Thresholds and resonances of Schroedinger operators on a lattice

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June. 28. 2017
RIMS Prof N. Obata 60 years birthday

1 Schrödinger operators on lattice (Review)

2 Non-local Schrödinger operators on lattice

3 Schrödinger operators with delta potentials on lattice

4 Concluding remarks

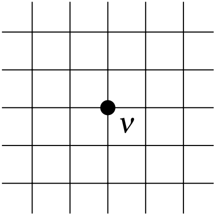
Lapalcian on $\ell^2(\mathbb{Z}^d)$:

$$L\psi(x) = -\frac{1}{2d} \sum_{|x-y|=1} (\psi(y) - \psi(x)).$$

Spectrum:

$$\sigma(L) = [0, 2].$$

Delta potential:



Potential $V(x) = v\delta_0(x)$

Definition

Schrödinger operators on d-dimensional lattice $\ell^2(\mathbb{Z}^d)$

$$L - V.$$

Here

$$V\psi(x) = v\delta_0(x)\psi(x).$$

(1) $v > 0$: coupling constant
(2) $\delta_0(x)$: the delta function with mass at $0 \in \mathbb{Z}^d$.

Fourier transformation F on torus

The d-dimensional torus: $\mathbb{T}^d = [-\pi, \pi]^d$
$F : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$ by $(F\psi)(\theta) = \sum_{x\in\mathbb{Z}^d} \psi(n)e^{-ix\cdot\theta}$.

$$F(L+V)F^{-1}\psi(\theta) = \left(1 + \frac{1}{d}\sum_{j=1}^{d} \cos\theta_j\right)\psi(\theta) + \frac{v}{(2\pi)^d}\int_{\mathbb{T}^d} \psi(\theta)d\theta.$$

Denote the right-hand side by $H = H(v)$.

$$H\psi = g\psi + v(\Omega, \psi)\Omega, \quad \Omega = (2\pi)^{-d/2}\mathbb{1},$$

where g is the multiplication by $g(\theta) = 1 + \frac{1}{d}\sum_{j=1}^{d} \cos\theta_j$.

Lemma 1. $\sigma_{\mathrm{p}}, \sigma_{\mathrm{ac}}, \sigma_{\mathrm{sc}}$

(1) $\sigma_{\mathrm{sc}}(H) = \emptyset$. (2) $\sigma_{\mathrm{p}}(H) \cap (0,2] = \emptyset$. (3)$\sigma_{\mathrm{ac}}(H) = [0,2]$.

Eigenvalues

$H\psi = E\psi$, i.e., $v(\Omega, \psi)\Omega = (E-g)\psi$.

The critical value is given by $v_c = (2\pi)^d \left(\int_{\mathbb{T}^d} \frac{1}{g(\theta)} d\theta\right)^{-1}$.

Lemma 2. Solution of $H\psi = E\psi$

$(d=1,2)$ $\psi = \frac{1}{E-g}$ and $E<0$ for each $v>0$.

$(d=3,4)$ $\psi = \frac{1}{E-g}$ and $E<0$ for $v>v_c$ and no solution for $v \le v_c$. $E=0$ is not e.v. for $v = v_c$.

$(d \ge 5)$ $\psi = \frac{1}{E-g}$ and $E \le 0$ for $v \ge v_c$ and no solution for $v < v_c$. $E=0$ is e.v. for $v = v_c$.

H. + Sasaki + Shirai + Suzuki 2012

Theorem

(1) $\sigma_{\text{ac}}(H) = \sigma_{\text{ess}}(H) = [0,2]$ for all $v \geq 0$.
(2) $\sigma_{\text{sc}}(H) = \emptyset$ for all $v \geq 0$.
(3) $\sigma_{\text{p}}(H)$
($d = 1,2$) For each $v > 0$, $\exists_1$ $E < 0$ st $\sigma_{\text{p}}(H) = \{E\}$.
($d = 3,4$)

$(v > v_c)$ $\exists_1$ $E < 0$ st $\sigma_{\text{p}}(H) = \{E\}$.

$(v \leq v_c)$ $\sigma_{\text{p}}(H) = \emptyset$.

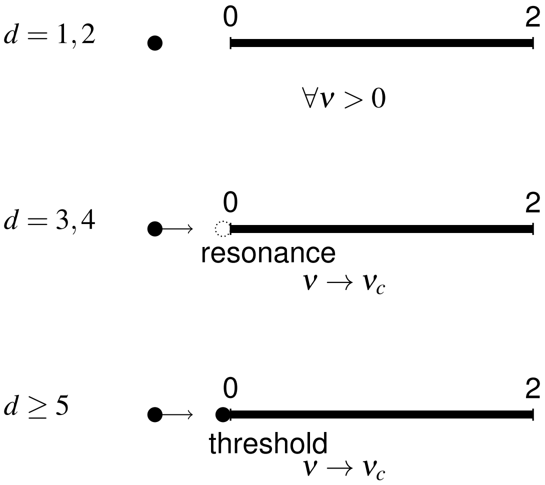
($d \geq 5$)

$(v > v_c)$ $\exists_1$ $E < 0$ st $\sigma_{\text{p}}(H) = \{E\}$.

$(v = v_c)$ $\sigma_{\text{p}}(H) = \{0\}$.

$(v < v_c)$ $\sigma_{\text{p}}(H) = \emptyset$.

Edge behavior



Non-local Schrödinger operator

We consider fractional Schrödinger operator $L^{\alpha/2}+V$.

Non-local Schrödinger operator

For $\Psi \in C^1((0,\infty))$ st $\Psi'(x)>0$,

$$H=\Psi(L)+v\delta_0(x).$$

- $\Psi(L)=F^{-1}\Psi(g)F$.
- Spectral mapping theorem yields that

$$\sigma(\Psi(L))=\Psi(\sigma(L))=[\Psi(0),\Psi(2)].$$

- Resonance and threshold:

If $\Psi(0)$ is e.v. $\in L^2(\mathbb{T}^d)$, we call it $\Psi(0)$-threshold.

If $\Psi(0)$ is e.v.$\notin L^2(\mathbb{T}^d)$, we call it $\Psi(0)$-resonance.

Eigenvalues

Let $H\Phi = E\Phi$. We introduce two integrals:

$$I(x) = \int_{\mathbb{T}^d} \frac{d\theta}{|x - \Psi(g(\theta))|^2}$$
$$J(x) = \int_{\mathbb{T}^d} \frac{d\theta}{x - \Psi(g(\theta))}.$$

Lemma 3.

(1) E is e.v. of $H \Longleftrightarrow I(E) < \infty$ and $J(E) \neq 0$.
(2) If E is e.v. of H, then v and E satisfy relation:

$$v = (2\pi)^d / J(E).$$

It is delicate to evaluate $I(x)$ and $J(x)$ at $x = \psi(2), \psi(0)$.

Density index

In order to study the case of $E=\Psi(2)$ and $E=\Psi(0)$, we introduce density index:

Definition of density index

We say that Ψ is of (a,b)-type or Ψ has density index (a,b) whenever

$$\lim_{x\to 0+}\frac{\Psi(x)-\Psi(0)}{x^a}\neq 0,\quad \lim_{x\to 0}\frac{\Psi(2)-\Psi(2-x)}{x^b}\neq 0.$$

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Lemma 4.

Let Ψ be of (a,b)-type.

$(E=\Psi(2))$ $I(E)<\infty \Longleftrightarrow d\geq 1+4a$, $J(E)<\infty \Longleftrightarrow d\geq 1+2a$.

$(E=\Psi(0))$ $I(E)<\infty \Longleftrightarrow d\geq 1+4b$, $J(E)<\infty \Longleftrightarrow d\geq 1+2b$.

Proof: Let Ψ be of (a,b)-type. Then we have at $\theta \approx (0,\ldots,0)$,

$$\Psi(2) - \Psi(g(\theta)) \approx \left(\frac{1}{2d}\sum_{j=1}^{d}\theta_j^2\right)^a$$

and at $\theta \approx (\pi,\ldots,\pi)$,

$$\Psi(g(\theta)) - \Psi(0) \approx \left(\frac{1}{2d}\sum_{j=1}^{d}(\theta_j-\pi)^2\right)^b.$$

Hence $I(\Psi(2)) \approx \int_0^1 \frac{r^{d-1}}{r^{4a}}dr$, and similarly $J(\Psi(2)) \approx \int_0^1 \frac{r^{d-1}}{r^{2b}}dr$.

$v > 0$	2-threshold	2-resonance
$d = 1,2$	$\times$	$\times$
$d = 3,4$	$\times$	$v = v_c$
$d \geq 5$	$v = v_c$	$\times$

Table: Thresholds and resonances of $L+V$ with $v > 0$

$v < 0$	0-threshold	0-resonance
$d = 1,2$	$\times$	$\times$
$d = 3,4$	$\times$	$v = v_c$
$d \geq 5$	$v = v_c$	$\times$

Table: Thresholds and resonances of $L+V$ with $v < 0$

H_+ Lorinczi 2014

Let $v_2 = (2\pi)^d / J(\Psi(2)) > 0$ and $v_0 = (2\pi)^d / J(\Psi(0)) < 0$.

$v > 0$	$\Psi(2)$-threshold	$\Psi(2)$-resonance
$d < 1 + 2b$	$\times$	$\times$
$1 + 2b \leq d < 1 + 4b$	$\times$	$v = v_2$
$d \geq 1 + 4b$	$v = v_2$	$\times$

$v < 0$	$\Psi(0)$-threshold	$\Psi(0)$-resonance
$d < 1 + 2a$	$\times$	$\times$
$1 + 2a \leq d < 1 + 4a$	$\times$	$v = v_0$
$d \geq 1 + 4a$	$v = v_0$	$\times$

Table: Thresholds and resonances of density index (a, b)

Normal type and fractional type

Normal type and fractional type

We call Ψ *normal type* if Ψ is $(1,1)$-type, and *fractional type* if Ψ is $(\alpha/2,1)$-type with $0<\alpha<2$.

1) Let Ψ be of normal type. In this case the spectral edge behaviour of $\Psi(L)+V$ is the same as that of $L+V$.
2)

$\Psi(u)$	u	$u^{\alpha/2}$	$(u+m^\alpha)^{\alpha/2}-m$	$u+bu^{\alpha/2}$	$\log(1+u^{\alpha/2})$
Type	$(1,1)$	$(\alpha/2,1)$	$(1,1)$	$(\alpha/2,1)$	$(\alpha/2,1)$

3) Let Ψ be a Bernstein function with vanishing right limits:

$$\Psi(u)=bu+\int_0^\infty(1-e^{-uy})\nu(dy),$$

where $b\geq 0$ and ν is a Lévy measure with mass on $(0,\infty)$ satisfying $\int_0^\infty(1\wedge y)\nu(dy)<\infty$. Then Ψ is of $(\alpha/2,1)$-type.

Massive vs massless

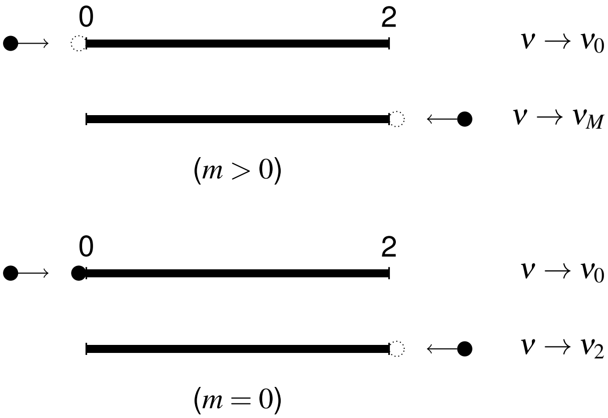
Theorem. Relativistic Schrödinger operator on lattice

1) Let

$$H_m = \sqrt{L+m^2} - m + V.$$

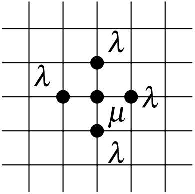
2) H_m is $(1,1)$-type for $m > 0$, and $(1/2,1)$-type for $m = 0$.
3) The edge behaviors of H_m for $m > 0$ are symmetric, but that of g is not symmetric.

Edge behavior for $d = 3$



Schrödinger operators with many delta potentials

V is the multi-delta function defined by



Potential V

Let $\lambda, \mu \geq 0$.

$$V(x) = \mu\delta_0(x) + \lambda \sum_{|s|=1} \delta_s(x), (V\psi)(x) = \begin{cases} \mu\psi(x), & \text{if} \quad x = 0 \\ \lambda\psi(x), & \text{if} \quad |x| = 1 \\ 0, & \text{if} \quad |x| > 1 \end{cases}$$

Schrödinger operators with many delta potentials

$$H_{\lambda\mu} = L - V,$$

Even part $H^{\mathrm{e}}_{\lambda\mu}$ and odd part H^{o}_{λ}

- ONS of $L^2(\mathbb{T}^d)$: $\left\{c_0 = \frac{1}{(2\pi)^{d/2}}, c_j = \frac{\sqrt{2}\cos\theta_j}{(2\pi)^{d/2}}, s_j = \frac{\sqrt{2}\sin\theta_j}{(2\pi)^{d/2}}\right\}$
- V is reduced to even part and odd part: $V = V^{\mathrm{e}}_{\lambda\mu} + V^{\mathrm{o}}_{\lambda}$ with

$$V^{\mathrm{e}}_{\lambda\mu} = \mu\langle\cdot, \mathrm{c}_0\rangle \mathrm{c}_0 + \frac{\lambda}{2}\sum_{j=1}^{d}\langle\cdot, \mathrm{c}_j\rangle \mathrm{c}_j, \quad V^{\mathrm{o}}_{\lambda} = \frac{\lambda}{2}\sum_{j=1}^{d}\langle\cdot, \mathrm{s}_j\rangle \mathrm{s}_j.$$

- By Fourier transformation F, $H_{\lambda\mu}$ is decomposed into

$$FH_{\lambda\mu}F^{-1} = H^{\mathrm{e}}_{\lambda\mu} \oplus H^{\mathrm{o}}_{\lambda} \ \textit{under}\ L^2(\mathbb{T}^d) = L^2_{\mathrm{e}}(\mathbb{T}^d) \oplus L^2_{\mathrm{o}}(\mathbb{T}^d).$$

$$H^{\mathrm{e}}_{\lambda\mu} = g - V^{\mathrm{e}}_{\lambda\mu}, \quad H^{\mathrm{o}}_{\lambda} = g - V^{\mathrm{o}}_{\lambda}.$$

- Estimate of the odd part is rather easier than that of even part.

Even part

1) $(g-z)^{-1}V^{\mathrm{e}}_{\lambda\mu}$ is a finite-rank operator.
2) $M_{d+1}=\mathscr{L}\{c_0,\cdots,c_d\}$
3) $\tilde{M}_{d+1}=(g-z)^{-1}M_{d+1}$ for $z\in\mathbb{C}\setminus[0,2]$
4) $C_2:M_{d+1}\to\mathbb{C}^{d+1}$, $C_1:\mathbb{C}^{d+1}\to\tilde{M}_{d+1}$ are the maps:

$$C_1:\mathbb{C}^{d+1}\ni\begin{pmatrix}w_0\\ \vdots\\ w_d\end{pmatrix}\mapsto(g-z)^{-1}\left(\mu w_0c_0+\frac{\lambda}{2}\sum_{j=1}^d w_jc_j\right)\in\tilde{M}_{d+1}$$

$$C_2:M_{d+1}\ni\phi\mapsto\begin{pmatrix}\langle\phi,c_0\rangle\\ \vdots\\ \langle\phi,c_d\rangle\end{pmatrix}\in\mathbb{C}^{d+1}.$$

5) $L^2_{\mathrm{e}}(\mathbb{T}^d)\supset M_{d+1}\cong\mathbb{C}^{d+1}\xrightarrow{C_2}\mathbb{C}^{d+1}\xrightarrow{C_1}\mathbb{C}^{d+1}\cong\tilde{M}_{d+1}\subset L^2_{\mathrm{e}}(\mathbb{T}^d)$

Matrix representation and BSP

Lemma 6.

$(g-z)^{-1}V^{\mathrm{e}}_{\lambda\mu} = C_1C_2 \oplus 0$ under $L^2_{\mathrm{e}}(\mathbb{T}^d) = M_{d+1} \oplus M^{\perp}_{d+1}$.

Define $G_{\mathrm{e}}(z) = C_2C_1 : \mathbb{C}^{d+1} \to \mathbb{C}^{d+1}$.

Matrix representation and BSP

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Define $G_{\mathrm{e}}(z) = C_2C_1 : \mathbb{C}^{d+1} \to \mathbb{C}^{d+1}$.

Lemma 7. BSP for $z \in \mathbb{C} \setminus [0,2]$

(a) z is e.v.of $H^{\mathrm{e}}_{\lambda\mu} \iff 1 \in \sigma(G_{\mathrm{e}}(z)) \iff \det(G_{\mathrm{e}}(z) - \mathrm{I}) = 0$.

(b) $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_d \end{pmatrix} \in \mathbb{C}^{d+1}$ satisfies $G_{\mathrm{e}}(z)Z = Z \iff H^{\mathrm{e}}_{\lambda\mu} f = zf$, i.e.

$$f(\theta) = \frac{1}{(2\pi)^{d/2}} \frac{1}{g(\theta)-z} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{d} w_j \cos\theta_j \right)$$

Proof:

(1) $H^{\mathrm{e}}_{\lambda\mu} f = zf \iff f = (g-z)^{-1} V^{\mathrm{e}}_{\lambda\mu} f$.

(2) z is e.v. of $H^{\mathrm{e}}_{\lambda\mu} \iff 1 \in \sigma((g-z)^{-1} V^{\mathrm{e}}_{\lambda\mu})$
$\iff 1 \in \sigma(C_1 C_2) \iff 1 \in \sigma(C_2 C_1)$.
C.f.

$$\sigma(C_1 C_2) \setminus \{0\} = \sigma(C_2 C_1) \setminus \{0\}.$$

(3) $C_2 C_1 Z = Z \iff f = (g-z)^{-1} V^{\mathrm{e}}_{\lambda\mu} f = C_1 C_2 f$, where

$$f = C_1 Z.$$

□

Extension: g^{-1} is not bounded in $L^2_{\mathrm{e}}(\mathbb{T}^d)$ as well as in $L^1_{\mathrm{e}}(\mathbb{T}^d)$. It is however obvious that

$$L^1_{\mathrm{e}}(\mathbb{T}^d) \ni f \mapsto g^{-1} V^{\mathrm{e}}_{\lambda\mu} f \in L^1_{\mathrm{e}}(\mathbb{T}^d) \quad d \geq 3.$$

For $d \geq 3$, C_1 and C_2 can be extended. $\overline{C}_1 \overline{C}_2 : L^1_{\mathrm{e}}(\mathbb{T}^d) \to L^1_{\mathrm{e}}(\mathbb{T}^d)$.

$$G_{\mathrm{e}}(0) = \overline{C}_2 \overline{C}_1 : \mathbb{C}^{d+1} \to \mathbb{C}^{d+1}.$$

Lemma 8. BSP for $z = 0$ for $d \geq 3$

(a) $H^{\mathrm{e}}_{\lambda\mu} f = 0$ has a solution in $L^1(\mathbb{T}^d) \iff 1 \in \sigma(G_{\mathrm{e}}(0))$.

(b) If $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_d \end{pmatrix} \in \mathbb{C}^{d+1}$ satisfies $G_{\mathrm{e}}(0) Z = Z$, then

$$f(\theta) = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{g(\theta)} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{d} w_j \cos\theta_j \right)$$

$G_{\mathrm{e}}(z)$ is also defined for $z \in (-\infty, 0]$ for $d \geq 3$. Let

$$a(z) = \langle c_0, (g-z)^{-1} c_0 \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{1}{g(\theta) - z} d\theta,$$

$$b(z) = \frac{1}{\sqrt{2}} \langle c_0, (g-z)^{-1} c_j \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\cos \theta_j}{g(\theta) - z} d\theta,$$

$$c(z) = \frac{1}{2} \langle c_j, (g-z)^{-1} c_j \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\cos^2 \theta_j}{g(\theta) - z} d\theta,$$

$$d(z) = \frac{1}{2} \langle c_i, (g-z)^{-1} c_j \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\cos \theta_i \cos \theta_j}{g(\theta) - z} d\theta, \quad i \neq j,$$

Factorization of $\det(G_e(z) - I)$

Lemma 9. Factorization of $\det(G_e(z) - I)$

$$\det(G_e(z) - I) = \gamma(z)\mathbb{H}_z(\lambda, \mu)\delta_c(\lambda; z),$$

$$\mathbb{H}_z(\lambda, \mu) = \left(\lambda - \frac{a(z)}{b(z)}\right)\Big(\mu - (d - z)\Big) - n,$$

$$\delta_c(\lambda; z) = (\lambda(c(z) - d(z)) - 1)^{d-1}.$$

- $\gamma(z) \neq 0$
- $\det(G_e(z) - I) = 0 \iff \mathbb{H}_z(\lambda, \mu) = 0$ or $\delta_c(\lambda; z) = 0$

Zeros of $\mathbb{H}_z(\lambda,\mu)$ and hyperbola

$\mathbb{H}_z(\lambda,\mu)$ can be extended from $z\in(-\infty,0)$ to $z\in(-\infty,0]$ for $d\geq 3$ as

$$\bar{\mathbb{H}}_z(\lambda,\mu)=\begin{cases}\mathbb{H}_z(\lambda,\mu), & z<0,\\ (\lambda-\frac{a(0)}{b(0)})(\mu-d)-d, & z=0.\end{cases}$$

We define the family of hyperbola $\mathfrak{H}_z$ indexed by $z\in(-\infty,0]$ by

$$\mathfrak{H}_z=\{(\lambda,\mu)\in\mathbb{R}^2|\bar{\mathbb{H}}_z(\lambda,\mu)=0\}.$$

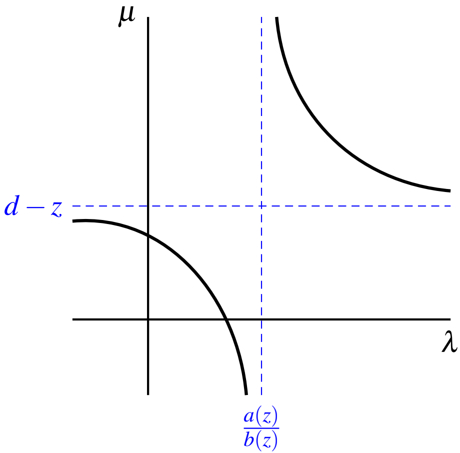
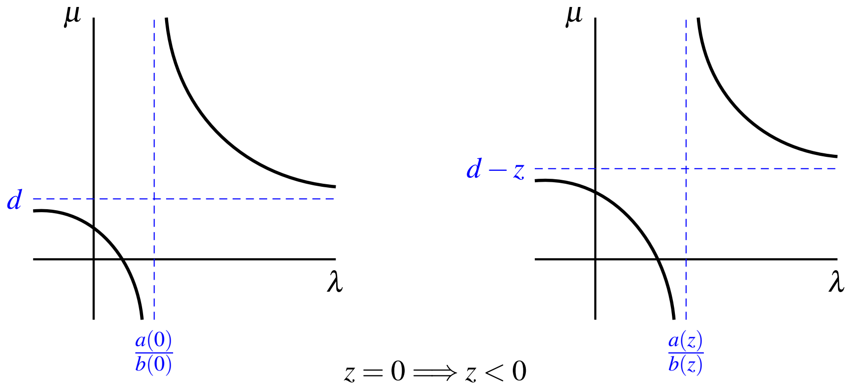


Figure: Hyperbola $\mathfrak{H}_z$



$$z = 0 \Longrightarrow z < 0$$

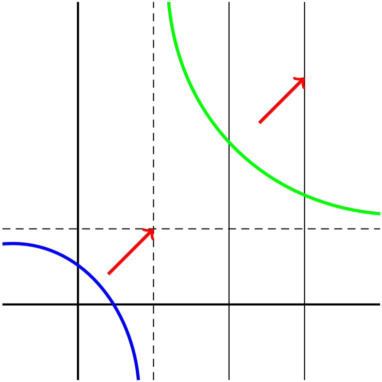


Figure: Hyperbola $\mathfrak{H}_z$ moves as $z : 0 \to -\infty$

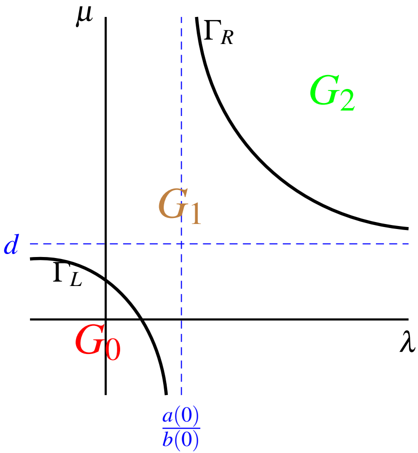


Figure: Hyperbola $\mathfrak{H}_z$ for $z = 0$

Lemma 10.

(1)

1. Let $(\lambda,\mu)\in G_0\cup\Gamma_L$. Then $\bar{\mathbb{H}}_z(\lambda,\mu)\neq 0$ for $z\in(-\infty,0)$.
2. Let $(\lambda,\mu)\in\Gamma_L$. Then $\bar{\mathbb{H}}_0(\lambda,\mu)=0$.
3. Let $(\lambda,\mu)\in G_0$. Then $\bar{\mathbb{H}}_0(\lambda,\mu)\neq 0$.

(2)

1. Let $(\lambda,\mu)\in G_1\cup\Gamma_R$. Then $\exists_1\ z\in(-\infty,0)$ st $\bar{\mathbb{H}}_z(\lambda,\mu)=0$.
2. Let $(\lambda,\mu)\in\Gamma_R$. Then $\bar{\mathbb{H}}_0(\lambda,\mu)=0$.

(3) Let $(\lambda,\mu)\in G_2$. Then $\exists z_j\in(-\infty,0)$ st $\bar{\mathbb{H}}_0(\lambda,\mu;z_j)=0, j=1,2$.

Zeros of $\delta_c(\lambda;z)$

Define $\bar{\delta}_c(\lambda;z)$ by

$$\bar{\delta}_c(\lambda;z) = \begin{cases} \delta_c(\lambda;z), & z \in (-\infty,0), \\ (\lambda\alpha-1)^{d-1}, & z=0. \end{cases}$$

Let $\alpha = \lim_{z\to 0-} c(z)-d(z)$. Note that $\alpha > 0$ and we set

$$\lambda_c = \frac{1}{\alpha}.$$

Lemma 11.

Let $\lambda \le \lambda_c$. Then $\bar{\delta}_c(\lambda;z) \neq 0$ for any $z \in (-\infty,0)$.
Let $\lambda = \lambda_c$. Then $\bar{\delta}_c(\lambda;0) = 0$, and $z=0$ has multiplicity $d-1$.
Let $\lambda > \lambda_c$. Then $\bar{\delta}_c(\lambda;\cdot)$ has the unique zero in $(-\infty,0)$ with multiplicity $d-1$.

Case of H^{o}_λ

In the case of H^{o}_λ we can proceed in a similar way to the the case of $H^{\mathrm{e}}_{\lambda\mu}$ and rather easier than that of $H^{\mathrm{e}}_{\lambda\mu}$. We can define the matrix $G_o(z)$ and show that

$$\det(G_o(z) - I) = (\lambda s(z) - 1)^d,$$

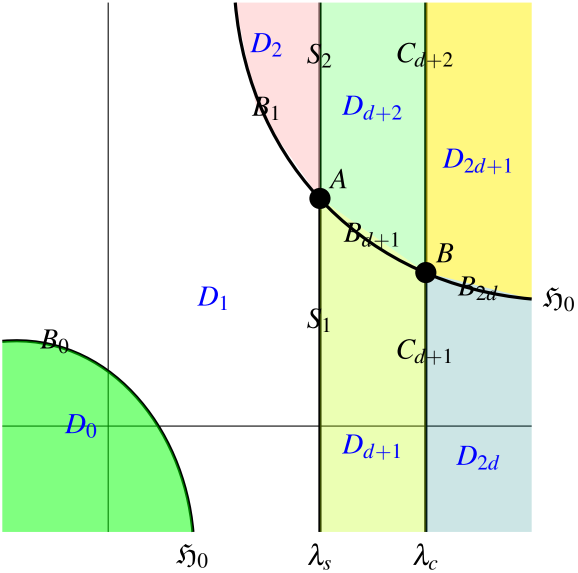
where $s(z) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \frac{\sin^2\theta_j}{g(\theta)-z} d\theta$. Let

$$\lambda_s = \frac{1}{s(0)}$$

Lemma

Let $\lambda \le \lambda_s$. Then $\det(G_o(z) - I) \neq 0$ for any $z \in (-\infty, 0)$.
Let $\lambda = \lambda_s$. Then $\det(G_o(0) - I) = 0$ and $z = 0$ has multiplicity d.
Let $\lambda > \lambda_s$. Then $\det(G_o(\cdot) - I)$ has the unique zero in $(-\infty, 0)$ with multiplicity d.

H+ Z. Muminov and U. Kuljanov 2016



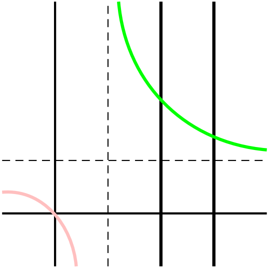
	D_0	D_1	D_2	D_{d+1}	D_{d+2}	D_{2d}	D_{2d+1}
E.v.< 0	0	1	2	$d+1$	$d+2$	$2d$	$2d+1$

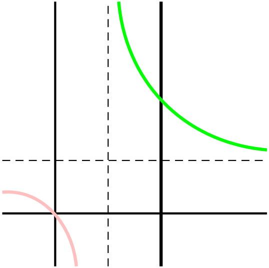
	B_k	S_k	C_k
E.v.< 0	k	k	k
Res.0	$d=2$ $-$ $d=3,4$ 1 $d\geq 5$ $-$	$d=2$ 2 $d\geq 3$ $-$	$d\geq 2$ $-$
Th.0	$d=2$ $-$ $d=3,4$ $-$ $d\geq 5$ 1	$d=2$ $-$ $d\geq 3$ d	$d\geq 2$ $d-1$

Table

	Point A		Point B	
E.v.< 0	1		$d+1$	
Res.0	$d=2$	$-$	$d=2$	$-$
	$d=3,4$	1	$d=3,4$	1
	$d\geq 5$	$-$	$d\geq 5$	$-$
Th.0	$d=2$	$-$	$d=2$	1
	$d=3,4$	d	$d=3,4$	$d-1$
	$d\geq 5$	$d+1$	$d\geq 5$	d

(1) In the case of $d = 1, 2$ point $(0, 0)$ is included in hyperbola $\mathfrak{H}_z$.





(2) Fractional Schrödinger operator with multi-delta function.

(3) Resonance appears for $d = 2$.

(4) $\exists \lambda$ st $H_{\lambda\mu}$ has only one negative eigenvalue for $\forall \mu$.