1936 Rabi

飛行機に乗り遅れる 2017/1/20

Rabi model 島根大学セミナー

1. Introduction, 2 Crossing 3 spectral zeta fct.
4. spin-boson model

interaction
photon 2013 Haroche etc
Nobel prize.

$2\Delta\{$ ⊗

· Bilbao, · Wakayam, · NcHO.

$$H = \Delta\sigma_z \otimes 1\!\!1 + 1\!\!1 \otimes a^\dagger a + g\sigma_x \otimes (a + a^\dagger)$$

$$\mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R})$$

· $\sigma_x, \sigma_y, \sigma_z$, · $\Delta \geq 0$, $g \in \mathbb{R}$, · $a = \frac{1}{\sqrt{2}}(x + \partial)$

$a^\dagger = \frac{1}{\sqrt{2}}(x - \partial)$

Euler. Gp.

Harmonic oscillators

$a^\dagger a = \frac{1}{2}\left(-\partial^2 + x^2 - \frac{1}{2}\right)$ $[a, a^\dagger] = 1\!\!1$

$\varphi_n = \underbrace{a^\dagger \cdots a^\dagger}_{n} \varphi_0$ $\varphi_0 = \pi^{-1/4} \exp\left(-\frac{|x|^2}{2}\right)$

$a^\dagger a\, \varphi_n = n\varphi_n$ $n = 0, 1, 2 \ldots$ $\mathrm{spec}(a^\dagger a) = \{0, 1, \cdots\}$

↑数論.

$$H \cong -\Delta\sigma_x + a^\dagger a + g\sigma_z(a + a^\dagger)$$

$$= \begin{pmatrix} a^\dagger a + \sqrt{2}g x & -\Delta \\ -\Delta & a^\dagger a - \sqrt{2}g x \end{pmatrix}$$

$\Delta = 0$ のとき

$$a^\dagger a + \sqrt{2}g x = \frac{1}{2}\left(-\partial^2 + (x + \sqrt{2}g)^2 - 1\right) - g^2$$
$$\cong a^\dagger a - g^2$$

$$\therefore H \cong \begin{pmatrix} a^\dagger a & 0 \\ 0 & a^\dagger a \end{pmatrix} - g^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore$ $\mathrm{spec}(H) = \{n - g^2\}_{n=0}^{\infty}$ multiplicity $= 2$

Bargmann - Rep.

$\mathcal{B} \ni f \iff$ ① $f: \mathbb{C} \to \mathbb{C}$ analytic

② $\pi^{-1}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dxdy < \infty$

$z = x + iy$

$\mathcal{B} \cong L^2$

$L^2(\mathbb{R}) \ni h(x) \to \sqrt{2}\int_{\mathbb{R}} h(x)\, e^{2\pi xz - \pi x^2 - \frac{\pi}{2}z^2} dx \in \mathcal{B}$

$-\pi(x-z)^2 + \frac{\pi}{2}z^2$

$= \sqrt{2}\, e^{\frac{\pi}{2}z^2}\int_{\mathbb{R}} h(x)\, e^{-\pi(x-z)^2} dx$

$f(z) = \sum_{n=0}^{\infty} a_n z^n$

$\to \sum_{n=0}^{\infty} a_n a^{\dagger n}\varphi_0 = \sum_{n=0}^{\infty} a_n \varphi_n$

- $a \sim \frac{d}{dz}$, $a^\dagger \sim z\times$
- L.H. $\{z^n\} = \mathcal{B}$.

$H\varphi = E\varphi$

$$\left[\Delta\sigma_x + z\frac{d}{dz} + g\sigma_z\left(z + \frac{d}{dz}\right)\right]\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = E\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

$$\begin{cases}(z+g)\frac{d\varphi_1}{dz} + (gz - E)\varphi_1 + \Delta\varphi_2 = 0\\ (z-g)\frac{d\varphi_2}{dz} - (gz+E)\varphi_2 + \Delta\varphi_1 = 0\end{cases}$$

$\Delta = 0$ のとき

$$\frac{d\varphi_1}{dz} + \frac{gz - E}{z+g}\varphi_1 = 0$$

$$\left(\frac{d\varphi_1}{dz} - \frac{E+g^2}{z+g} + g\right)\varphi_1 = 0 \quad \therefore \varphi_1(z) = e^{-gz}(z+g)^{E+g^2}$$

similarly

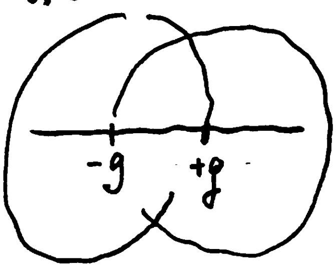
$\varphi_2 = e^{gz}(z-g)^{E+g^2} \in \mathcal{B} \Leftrightarrow E+g^2 \in \mathbb{N}^\times$ $E+g^2 \in \mathbb{N}^\times$ のとき solになる

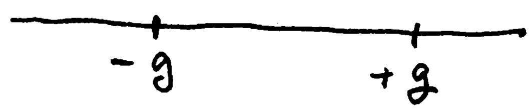
2. Crossing between E_{2n} and E_{2n+1}

$\Delta \neq 0$ spec$(H) = \{E_0, E_1, E_2 \cdots\}$

$H\varphi = E\varphi$

$\frac{d}{dz}\varphi + A(z)\varphi = 0$ where $A(z) = \begin{pmatrix} \frac{gz-E}{z+g} & \Delta \\ \Delta & -\frac{gz+E}{z-g} \end{pmatrix}$

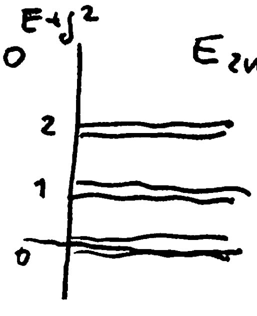




∘ 解空間の次元 ≤ 2

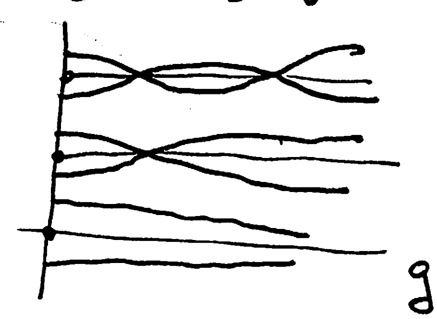
∘ $\mathbb{R} \ni g \mapsto E_n(g)$ spectral curves

$\Delta = 0$



$E_{2n}+g^2 = E_{2n+1}+g^2 = n$,

$\Delta \neq 0$ $E+g^2$



$E_{2n} = E_{2n+1} = n + g^2$

Thm (H+ Hirokawa)
$E_0(g)$ has no-crossing.

$a^\dagger a = h_0$ $(X_t)_{t \geq 0}$ OU process

$$(f, e^{-th_0'} g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} dx\, \varphi_0(x)^2\, \mathbb{E}^x\left[\overline{f(X_0)}\, g(X_t)\right] \quad \text{FKF}$$

where $h_0' = \frac{1}{\varphi_0} h_0 \varphi_0 = \frac{1}{2}(-\partial^2 + x\partial)$ unitary.

$$\frac{1}{\varphi_0} H \varphi_0 = h_0' + \Delta\sigma_x + \sqrt{2} g \sigma_x x = H'$$

$$(H'f)(x, \sigma) = (h_0' + \sqrt{2} g x) f(x, \sigma) + \Delta f(x, -\sigma)$$

$$\mathbb{C}^2\otimes L^2(\mathbb{R}) \cong L^2(\mathbb{R}\times\{\pm 1\}) \cong \in L^2(\mathbb{R}\times\{\pm1\}, \varphi_0^2 dx)$$

$$(f, e^{-tH'}g) = \sum_{\sigma=\pm1}\int dx\, \varphi_0(x)^2\, \mathbb{E}^{x\sigma}\left[f(x_0\sigma_0)g(x_t\sigma_t)S\right]$$

$$S = e^{\sqrt{2}g\int_0^t X_s ds}\Delta^{N_t} \qquad (N_t)_{t\geq 0} \text{ Poisson Process}$$

$$\sigma_t = (-1)^{N_t}$$

もし $\Delta = 0$ のときは

$N_t = 0$ のところだけ生き残る $\qquad \mathbb{E}[N_t = n] = \frac{t^n}{n!}e^{-t}$

$$\sigma_0 = (-1)^{N_0} = \sigma \quad \Big| \quad (f, e^{-tH'}g) = \sum_{\sigma=\pm1}\int dx\, \varphi_0^2(x)\, \mathbb{E}^{x}\left[f(x_0\sigma)g(x_t\sigma)e^{S}\right]$$
$$\sigma_t = (-1)^{N_t} = \sigma$$

$$(f, e^{-tH'}g) \underset{\neq}{\geq} 0 \quad \text{if } f\geq 0,\ g\geq 0$$

Perron – Frobenius Thm $\Rightarrow$ E_0 is simple.

Thm 2. (Kuś, Wakayama-Yamazaki)

Let $0<\Delta<1$. Then E_{2n} と E_{2n+1} は

n 回 ~~cross~~ crossing する.

Remark: 1) $E_{2n} = E_{2n+1} = n - g^2$ の形になる.

2) 縮退しているときは Parity にわけられる

$$P = (-1)^{a^\dagger a}\otimes\sigma_x$$

3 Spectral zeta function (Hurwitz)

$$\zeta(s;\tau) = \sum_{n=0}^{\infty} \frac{1}{(n+\tau)^s} \quad (s>1)$$

$s \in \mathbb{C}$ meromorphic ft.

$$\mathrm{Spec}(H) = \{E_j(g)\}_{j=0}^{\infty}$$

$\inf \mathrm{Spec}(H) \geq -g^2 - \Delta$ なので

$$\zeta_g(s;\tau) = \zeta_g(s) = \sum_{j=0}^{\infty} \frac{1}{(E_j + g^2 + \tau)^s}$$

$$(\tau > \Delta)$$

- $^{\exists}c j \leq E_j \leq {}^{\exists}C j \quad \forall j = 0,1,2\ldots$
- $\zeta_g(s)$ conv. for $s>1$.

Thm 3 (S. Sugiyama) $\exists$ meromorphic ext. of $\zeta_g(s)$ for any g.

$$H \cong \begin{pmatrix} a^+a - g^2 & \Delta e^{i2\sqrt{2}gx} \\ \Delta e^{-i2\sqrt{2}gx} & a^+a - g^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^+a & 0 \\ 0 & a^+a \end{pmatrix} - g^2 I + T_g \qquad T_g = \begin{pmatrix} 0 & \Delta e^{i2\sqrt{2}gx} \\ \Delta e^{-i2\sqrt{2}gx} & 0 \end{pmatrix}$$

@ $e^{igx} \xrightarrow{w} 0 \quad (g \to \pm\infty)$

$$\zeta_g(s) = \mathrm{Tr}\left(\begin{pmatrix} a^+a & 0 \\ 0 & a^+a \end{pmatrix} + T_g + \tau \right)^{-s}$$

$$\to 2\,\mathrm{Tr}(a^+a + \tau)^{-s} = 2\zeta(s,\tau)$$

Thm 4 $\displaystyle\lim_{g\to\infty} \zeta_g(s) = 2\zeta(s;\tau)$. $\tau > \Delta$

Spin-boson model $\qquad a^{+} \to a^{+}(f) \quad f \in L^2$

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n L^2(\mathbb{R}^d) \right], \quad a^{+}(f), \quad a(f)$$

- CCR $[a(f), a^{+}(g)] = (\bar{f}, g)$.
- $H_f \, a^{+}(f_1) \cdots a^{+}(f_n) \Omega = \sum_{j=1}^{n} a^{+}(f_1) \cdots a^{+}(\omega f_j) \cdots a^{+}(f_n) \Omega$

$$H = \Delta \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes H_f + g \sigma_x \otimes \phi(f)$$

<u>Thm</u> $f/\omega \in L^2 \Rightarrow H$ has the unique ground state

Embeded e.v. $\curvearrowright$ perturbation theory.