Analysis of a scalar field model without ultraviolet cutoff by path measures.

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2017/9/4-9/8
Tosio Kato Centennial Conference
University of Tokyo

The method of renormalization of the Nelson Hamiltonian

- Operator theory
- Stochastic method
- Perturbation theory

A belief history of renormalization of the Nelson Hamiltonian

- 1963: Nelson, renormalization by stochastic method
- 1964: Nelson, renormalization by operator theory
- 1973: Fröhlich, asymptotic field of fiber Hamiltonian
- 2001: Ammari, HZV-type theorem for massive case
- 2005: Hirokawa+FH+Spohn, the existence of ground state
- 2012: Gérard-FH-Panati-Suzuki, renormalization on a manifold
- 2014: Gubinelli-FH-Lorinczi, renormalization by stochastic method
- 2016: Ammari-Falconi, semi-classical analysis
- 2017: Matte+Møller, FH, FKF and properties of ground state $\leftarrow$ Todays talk.....

►Gaussian random variable $\phi(f)$, $f \in L^2_{\mathbb{R}}(\mathbb{R}^3)$, on (Q, Σ, μ)

$$E_\mu[\phi(f)] = 0, \quad E_\mu[\phi(f)\phi(g)] = \frac{1}{2}(f, g)_{L^2(\mathbb{R}^3)}$$

Cf.

$$\phi(f) = \frac{1}{\sqrt{2}} \int a^\dagger(k)\hat{f}(k) + a(k)\hat{f}(-k)dk \quad CCR \ \ [a(k), a^\dagger(k)] = \delta(k - k').$$

►Boson Fock space $\mathscr{F} = L^2(Q, \mu)$

►Total Hilbert space of the Nelson model is $\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathscr{F}$

►Nelson Hamiltonian with UV(ultraviolet) cutoff parameter $\varepsilon > 0$:

$$H_\varepsilon = H_p \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + g\phi(\rho_\varepsilon(\cdot - x))$$

►$H_p = -\frac{1}{2}\Delta + V(x)$ is Schrödinger op. e.g., V is Kato class potential.
►$H_{\rm f} = d\Gamma(\omega(-i\nabla))$ is free field Hamiltonian. I.e.,

$$H_{\rm f} : \phi(f_1)\cdots\phi(f_n) := \sum_j : \phi(f_1)\cdots\phi(\omega(-i\nabla)f_j)\cdots\phi(f_n) :$$

Formally

$$H_{\rm f} = \int \omega(k)a^\dagger(k)a(k)dk$$

with the dispersion relation $\omega(k) = |k|$.
►By Kato-Rellich theorem H_ε is s.a. on $D(H_p) \cap D(H_{\rm f})$ and $H_\varepsilon > -\exists C_\varepsilon$.
►Kato theory was introduced by Kato-Mugibayashi, Høegh Krohn, etc.

►UV cutoff function ρ_ε with UV cutoff parameter $\varepsilon > 0$. The Fourier transform of ρ_ε is given by

$$\hat{\rho}_\varepsilon(k) = \frac{e^{-\varepsilon|k|^2/2}}{\sqrt{\omega(k)}} 1\!\!1_{|k|>\lambda} \in L^2(\mathbb{R}^3) \quad \varepsilon > 0$$

►Removal of UV cutoff:

$$\lim_{\varepsilon \downarrow 0} \hat{\rho}_\varepsilon(k) = \frac{1}{\sqrt{\omega(k)}} 1\!\!1_{|k|>\lambda} \notin L^2(\mathbb{R}^3)$$

THEOREM (E. NELSON, 1964)

Let $E_\varepsilon = -\frac{1}{2}\int_{|k|>\lambda} |\hat{\rho}_\varepsilon(k)|^2 \beta(k) dk$ with $\beta(k) = \frac{1}{\omega(k)+|k|^2/2}$, where $E_\varepsilon \to -\infty$ as $\varepsilon \downarrow 0$. Then a self-adjoint operator $\exists H_{\rm ren}$ such that

$$\lim_{\varepsilon \downarrow 0} e^{-T(H_\varepsilon - g^2 E_\varepsilon)} = e^{-i\pi} e^{-TH_{\rm ren}} e^{i\pi},$$

where $e^{i\pi}$ is a unitary operator called Gross transform.

$[[H_p]]$

►$(B_t)_{t\in\mathbb{R}}$ is BM on whole real line $\mathbb{R}$ on wiener space $(\Omega, \mathscr{F}, W)$.

►$(f, e^{-TH_p} g)_{L^2(\mathbb{R}^3)} = \int dx E_W^x[\overline{f(B_{-T})} e^{-\int_{-T}^T V(B_s)ds} g(B_T)]$

$[[H_{\rm f}]]$ By B.Simon [$P(\phi)_2$ Euclidean quantum field theory]

►$\mathrm{J}_T = \Gamma(j_T)$ and $j_T : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^4)$, $T \in \mathbb{R}$, is the family of isometries:

►$j_t^* j_s = e^{-|t-s|\omega(-i\nabla)}$ and $J_T^* J_S = e^{-|T-S|H_{\rm f}}$.

►$(\Psi, e^{-2TH_{\rm f}}\Phi)_{\mathscr{F}} = E_\mu[\overline{\Psi \mathrm{J}_{-T}^*} \mathrm{J}_T \Phi]$.

$[[H_\varepsilon]]$ By 

(Lőrinczi-FH-Betz,11) for $F, G \in \mathscr{H}$ we have FKF for $0 < \varepsilon$

$$(F, e^{-2TH_\varepsilon} G)_{\mathscr{H}} = \int dx E_{W\times\mu}^x \left[\overline{F(B_{-T})} L_T G(B_T) \right]$$

►$L_T = e^{-\int_{-T}^T V(B_s)ds} \mathrm{J}_{-T}^* e^{-g\phi(\int_{-T}^T j_s \rho_\varepsilon(\cdot - B_s)ds)} \mathrm{J}_T$

►Fock vacuum $\mathbb{1} \in \mathscr{F}$ and $f, h \in L^2(\mathbb{R}^3)$ are fixed.

►$(f \otimes \mathbb{1}, e^{-2TH_\varepsilon} h \otimes \mathbb{1}) = \int dx E_W^x \left[\overline{f(B_{-T})} h(B_T) e^{\frac{g^2}{2} S_\varepsilon} e^{-\int_{-T}^T V(B_s) ds} \right]$

►Pair interaction

$$S_\varepsilon = \int_{-T}^T ds \int_{-T}^T dt W_\varepsilon(B_t - B_s, t - s)$$

►Pair potential

$$W_\varepsilon(B_t - B_s, t - s) = \frac{1}{2} \int |\hat{\rho}_\varepsilon(k)|^2 e^{-i(B_t - B_s) \cdot k} e^{-\omega(k)|t-s|} dk.$$

►The diagonal part of S_ε is singular at $\varepsilon = 0 \Longrightarrow E_\varepsilon$. Then

$$S_\varepsilon = S_{OD} + S_D$$

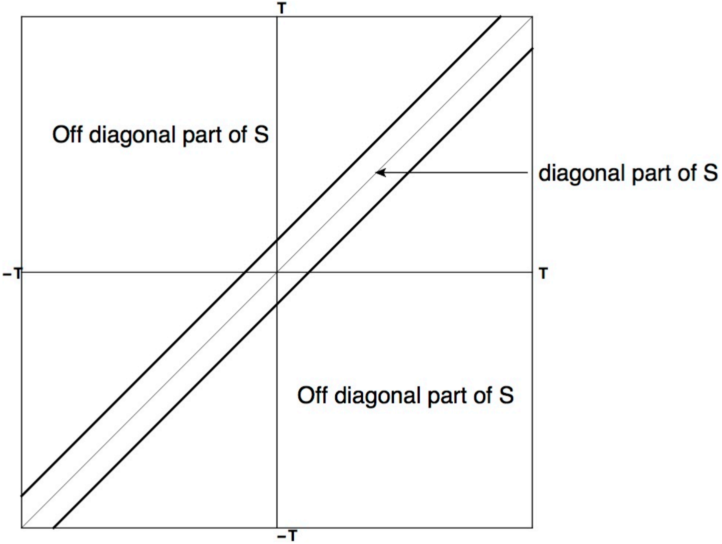

FIGURE: S_ε

►Let $\varphi_\varepsilon(x,t) = \dfrac{1}{2}\displaystyle\int |\hat\rho_\varepsilon(k)|^2 \frac{e^{-ik\cdot x - \omega(k)|t|}}{\omega(k)+|k|^2/2} dk.$

►Ito formula:

$$f(t,B_t) - f(0,0) = \int_0^t \nabla f(s,B_s)dB_s + \int_0^t (\partial s + \tfrac{1}{2}\Delta_x) f(s,B_s) ds$$

$$\varphi_\varepsilon(B_T - B_s, T-s) - \varphi_\varepsilon(0,0) = \int_s^T \nabla\varphi_\varepsilon(B_t - B_s, t-s)dB_t - \int_s^T W_\varepsilon(B_t - B_s, t-s$$

►$-\varphi_\varepsilon(0,0) = E_\varepsilon$

►$S_\varepsilon = S_{\mathrm{OD}} + \underbrace{Y_\varepsilon + Z_\varepsilon + 4T\varphi_\varepsilon(0,0)}_{\text{diagonal part}=\mathrm{S_D}}$ with

$$Y_\varepsilon = 2\int_{-T}^T ds \int_{s-\tau}^s \nabla\varphi_\varepsilon(B_t - B_s, t-s)\cdot dB_t \leftarrow \text{dangerous part}$$

$$Z_\varepsilon = -2\int_{-T}^T \varphi_\varepsilon(B_{s+\tau} - B_s, (s+\tau)-s)ds$$

$$S_{\mathrm{OD}} = 2\int_{-T}^T ds \int_{s+\tau}^T dt W_\varepsilon(B_t - B_s, t-s)$$

$$S_\varepsilon^{ren} = S_\varepsilon + 4TE_\varepsilon = S_{\mathrm{OD}} + Y_\varepsilon + Z_\varepsilon$$

for any $\tau > 0$. τ is the width of the diagonal part of $[-T,T]\times[-T,T]$.

Let $K_\varepsilon = (f\otimes 1\!\!1, e^{-2T(H_\varepsilon - g^2 E_\varepsilon)} h\otimes 1\!\!1) = \int_{\mathbb{R}^3} E_W^x \left[\overline{f(B_{-T})} h(B_T) e^{\frac{g^2}{2} S_\varepsilon^{ren}} \right] dx$

THEOREM (GUBINELLI+FH+LORINCZI, JFA 14)

► $\exists S_{\rm ren}\ st \lim_{\varepsilon\downarrow 0} K_\varepsilon = \int_{\mathbb{R}^3} E_W^x \left[\overline{f(B_{-T})} h(B_T) e^{\frac{g^2}{2} S_{\rm ren}} \right] dx$

► $\exists C\ st\ |K_\varepsilon| \le \|f\| \|h\| e^{CT}$

► $\exists C'\ st \inf \sigma(H_\varepsilon - g^2 E_\varepsilon) \ge C'$

► $\exists \lim_{\varepsilon\downarrow 0}(F, e^{-2T(H_\varepsilon - g^2 E_\varepsilon)} G)\ for\ F, G \in \exists Dense\ \mathscr{D}.$

► The uniform lower bound implies that

$$\exists \lim_{\varepsilon\downarrow 0}(F, e^{-2T(H_\varepsilon - g^2 E_\varepsilon)} G) \quad \forall F, G \in \mathscr{H}.$$

THEOREM (COROLLARY)

$\exists H_{\rm ren}\ such\ that \lim_{\varepsilon\downarrow 0} e^{-T(H_\varepsilon - g^2 E_\varepsilon)} = e^{-TH_{\rm ren}}$

► $V = 0$ and $\inf \sigma(H_\varepsilon) = E(g^2) = E(0) + g^2 E_\varepsilon + O(g^3)$

THEOREM (FH2015, PREPRINT)

$$\lim_{g \to 0} \frac{E(g^2) - E(0)}{g^2} = E_\varepsilon$$

$$E(g^2) = -\frac{1}{2T} \lim_{T \to \infty} \log(f \otimes 1\!\!1, e^{-2TH_\varepsilon} f \otimes 1\!\!1)$$

The most crucial fact is that we do not know the explicit form of UV renormalized Hamiltonian H_{ren} and the domain!

▶ $(F, e^{-2TH_\varepsilon} G) =$

$$\int dx E^x_{W\times\mu}\left[e^{-\int_{-T}^T V(B_s)ds}\overline{F(B_{-T})}\mathrm{J}^*_{-T} e^{-g\phi(\int_{-T}^T j_s\rho_\varepsilon(\cdot - B_s)ds)}\mathrm{J}_T G(B_T)\right]$$

▶ The Baker-Campbell-Hausdorff formula: $e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]}$ if $[X, Y]$ commutes with X and Y. We have

$$e^{2TE_\varepsilon}\mathrm{J}^*_{-T} e^{-g\phi(\int_{-T}^T j_s\rho_\varepsilon(\cdot - B_s)ds)}\mathrm{J}_T$$

$$= e^{\frac{g^2}{2}S_\varepsilon + 2TE_\varepsilon}\mathrm{J}^*_{-T} e^{-a^\dagger(\frac{g}{\sqrt{2}}\int_{-T}^T j_s\rho_\varepsilon(\cdot - B_s)ds)} e^{-a(\frac{g}{\sqrt{2}}\int_{-T}^T j_s\rho_\varepsilon(\cdot - B_s)ds)}\mathrm{J}_T$$

$$= e^{\frac{g^2}{2}S_\varepsilon + 2TE_\varepsilon}\underbrace{e^{-a^\dagger(\frac{g}{\sqrt{2}}\int_{-T}^T e^{-|s+T|\omega}\hat\rho_\varepsilon e^{-ikB_s}ds)} e^{-TH_{\rm f}}}_{\text{bounded op.}}\underbrace{e^{-TH_{\rm f}} e^{-a(\frac{g}{\sqrt{2}}\int_{-T}^T e^{-|s-T|\omega}\hat\rho_\varepsilon e^{ikB_s}ds)}}_{\text{bounded op.}}$$

$$\stackrel{\varepsilon\to 0}{\to} e^{\frac{g^2}{2}S_{\rm ren}} e^{a^\dagger(U_T)} e^{-2TH_{\rm f}} e^{a(\bar U_T)}$$

$$U_T = -\frac{g}{\sqrt{2}}\int_{-T}^T \frac{e^{-|s+T|\omega(k)}}{\sqrt{\omega(k)}} e^{-ikB_s}ds \in L^2(\mathbb{R}^3_k) \quad a.s.(*)$$

To prove (*) is non-trivial!

Theorem (Matte and Møller 2017)

Let $F, G \in \mathscr{H}$. Then

$$(F, e^{-TH_{\rm ren}}G)$$
$$= \int dx E_W^x[e^{-\int_{-T}^T V(B_s)ds} e^{\frac{g^2}{2}S_{\rm ren}}(F(B_0), e^{a^\dagger(U_T)}e^{-2TH_{\rm f}}e^{a(\bar{U}_T)}G(B_t))].$$

[Ground state of QFT model with UV cutoff]

- Bach,Fröhlich and Sigal, Adv Math 1997, CMP 1999
- Arai and Hirokawa, JFA 1997
- Griesemer,Lieb and Loss, Inventiones 2001

Corollary (Hirokawa+FH+Spohn Adv Math 05, MM17)

There exists a $g_0 > 0$ such that $H_{\rm ren}$ has the ground state and it is unique providing that $|g| < g_0$.

Proof: The existence of ground state is due to [HHS 05] and the uniqueness can be derived from [MM17] and the Perron-Frobenius theorem ☐

Gibbs measure is useful to estimate the ground state expectations. Let $\varphi_{\rm g}$ be the ground state of $H_{\rm ren}$.

►Let $0 \le \phi \in L^2(\mathbb{R}^3)$ and since $(\phi \otimes 1\!\!1, \varphi_{\rm g}) \neq 0$,

$$\phi_T^\varepsilon = \frac{e^{-TH_\varepsilon}\phi \otimes 1\!\!1}{\|e^{-TH_\varepsilon}\phi \otimes 1\!\!1\|} \stackrel{\varepsilon\downarrow 0}{\to} \phi_T = \frac{e^{-TH_{\rm ren}}\phi \otimes 1\!\!1}{\|e^{-TH_{\rm ren}}\phi \otimes 1\!\!1\|} \stackrel{T\to\infty}{\to} \varphi_{\rm g}$$

►$(\varphi_{\rm g}, O\varphi_{\rm g}) = \lim_{T\to\infty}\lim_{\varepsilon\downarrow 0}(\phi_T^\varepsilon, O\phi_T^\varepsilon) = \lim_{T\to\infty}\lim_{\varepsilon\downarrow 0}\frac{(e^{-TH_\varepsilon}\phi \otimes 1\!\!1, Oe^{-TH_\varepsilon}\phi \otimes 1\!\!1)}{\|e^{-TH_\varepsilon}\phi \otimes 1\!\!1\|^2}$

►On the Wiener space $(\Omega, \mathscr{F}, W)$ the finite volume Gibbs measure is defined by

$$\mu_T(A) = \frac{1}{Z_T}\int_{\mathbb{R}^3} dx E_W^x \left[1\!\!1_A \phi(B_{-T})\phi(B_T) e^{\frac{g^2}{2} S_{\rm ren}} \right]$$

for $A \in \mathscr{F}$.

(1) Number of bosons $d\Gamma(\mathbb{1}_{|k|<\Lambda}) = N_\Lambda$

►$\lim_T (\phi_T, e^{-\beta N_\Lambda}\phi_T) = \lim_T E_{\mu_T}[e^{-(1-e^{-\beta})\int_{-T}^0 ds \int_0^T dt W_\Lambda}]$

►$W_\Lambda = \int_{\lambda<|k|<\Lambda} \frac{1}{\omega(k)} e^{-|t-s|\omega(k)} e^{-ik(B_t-B_s)} dk$

(2) Gaussian domination

►$\lim_T (\phi_T, e^{ik\phi(f)}\phi_T) = \lim_T e^{-\frac{|k|^2}{4}\|f\|^2} E_{\mu_T}[e^{+ikS_T}]$

►$\lim_T (\phi_T, e^{-\beta\phi(f)^2}\phi_T) = \lim_T \frac{1}{\sqrt{1+\beta\|f\|^2/2}} E_{\mu_T}\left[e^{-\frac{\beta S_T^2}{2(1+\beta\|f\|^2/2)}}\right]$

►$S_T = \frac{1}{2}\int_{-T}^T ds \int_{\lambda<|k|} \frac{e^{-|s|\omega(k)}}{\omega(k)} e^{-ikB_s} f(k) dk.$

▶Question:

$$\mu_T \to \mu \quad (T \to \infty)?$$

[Existence of infinite volume Gibbs measure with UV cutoff]

- Spohn, CMP 90, SB model
- Osada-Spohn, Ann Prob 99, Nelson model (some approximation)
- Betz-FH-Lorinzi-Minlos-Spohn, RMP 01, Nelson model
- Betz-Spohn 02, translation invariant Nelson model
- Hirokawa-FH-Lorinczi, Math Z 14, SB model
- FH., Adv Math 14, semi-relativistic PF model

$\mathscr{F}_{[-S,S]} = \sigma(B_r, r \in [-S,S])$ and we set $\mathscr{G} = \sigma(\cup_{S\geq 0}\mathscr{F}_{[-S,S]})$

THEOREM (GIBBS MEASURE FH2017)

Let μ_T be the finite volume Gibbs measure on $(\Omega, \mathscr{G})$. There exists a prob. measure μ_∞ on $(\Omega, \mathscr{G})$ such that $\mu_T \to \mu_\infty$ as $T \to \infty$ in the local weak sense. I.e., $\mu_T(A) \to \mu_\infty(A)$ for $A \in \mathscr{F}_{[-S,S]} \forall S$.

Proof: The ground state $\varphi_{\rm g}$ exists and $\varphi_{\rm g} > 0$. Then $\phi_T = e^{-T(H_{\rm ren}-E)}\phi \otimes 1\!\!1$ with $E = \inf \sigma(H_{\rm ren})$ and $\phi_T \to \varphi_{\rm g}$.

$$\mu_T(A) = e^{2ES} \int dx E_W^x \left[1\!\!1_A\left(\frac{\phi_{T-S}(B_{-S})}{\|\phi_T\|}, \mathrm{K}_S \frac{\phi_{T-S}(B_S)}{\|\phi_T\|}\right)\right] \quad \textit{for} \quad A \in \mathscr{F}_{[-S,S]}$$

where

$$\mathrm{K}_S = e^{\frac{g^2}{2}S_{\rm ren}} e^{-\int_{-S}^{S} V(B_s)ds} e^{a^\dagger(U_S)} e^{-2SH_{\rm f}} e^{a(\bar{U}_S)}.$$

We then have

$$\lim_T \mu_T(A) = \mu_\infty(A) = e^{2ES} \int dx E_W^x \left[1\!\!1_A(\varphi_{\rm g}(B_{-S}), \mathrm{K}_S \varphi_{\rm g}(B_S))\right] \textit{ for } A \in \mathscr{F}_{[-S,S]}$$

(1) Number of bosons

► $(\varphi_{\rm g}, e^{-\beta N_\Lambda}\varphi_{\rm g}) = E_{\mu_\infty}[e^{-(1-e^{-\beta})\int_{-\infty}^0 ds \int_0^\infty dt W_\Lambda}]$

► $(\varphi_{\rm g}, e^{+\beta N_\Lambda}\varphi_{\rm g}) = E_{\mu_\infty}[e^{-(1-e^{+\beta})\int_{-\infty}^0 ds \int_0^\infty dt W_\Lambda}] < \infty$ for $\beta > 0$ by an analytic continuation on $\beta \in \mathbb{C}$

(2) Gaussian domination

► $(\varphi_{\rm g}, e^{+\beta\phi(f)^2}\varphi_{\rm g}) = \frac{1}{\sqrt{1-\beta\|f\|^2/2}} E_{\mu_\infty}\left[e^{+\frac{\beta S_\infty^2}{2(1-\beta\|f\|^2/2)}}\right]$

► $(\varphi_{\rm g}, e^{\beta\phi(f)^2}\varphi_{\rm g}) < \infty$ for $\beta < 1/(2\|f\|^2)$

► $\lim_{\beta\uparrow 1/(2\|f\|^2)} (\varphi_{\rm g}, e^{\beta\phi(f)^2}\varphi_{\rm g}) = \infty$

Let $\omega(k) = \sqrt{|k|^2 + \nu^2}$.

▶ N-body model $H_\varepsilon = h_p + H_{\mathrm{f}} + \sum_{j=1}^{N} \phi(\rho_\varepsilon(\cdot - x_j))$, where

$$h_p = \sum_{j=1}^{N} (-\frac{1}{2}\Delta_j) + V(x_1, ..., x_N)$$

▶ $H_\varepsilon(\kappa) = h_p + \kappa^2 H_{\mathrm{f}} + \kappa \sum_{j=1}^{N} \phi(\rho_\varepsilon(\cdot - x_j))$, $\kappa > 0$

▶ $E_\varepsilon(\kappa) = -\frac{g^2 N}{2} \int_{\mathbb{R}^3} |\hat{\rho}_\varepsilon(k)|^2 \frac{\kappa^2}{\kappa^2 \omega(k) + |k|^2/2} dk.$

Theorem (GHL 14, Effective potential)

$\lim_{\kappa \to \infty} \lim_{\varepsilon \downarrow 0} (f \otimes \mathbb{1}, e^{-T(H_\varepsilon(\kappa) - E_\varepsilon(\kappa))} h \otimes \mathbb{1}) = (f, e^{-T h_{\mathrm{eff}}} h)$, *where*

$$h_{\mathrm{eff}} = -\frac{1}{2} \sum_{j=1}^{N} \Delta_j + V(x^1, ..., x^N) - \frac{g^2}{4\pi} \sum_{i<j} \frac{e^{-\nu |x_i - x_j|}}{|x_i - x_j|}.$$

1. Relativistic Nelson model: for $d=2$ by Møller etc 2016.
   cf Fröhlich (1973)
2. Non-trivial lower bound of $\inf\sigma(H_{\rm ren})$ by G. Bley etc 2016
3. Translation invariant case

► $V=0 \Longrightarrow H_\varepsilon = \int_{\mathbb{R}^3}^{\oplus} H_\varepsilon(P)dP$

►Nelson model with total momentum P

$$H_\varepsilon(P) = \frac{1}{2}(P-P_f)^2 + \phi(\hat\rho_\varepsilon) + H_{\rm f}, \quad P\in\mathbb{R}^3$$

THEOREM (UV RENORMALIZATION)

$\exists H_{\rm ren}(P)$ such that $\lim_{\varepsilon\downarrow 0} e^{-T(H_\varepsilon(P)-E_\varepsilon)} = e^{-TH_{\rm ren}(P)}$ for $\forall P\in\mathbb{R}^3$.