Rennes I unive Conf. 2017/3/16

① $H = -\frac{1}{2}\Delta + V$ in $L^2(\mathbb{R}^d)$ Schrödinger op.

$V = 0 \cdot (f, e^{-tH} g) = \int dx\, \mathbb{E}^x[\overline{f(B_0)} g(B_t)]$

where $(B_t)_{t\geq 0}$ BM. on $(\Omega, \mathcal{F}, P^x)$

$V \neq 0$ $(f, e^{-tH} g) = \int dx\, \mathbb{E}^x[\overline{f(B_0)} g(B_t) e^S]$, where $P^x(B_0 = x) = 1$

$S = -\int_0^t V(B_s) ds$. $\underline{H\varphi = E\varphi}$ ~~φ~~ $\varphi = e^{-t(H-E)}\varphi$

$(e^{-tH} g)(x) = \mathbb{E}^x[g(B_t) e^S]$

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② $H = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2 - \frac{1}{2}$. $\sigma(H) = \{n\}_{n=0}^\infty$.

$H\varphi_n = n\varphi_n$ ~~$\varphi_n = \Pi(x_j - \frac{d}{dx_j})^n \varphi_0$~~ $\underline{\varphi_0 = (\pi)^{-1/4} e^{-\frac{1}{2}|x|^2}}$

$T : L^2(\mathbb{R}^d, dx) \longrightarrow L^2(\mathbb{R}^d, \varphi_0^2 dx)$ $\underline{\varphi_0 > 0}$.

$f \longmapsto Tf = \frac{1}{\varphi_0} f$

$H \longmapsto THT^{-1} = \frac{1}{\varphi_0} H \varphi_0 = -\frac{1}{2}\Delta + x\cdot\partial$

$\exists (X_t)_{t\geq 0}$ Ornstein Uhlenbeck process on $(\Omega, \mathcal{F}, P^x)$

s.t. $(f, e^{-tH} g) = (Tf, e^{-t\tilde{H}} Tg)_{L^2(\mathbb{R}^d, \varphi_0^2 dx)} = \int \varphi_0^2 dx\, \mathbb{E}^x[\overline{f(X_0)} g(X_t)]$

$= \int dx\, \mathbb{E}^x[\overline{f(B_0)} g(B_t) e^{-\int_0^t |B_s|^2 ds}]$

$\underline{V: \text{Kato class}.\ \varphi > 0}$ $H = -\frac{1}{2}\Delta + V$ φ.

$\exists (X_t)_{t\geq 0}$, $(\Omega, \mathcal{F}, P^x)$ s.t.

$(f, e^{-tH} g) = \int \varphi^2\, \mathbb{E}^x[\overline{f(X_0)} g(X_t)]$ $P(\phi)_1$-process.

(3) $\sqrt{-\Delta + m^2} - m + V = H.$

$\sqrt{-\Delta + m^2} - m = \psi(-\frac{1}{2}\Delta)$ where $\psi(u) = \sqrt{2u + m^2}$

$\psi \in C^{\infty}((0,\infty))$, $\psi(0) = 0$, $\psi^{(n)}(-1)^n \le 0$ e.g. u^{α}, ($0 < \alpha < 1$) $1 - e^{-\beta u}$

Bernstein function

$\psi(u) = \int_{(0,\infty)} (1 - e^{-\beta u}) d\rho(u) + bu$

Lévy measure $= \int_0^{\infty} y \wedge 1 \; \lambda(dy) < \infty$

$(T_t)_{t \ge 0}$ subordinator $\Leftrightarrow$ · 1-dim Lévy process

· $T_t \uparrow$ $(t \uparrow \infty)$

$\mathbb{E}[e^{-uT_t^{\psi}}] = e^{-t\psi(u)}$

$B \leftrightarrow T$

1:1 onto

$\psi \mapsto T_t^{\psi}$

$(f, e^{-t(\sqrt{-\Delta + m^2} - m)} g) = \mathbb{E}\left[(f, e^{-T_t (-\frac{1}{2}\Delta)} g)\right]$

$= \int dx \, \mathbb{E}\mathbb{E}^x[f(B_0) g(B_{T_t})]$

$(f, e^{-tH} g) = \int dx \, \mathbb{E}\mathbb{E}^x[f(B_0) g(B_{T_t}) e^{-S}]$

$S = \int_0^t V(B_s) ds$

Fractional $P(\phi)_1$ process $H = (-\Delta)^{\frac{\alpha}{2}} + V$

spin, vector potential

$\frac{1}{2}[\sigma \cdot (p - A)]^2 + V$, $\sqrt{[\sigma \cdot (p - A)]^2 + m^2} - m + V$. etc.

Decay properties $e^{tE} e^{-\int_0^t V(B_s + x)} \varphi(B_t + x) = X_t(x)$

is martingale ∴ $\mathbb{E}[X_0(x)] = \varphi(x) = \mathbb{E}[X_{t \wedge \tau}(x)]$

④ Q F T. (application)

$$H = H_S \otimes \mathbb{1} + \mathbb{1}\otimes H_f + g\phi.$$

acts on $L^2(\mathbb{R}^d)\otimes \mathcal{F}$

$$H_f = \int \omega a^+ a \, dk. \quad \phi(x) = \frac{1}{\sqrt{2}}\int a^+(k)\frac{\hat{\varphi}}{\sqrt{\omega}} e^{-ikx} + h.c.$$

$\omega = \sqrt{|k|^2+m^2}$ $\quad m \geq 0$ $\quad$ $m=0$ massless (← difficult), $m>0$ massive

$\hat{\varphi}/\sqrt{\omega} \in L^2$ $\quad$ UV cutoff.

(1) H has a ground state — Bach-Fröhlich $|g|\ll 1$ — Sigal 1999 — Griesemer-Lieb-Loss $\forall g$ 2001

$\int |\hat{\varphi}|^2/\omega^3 < \infty.$

(2) $H - E_{ren} \to H_\infty$

where $E_{ren} = -\int \frac{1}{|k|+|k|^2/2}\frac{|\hat{\varphi}|^2}{|\omega|} g^2$

(Nelson 1964, Gubinelli-H-Lörinczi 2014)

H_∞ has a ground state $\quad$ Hiroshima-H-Spohn 2005

$$\frac{e^{-TH} f\otimes\Omega_F}{\|e^{-TH} f\otimes \Omega_F\|} \to \varphi \quad (\text{as } T\to\infty) \qquad \sigma = e^{-\beta N}, e^{-\beta\phi^2}, \ldots$$

ground state of H_S ↓ $f=\varphi$ $\quad$ ①

$$(\varphi, \sigma\varphi) = \lim_T (\varphi^T, \sigma\varphi^T)$$

$$= \frac{1}{N_T}\int dx\, \mathbb{E}^x\left[\overline{f(B_{-T})} f(B_T) e^{-\int_{-T}^T V} e^{-\int_{-T}^T\int_{-T}^T W} e^{(1-e^{-\beta})\int_{-T}^0\int_0^T W}\right]$$

Ω

$$= \frac{1}{N_T}\int dx\, \varphi_0(x)^2\, \mathbb{E}^x\left[e^{-\int_{-T}^T\int_{-T}^T W(X_t-X_s,\, t-s)} e^{(1-e^{-\beta})\int\int W}\right]$$

$$= \mathbb{E}_{\mu_T}\left[e^{(1-e^{-\beta})\int_{-T}^0\int_0^T W}\right] \to \mathbb{E}_{\mu_\infty}\left[e^{(1-e^{-\beta})\int_{-\infty}^0\int_0^\infty W}\right]$$

Betz-H-Lörinczi-Minlos-Spohn. (2001)

Lemma $\quad \varphi \in D(e^{+\beta N})\ \forall\beta$

$\varphi \in D(e^{\frac{\beta}{2}\phi(f)^2})$ for $\beta \leq \frac{1}{\|f\|^2}$

$\|e^{+\frac{\beta}{2}\phi(f)^2}\varphi\| \to 0 \quad (\beta \uparrow \frac{1}{\|f\|^2})$

FKF of $(F, e^{-tH_\infty} G)$

Gubinelli - H - Lőrinczi 2014,
Matte + ~~θ~~ Møller 2017

$\exists$ Gibbs meas (H - Matte in preparating)

Cor. $\varphi_\infty \in D(e^{+\beta N})$ $\forall \beta \in \mathbb{C}$.

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