北大数物セミナー
2017/10/5

Nelson model $\mathcal{H} = L^2(\mathbb{R}^3) \otimes L^2(Q)$

$H_\varepsilon = H_p + H_f + g\,\phi(\tilde{\varphi}_\varepsilon(\cdot - x))$ / $\tilde{\varphi}_\varepsilon = \left(\frac{\hat{\varphi}_\varepsilon}{\sqrt{\omega}}\right)^\vee 1\!\!1_{|k|>\lambda}$

$H_p = -\frac{1}{2}\Delta + V$, $H_f = d\Gamma(\omega)$ $\omega = |k|$

$\phi(f)$, $f \in L^2_{\mathbb{R}}(\mathbb{R}^3)$ Gaussian r.v. (Q, Σ, μ)

$\mathbb{E}_\mu[\phi(f)] = 0$ $\mathbb{E}_\mu[\phi(f)\phi(g)] = \frac{1}{2}(f, g)$.

$\sim$ $\mathbb{E}[e^{z\phi(f)}] = e^{\frac{z^2}{4}\|f\|^2}$ / $\varphi_\varepsilon = e^{-\frac{\varepsilon}{2}|k|^2} \to 1$ $(\varepsilon \downarrow 0)$

$\sim$ $\mathbb{E}[e^{-\phi(f)^2}] = \frac{1}{\sqrt{1 + \|f\|^2/2}}$

$\phi(f) \cong \frac{1}{\sqrt{2}}\left[a^*(\hat{f}) + a(\tilde{\hat{f}})\right]$ $f \in L^2_{\mathbb{R}}(\mathbb{R}^3)$

[E. Nelson (1964) $E_\varepsilon = -\frac{g^2}{2}\int \frac{1}{\omega + |k|^2/2}\left(\frac{\hat{\varphi}_\varepsilon}{\sqrt{\omega}}\right)^2 dk$ $(\to -\infty)$

$e^{-T(H_\varepsilon - E_\varepsilon)} \to e^{-T H_{ren}}$ strongly]

H_{ren}? H_{ren}はH_fを含むか? H_{ren}のground state? locality?

FKF: $(B_t)_{t\in\mathbb{R}}$ BM. on $(\Omega, \mathcal{F}, P)$

$$(F, e^{-TH_\varepsilon} G) = \int dx\, \mathbb{E}_P^x\left[e^{-\int_0^T V(B_s)ds}\left(J_0 F(B_0), e^{-\phi(A)} J_T G(B_T)\right)\right]$$

$J_T^* J_s = e^{-|T-s|H_f}$, $A = \int_0^T j_s \tilde{\varphi}_\varepsilon(\cdot - B_s)\,ds$

$j_s^* j_t = e^{-|t-s|\omega(-i\nabla)}$

GHL (2014)

一般論 $e^X e^Y e^{-X} = e^{ad_X}(e^Y)$

$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]}$ ($[X,Y]$ scalar)

$$e^{-\phi(A)} = e^{-\frac{1}{\sqrt{2}}[a^*(\hat{A}) + a(\tilde{\hat{A}})]} = e^{a^*(\hat{A})} e^{a(\tilde{\hat{A}})} e^{\frac{1}{4}\|\hat{A}\|^2}$$

$$(F_1\ e^{-2TH_\varepsilon} G) = \int dx\, \mathbb{E}^x\left[e^{-\int_{-T}^{T} V} \left(J_{-T} F(B_{-T}),\ e^{a^+(U)} e^{a(\tilde U)} J_T G(B_T)\right) e^{\frac{1}{4}\Box} \right]$$

$$\Box = \int_{-T}^{T} dt \int_{-T}^{T} ds\ W(B_t - B_s\ \ t-s) \qquad W(X, t) = \int \frac{|\hat\varphi_\varepsilon|^2}{\omega} e^{-|t|\omega} e^{-ikB_t} dk$$

$$J_{-T}^* e^{a^+(U)} e^{a(\tilde U)} J_T = \overbrace{e^{a^+(K)} e^{-2TH_f}}\ \overbrace{e^{a(\tilde K)}} = \underline{A^* A}$$

$$K_\varepsilon = \int_{-T}^{T} e^{-|T+s|\omega} \frac{\hat\varphi_\varepsilon}{\sqrt\omega} e^{-ik\cdot B_s} dk, \quad \tilde K_\varepsilon = \int_{-T}^{T} e^{-|T-s|\omega} \frac{\hat\varphi_\varepsilon}{\sqrt\omega} e^{ik\cdot B_s} dk$$

<u>Lemma</u>. $K_0 = \int_{-T}^{T} e^{-|T+s|\omega} \frac{1}{\sqrt\omega} e^{-ikB_s} \in L^2$ a.s.

$\tilde K_0 \in L^2$

$\because$ $\mathbb{E}\int |K_0|^2 dk < \infty$ / path に一様ではない.

Thm (Matte - Møller 2017).

$$(F_1\ e^{-2T(H_\varepsilon - E_\varepsilon)} G) \Rightarrow \int dx\ \mathbb{E}^x\left(e^{-\int_{-T}^{T} V} (F(B_{-T}), A_0^* A_0 G(B_T)) e^{\exists S_{ren}} \right]$$

$A_0 = e^{-TH_f} e^{a(\tilde K_0)}$.

<u>Cor.</u> $\exists_1$ g.s.

$\because$ $e^{a(\tilde K_0)}$ は shift op.

Cor. $F = \cancel{\mathbb{1}\otimes\phi}\ \phi\otimes\mathbb{1}$

$$(F, e^{-2TH_{ren}} G) = \int dx\, \mathbb{E}^x\left[e^{-\int_{-T}^{T} V} e^{S_{ren}} \right]$$

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(1) $N_\Lambda = \int \mathbb{1}_{|k|<\Lambda}\, a^+(k) a(k) dk$. $\qquad \mathcal{O} = e^{-\beta N_\Lambda}$

$$(\varphi_g, \mathcal{O}\varphi_g) = \lim_T (\varphi_g^T\ \mathcal{O}\varphi_g^T) = \lim_T \lim_\varepsilon (\varphi_g^{T\varepsilon}\ \mathcal{O}\ \varphi_g^{T\varepsilon})$$

$$\varphi_g^{T\varepsilon} = e^{-TH_\varepsilon} \phi\otimes\mathbb{1} / \| e^{-TH_\varepsilon} \phi\otimes\mathbb{1} \|$$

$$\to e^{-TH} \phi\otimes 1 / \| e^{-TH} \phi\otimes 1 \| \to \varphi_g$$

$$(\varphi_g^T, \mathcal{O}\varphi_g^T) = \mathbb{E}_{\mu_T}\left[e^{-2(1-e^{-\beta}) \int_{-T}^{0} dt \int_0^T ds\, W} \right]$$

$$W = \int_{\lambda \le |k| \le \Lambda} \frac{1}{\omega} e^{-|t|\omega} e^{-ikB_t} dk$$

$$\mu_T(A) = \int dx\, \mathbb{E}^x\left[e^{-\int_{-T}^{T} V} \mathbb{1}_A \, \phi(B_{-T})\phi(B_T)\, e^{S_{ren}} \right]$$

Lemma (HM, H, HHL, BS)

$\mathcal{F}_{[-T,T]} = \sigma(B_r,\ r \in [-T,T])$

$\bigcup_{T \ge 0} \mathcal{F}_{[-T,T]} = \mathcal{G} \to \sigma(\mathcal{G})$

$\exists \mu_\infty$ on $(\Omega, \sigma(\mathcal{G}))$ st $\mu_T \to \mu_\infty$
local weak conv.

i.e. $\forall A \in \mathcal{F}_{[-S,S]}$, $\mu_T(A) \to \mu_\infty(A)$.

ちなみに
$$\mu_\infty(A) = e^{2ES} \int dx\, \mathbb{E}^x\left[\mathbb{1}_A \left(\varphi_g(B_{-S}), \underbrace{K_S}_{A_S^* A_S} \varphi_g(B_S) \right) \right]$$
$A \in \mathcal{F}_{[-S,S]}$

$$\mathrm{Cov}(\varphi_g, e^{-\beta N_\Lambda} \varphi_g) = \mathbb{E}_{\mu_\infty}\left[e^{-2(1-e^{-\beta})\int_{-\infty}^{0}\int_{0}^{\infty} W} \right] < \infty \quad \forall \beta \in \mathbb{R}.$$

同様 (というかもっと easy)

$$(\varphi_g, e^{-\beta \phi(f)^2} \varphi_g) = \frac{1}{\sqrt{1+\beta\|f\|^2}} \mathbb{E}_{\mu_\infty}\left[e^{-\frac{S_\infty}{2(1+\beta\|f\|^2)}} \right]$$

同様に exp decay for x も示せる