m+1} + o(x^{2n+2}) \quad \text{odd}$

注

左は $\tan x$ は無構のとくにうるさい

Prop 4.6.2 f が $x \rightarrow 0$ で ∞ または 0 に収束する ($= \infty$)

$$(1) \quad f(x) = \sum_{k=0}^n a_k x^k + o(x^n) \quad (x \rightarrow 0)$$

$$\rightarrow a_k = \frac{f^{(k)}(0)}{k!} \quad (\text{Taylor 級数} = \sum \frac{f^{(k)}(0)}{k!} x^k)$$

(2) f が even $\rightarrow T_n(x)$ は 偶数べきのみ

(3) odd $\rightarrow T_n(x)$ は 奇数べきのみ

$$\therefore (1) \quad f(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k + o(x^n)$$

$$= \sum_{k=0}^n a_k x^k + o(x^n)$$

$$\therefore \sum_{k=1}^n \left(\frac{f^{(k)}(0)}{k!} - a_k \right) x^k = o(x^n) \quad (= 2 \text{ つ})$$

$$x \rightarrow 0 \text{ のとき } a_0 = f(0)$$

$x \rightarrow 0$ のとき

$$\sum_{k=1}^n \left(\frac{f^{(k)}(0)}{k!} - a_k \right) x^{k-1} = \frac{o(x^n)}{x} = o(x^{n-1})$$

$$x \rightarrow 0 \text{ のとき } a_1 = \frac{f'(0)}{1!} \quad (= 1 \text{ つ})$$

$$(2) \quad f(x) = f(-x) \quad \therefore f'(x) = -f'(-x) \quad \text{なぜ } f'(0) = -f'(0) \\ \therefore f'(0) = 0$$

$$f''(x) = f''(-x) \quad \therefore f''(x) = -f''(-x) \quad \therefore f''(0) = 0$$

\therefore 1つ目

二整系数

$$n C_k = \frac{n(n-1)\cdots(n-k+1)}{k!} \rightarrow \text{自然数}$$

Def 4.66

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad \begin{matrix} \alpha \in \mathbb{R} \\ k=1, 2, \dots \end{matrix}$$

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

$$(1+x)^{-1} = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \sum_{n=0}^{\infty} \binom{-1}{n} \frac{x^n}{n!}$$

例 - 算法

Th 4.69 $\forall \alpha \in \mathbb{R}, \forall n \in \mathbb{N}$

$$(1+x)^\alpha = \sum_{k=0}^n \binom{\alpha}{k} x^k + o(x^n) \quad (x \rightarrow 0)$$

$$\because (1+x)^\alpha = \alpha(\alpha-1)\cdots(\alpha-k+1) (x+1)^{\alpha-k}$$

$k \neq 3$,

137 4.72 T_n 102 級で T_n 0 の $\frac{1}{2}$ の 答え

$$\log \cos x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + o(x^5) \quad x \rightarrow 0$$

$$\because \log(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k + o(x^n)$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} + o(x^{2n+1})$$

左の式 $\log(\cos x - 1 + 1)$ で思って

$$\left\{ \begin{array}{l} \cos x - 1 = -\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5) \\ \log(u+1) = u - \frac{1}{2}u^2 + o(u^3) \end{array} \right.$$

$$\therefore \log(\cos x) = \left(-\frac{1}{2}x^2 + \frac{1}{4!}x^4 + o(x^5) \right) - \frac{1}{2} \left(-\frac{1}{2}x^2 + \frac{1}{4!}x^4 + o(x^5) \right)^2 + o(x^6)$$

$$= -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{2} \left(-\frac{1}{2}x^2 \right)^2 + o(x^5)$$

$$= -\frac{1}{2}x^2 - \frac{1}{12}x^4 + o(x^5)$$

→ 左の式がいいです。

$$(2k-1)!! = (2k-1) \cdots 3 \cdot 1$$

$$0!! = 1$$

$$(2k)!! = (2k)(2k-2) \cdots 2$$

$$(-1)!! = 1$$

と統一する

例)

$$\sqrt{x+1} = (x+1)^{\frac{1}{2}} = \sum_{k=0}^n \left(\frac{1}{2}\right)_k x^k + o(x^n)$$

$$\left(\frac{1}{2}\right)_k = \frac{\frac{1}{2}(\frac{1}{2}-1) \cdots (\frac{1}{2}-k+1)}{k!} \quad (k \geq 1)$$

$$= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{2k-3}{2})}{k!}$$

$$= \frac{(-1) \left(\frac{1}{2}\right)^k 1 \cdot 3 \cdots (2k-3)}{k!} = \frac{(-1)^{k-1} \left(\frac{1}{2}\right)^k (2k-3)!!}{k!} \quad (k \geq 1)$$

$$\sqrt{x+1} = \sum_{k=0}^n \frac{(-1)^{k-1} (2k-3)!!}{(2k)!!} x^k + o(x^n)$$

$$\text{つまり } \sqrt{x+1} = 1 + \frac{1}{2}x + o(x)$$

$$- \ln \frac{1}{2} x \quad (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + o(x)$$

例) $\arctan x$ の Taylor 級数

$$(\arctan x)' = \frac{1}{1+x^2} \quad \therefore \arctan x = \int_0^x \frac{1}{1+t^2} dt \quad -(*)$$

$$\begin{aligned} \frac{1}{1+t^2} &= \sum_{k=0}^{\infty} (-t^2)^k \quad t \in \mathbb{R} \quad (|t| < 1) \\ &= \sum_{k=0}^n (-t^2)^k + \frac{(-t^2)^{n+1}}{1+t^2} \end{aligned}$$

= 余弦 (x), I = 1^{\circ} \sim 12^{\circ}

$$\begin{aligned} \arctan x &= \sum_{k=0}^n \int_0^x (-t^2)^k dx + \underbrace{\int_0^n \frac{(-t^2)^{n+1}}{1+t^2} dt}_J \\ &= \sum_{k=0}^n (-1)^k \frac{1}{2k+1} x^{2k+1} + S_n(x) \end{aligned}$$

$$\frac{S_n(x)}{x^{2n+2}} \underset{x \rightarrow 0}{\lim} = 0 \quad (= 2 \pi / 2 \quad (\text{平行} \sim t^{2n+2} \sim (2\pi)^{2n+3}))$$

Taylor 級数の - $\frac{\pi}{2} \sim \frac{\pi}{2}$ の式

$$\arctan x = \sum_{k=0}^n \frac{(-1)^k}{2k+1} x^{2k+1} + o(x^{2n+2})$$

($|x| < 1$) の場合で x^2

$$f(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n + R_N(x)$$

n が 固定した $\lim_{N \rightarrow \infty} R_N(x) = 0$ である

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \text{ と書く}$$

$$\text{例} e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\therefore R_n(n) = \frac{e^{\theta x} x^{(n+1)}}{(n+1)!} \rightarrow 0 \quad (n \rightarrow \infty)$$

つまり $x = 3\pi - 2\sqrt{2}$

2018.7.10

11回目

§§ 4.8 不定形の下限問題

不等式による証明

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^n \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{1}{k!}$$

$$\leq \sum_{k=0}^{n+1} \left(1 - \frac{1}{n+1}\right) \cdots \left(1 - \frac{k-1}{n+1}\right) \frac{1}{k!} \quad \therefore a_n < a_{n+1}$$

$$\left(1 + \frac{1}{n}\right)^n \leq \sum_{k=0}^n \frac{1}{k!} = 1 + 1 + 1 \leq 3$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \text{ は収束かいかん。}$$

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \xi_n \frac{1}{k!} \text{ とする}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^n \xi_n \frac{1}{k!} &= \left(\sum_{k=0}^M \frac{1}{k!} - \sum_{k=0}^M \xi_n \frac{1}{k!} \right) + \left(\sum_{k=M+1}^{\infty} \frac{1}{k!} - \sum_{k=M+1}^{\infty} \xi_n \frac{1}{k!} \right) \\ &\quad + \left(\sum_{k=M+1}^M \xi_n \frac{1}{k!} - \sum_{k=M+1}^M \xi_n \frac{1}{k!} \right) = ① + ② + ③ \end{aligned}$$

 $n > M$ (= たとえ大事)

$$① \leq \varepsilon \quad (M \gg 1) \quad ③ \sum_{k=M+1}^n \xi_n \frac{1}{k!} \leq \sum_{k=M+1}^{\infty} \frac{1}{k!} \leq \varepsilon \quad (M \gg 1)$$

$$\lim_n (②) = 0$$

$$\therefore \lim_n (① + ② + ③) \leq 2\varepsilon \quad //$$

$$(3) \lim_{n \rightarrow 0} \frac{1}{\sin^2 x} - \frac{1}{x^2} = \left(\frac{n + \sin n}{n} \right) \left(\frac{n - \sin n}{x} \right) \left(\frac{x}{\sin x} \right)^2$$

$\rightarrow \frac{1}{3}$

$$\sin x \sim x - \frac{1}{3!} x^3 + o(x^4)$$

(3) $\frac{1}{\sin^2 x} - \frac{1}{x^2} = \frac{1}{x^2} \left(\frac{1}{\left(1 - \frac{1}{3!} x^2 + o(x^4) \right)^2} - 1 \right)$

$$\sim \frac{1}{x^2} \left(1 + \frac{1}{3} x^2 - 1 \right) \sim \frac{1}{3}$$

L'Hospital の 定理

f, g diff on (a, b)	$g' \neq 0$ on (a, b)
$\exists f, g$ cont on $[a, b]$	$f(a) = g(a) = 0$
$\exists \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = l$	$\exists \lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = l$

\therefore Cauchy の 定理.

例題 1) 成り立つ: $f(x) = x^2 \sin \frac{1}{x}, g(x) = x$

$$\frac{f'(x)}{g'(x)} = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \rightarrow 0 \text{ に束} \quad (x \rightarrow 0)$$

2018 · 7 · 2k

、 定理 4.5.3

、 $\left(1 + \frac{1}{n}\right)^n$ の 单增, 有界性

、 命題 4.1 $\exists \epsilon - \delta$ 使得 $\forall x$