格子上のシュレディンガー作用素の resonance と threshold

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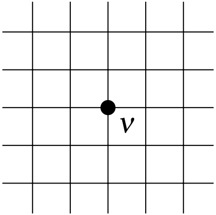
Lapalcian on $\ell^2(\mathbb{Z}^d)$:

$$L\psi(x) = -\frac{1}{2d}\sum_{|x-y|=1}(\psi(y)-\psi(x)).$$

Spectrum: $\sigma(L) = [0,2]$.
Single delta potential:



Potential $V(x) = v\delta_0(x)$

Schrödinger operators on d-dimensional lattice $\ell^2(\mathbb{Z}^d)$

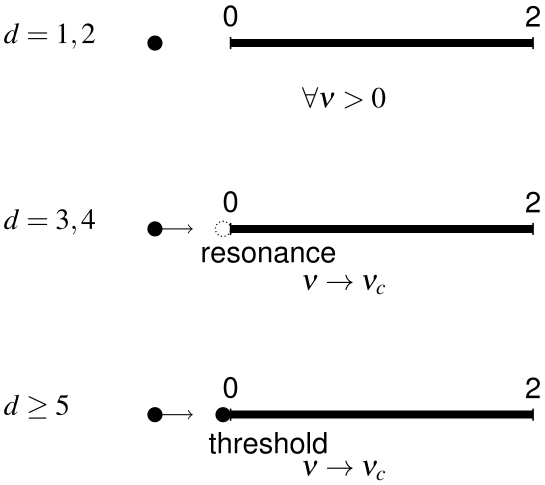
$$L - V.$$

Here

$$V\psi(x) = v\delta_0(x)\psi(x).$$

(1) $v > 0$: coupling constant
(2) $\delta_0(x)$: the delta function with mass at $0 \in \mathbb{Z}^d$.

Edge behavior



Massive vs massless

$\Psi(L)+V$, (a,b)-type.

$$\lim_{x\to 0+}\frac{\Psi(x)-\Psi(0)}{x^a}\neq 0,\quad \lim_{x\to 0+}\frac{\Psi(x+2)-\Psi(2)}{x^b}\neq 0.$$

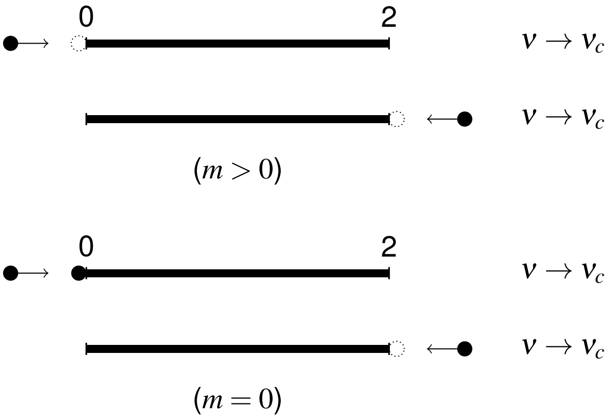
Thm.(FH+Lorinczi 14). Relativistic Schrödinger op. on lattice

1) Let

$$H_m=\sqrt{L+m^2}-m+V.$$

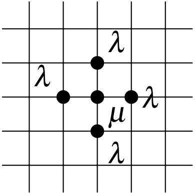
2) H_m is $(1,1)$-type for $m>0$, and $(1/2,1)$-type for $m=0$.
3) The edge behaviors of H_m for $m>0$ are symmetric, but that of H_0 is not symmetric.

Edge behavior for $d = 3$



Schrödinger operators with multi delta potentials

V is the multi-delta function defined by



Potential V

Let $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}$.

$$V(x) = \mu \delta_0(x) + \lambda \sum_{|s|=1} \delta_s(x), (V\psi)(x) = \begin{cases} \mu\psi(x), & \text{if} \quad x = 0 \\ \lambda\psi(x), & \text{if} \quad |x| = 1 \\ 0, & \text{if} \quad |x| > 1 \end{cases}$$

Schrödinger operators with multi delta potentials

$$H_{\lambda\mu} = L - V,$$

Even part $H^{\mathrm{e}}_{\lambda\mu}$ and odd part H^{o}_{λ}

- ONS: $\left\{c_0 = \frac{1}{(2\pi)^{d/2}}, c_j = \frac{\sqrt{2}\cos\theta_j}{(2\pi)^{d/2}}, s_j = \frac{\sqrt{2}\sin\theta_j}{(2\pi)^{d/2}}, j = 1, \dots, d\right\}$
- V is reduced to even part and odd part:

$$V^{\mathrm{e}}_{\lambda\mu} = \mu\langle\cdot, \mathrm{c}_0\rangle\mathrm{c}_0 + \frac{\lambda}{2}\sum_{j=1}^{d}\langle\cdot, \mathrm{c}_j\rangle\mathrm{c}_j, \quad V^{\mathrm{o}}_{\lambda} = \frac{\lambda}{2}\sum_{j=1}^{d}\langle\cdot, \mathrm{s}_j\rangle\mathrm{s}_j.$$

- By Fourier transformation F, $H_{\lambda\mu}$ is decomposed into

$$FH_{\lambda\mu}F^{-1} = H^{\mathrm{e}}_{\lambda\mu} \oplus H^{\mathrm{o}}_{\lambda}$$

$$H^{\mathrm{e}}_{\lambda\mu} = g - V^{\mathrm{e}}_{\lambda\mu}, \quad H^{\mathrm{o}}_{\lambda} = g - V^{\mathrm{o}}_{\lambda}.$$

Matrix representation and BSP

$H^{\mathrm{e}}_{\lambda\mu}\phi = z\phi \iff (g-z)\phi = V^{\mathrm{e}}_{\lambda\mu}\phi \iff \frac{1}{g-z}V^{\mathrm{e}}_{\lambda\mu}\phi$ BP operator

- $\mathscr{L} = \mathscr{L}\{c_0, \cdots, c_d\}$

$\mathscr{L} \ni \phi \mapsto f\sum_{j=0}^{d}\langle c_j, \phi\rangle c_j \ni \mathscr{L}$ can be regarded as

$$\mathscr{L} \ni \phi \xrightarrow{C_2} \begin{pmatrix} \langle c_0, \phi\rangle \\ \vdots \\ \langle c_d, \phi\rangle \end{pmatrix} = \begin{pmatrix} z_0 \\ \vdots \\ z_d \end{pmatrix} \xrightarrow{C_1} f\sum_{j=1}^{n} z_j c_j \in f\mathscr{L}$$

- $\frac{1}{g-z}V^{\mathrm{e}}_{\lambda\mu} = C_1C_2 \oplus 0$ under $L^2_{\mathrm{e}}(\mathbb{T}^d) = \mathscr{L} \oplus \mathscr{L}^{\perp}$. Define

$$G_{\mathrm{e}}(z) = C_2C_1 : \mathbb{C}^{d+1} \to \mathbb{C}^{d+1}.$$

Lemma. BSP for $z \in \mathbb{C} \setminus [0,2]$

(a) z is e.v.of $H^{\mathrm{e}}_{\lambda\mu} \iff 1 \in \sigma(G_{\mathrm{e}}(z)) \iff \det(G_{\mathrm{e}}(z) - \mathrm{I}) = 0$.

(b) $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_d \end{pmatrix} \in \mathbb{C}^{d+1}$ satisfies $G_{\mathrm{e}}(z)Z = Z \iff H^{\mathrm{e}}_{\lambda\mu} f = zf$ with

$$f(\theta) = \frac{1}{(2\pi)^{d/2}} \frac{1}{g(\theta) - z} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{d} w_j \cos\theta_j \right)$$

For $d \geq 3$, C_1 and C_2 can be extended. $\overline{C}_1\overline{C}_2 : L^1_{\mathrm{e}}(\mathbb{T}^d) \to L^1_{\mathrm{e}}(\mathbb{T}^d)$.

$$G_{\mathrm{e}}(0) = \overline{C}_2\overline{C}_1 : \mathbb{C}^{d+1} \to \mathbb{C}^{d+1}.$$

Lemma. BSP for $z = 0$ for $d \geq 3$

(a) $H^{\mathrm{e}}_{\lambda\mu} f = 0$ has a solution in $L^1(\mathbb{T}^d) \Longleftrightarrow 1 \in \sigma(G_{\mathrm{e}}(0))$.

(b) If $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_d \end{pmatrix} \in \mathbb{C}^{d+1}$ satisfies $G_{\mathrm{e}}(0)Z = Z$, then

$$f(\theta) = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{g(\theta)} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{d} w_j \cos\theta_j \right)$$

Matrix representation of $G_e(z)$

Let

$$a = \langle c_0, \frac{1}{g-z} c_0 \rangle, \quad b = \frac{1}{\sqrt{2}} \langle c_0, \frac{1}{g-z} c_j \rangle,$$
$$c = \frac{1}{2} \langle c_j, \frac{1}{g-z} c_j \rangle, \quad h = \frac{1}{2} \langle c_i, \frac{1}{g-z} c_j \rangle.$$

From the definition of $G_{\mathrm{e}}(z)$, we have

$$G_{\mathrm{e}}(z) = \begin{pmatrix} \mu a & \frac{\lambda}{\sqrt{2}} b & \dots & \dots & \frac{\lambda}{\sqrt{2}} b \\ \sqrt{2}\mu b & \lambda c & \lambda h & \dots & \lambda h \\ \vdots & \lambda d & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \lambda h \\ \sqrt{2}\mu b & \lambda h & \dots & \lambda h & \lambda c \end{pmatrix}$$

Factorization of $\det(G_{\mathrm{e}}(z) - \mathrm{I})$

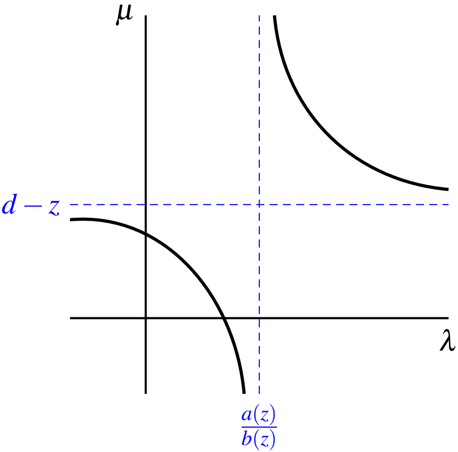
Lemma. Factorization of $\det(G_{\mathrm{e}}(z) - \mathrm{I})$

$$\det(G_{\mathrm{e}}(z) - \mathrm{I}) = \gamma \mathbb{H}_z(\lambda, \mu) \delta_z(\lambda),$$
$$\mathbb{H}_z(\lambda, \mu) = \left(\lambda - \frac{a}{b}\right)\Big(\mu - (d - z)\Big) - d,$$
$$\delta_z(\lambda) = ((c-h)\lambda - 1)^{d-1}.$$

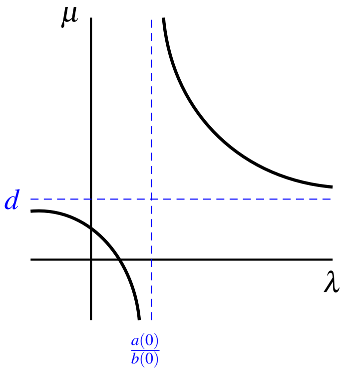
- $\gamma \neq 0$
- $\det(G_{\mathrm{e}}(z) - \mathrm{I}) = 0 \Longleftrightarrow \mathbb{H}_z(\lambda, \mu) = 0$ or $\delta_z(\lambda) = 0$

Zeros of $\mathbb{H}_z(\lambda, \mu)$

We define the family of hyperbola $\mathfrak{H}_z$ indexed by $z \in (-\infty, 0]$ by

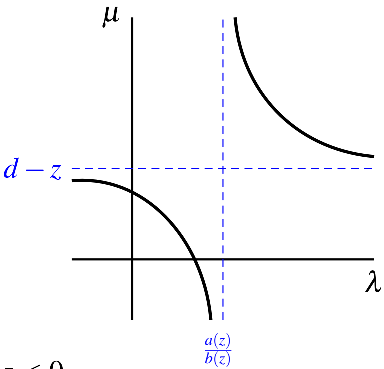


$$\mathfrak{H}_z = \{(\lambda, \mu) \in \mathbb{R}^2 | \bar{\mathbb{H}}_z(\lambda, \mu) = 0\}.$$



Hypabola $\mathfrak{H}_0$

$$z = 0 \Longrightarrow z < 0$$



Hypabola $\mathfrak{H}_z$

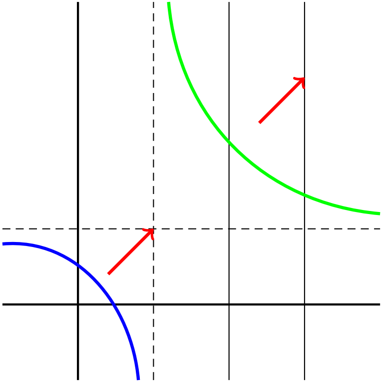


Figure: Hyperbola $\mathfrak{H}_z$ moves as $z : 0 \to -\infty$

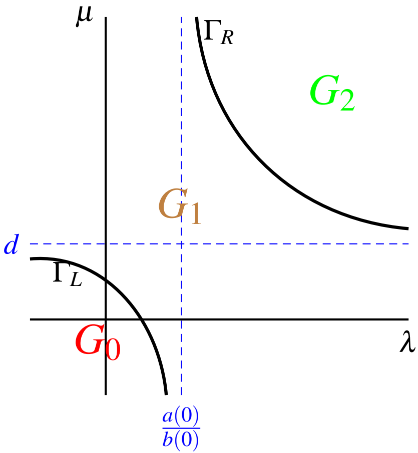


Figure: Hyperbola $\mathfrak{H}_z$ for $z = 0$

Lemma. Regions G_0, G_1 and G_2

(G_0)

1. Let $(\lambda,\mu) \in G_0 \cup \Gamma_L$. Then $\bar{\mathbb{H}}_z(\lambda,\mu) \neq 0$ for $z \in (-\infty,0)$.
2. Let $(\lambda,\mu) \in \Gamma_L$. Then $\bar{\mathbb{H}}_0(\lambda,\mu) = 0$.
3. Let $(\lambda,\mu) \in G_0$. Then $\bar{\mathbb{H}}_0(\lambda,\mu) \neq 0$.

(G_1)

1. Let $(\lambda,\mu) \in G_1 \cup \Gamma_R$. Then $\exists_1\ z \in (-\infty,0)$ st $\bar{\mathbb{H}}_z(\lambda,\mu) = 0$.
2. Let $(\lambda,\mu) \in \Gamma_R$. Then $\bar{\mathbb{H}}_0(\lambda,\mu) = 0$.

(G_2) Let $(\lambda,\mu) \in G_2$. Then $\exists z_j \in (-\infty,0)$ st $\bar{\mathbb{H}}_{z_j}(\lambda,\mu) = 0, j = 1,2$.

Zeros of $\delta_z(\lambda)$

Define $\bar{\delta}_z(\lambda)$ by

$$\bar{\delta}_z(\lambda) = \begin{cases} \delta_z(\lambda), & z \in (-\infty, 0), \\ (\alpha\lambda - 1)^{d-1}, & z = 0. \end{cases}$$

Let $\alpha = \lim_{z \to 0-} c(z) - h(z)$. Note that $\alpha > 0$ and we set $\lambda_c = \frac{1}{\alpha}$.

Lemma.

Let $\lambda \leq \lambda_c$. Then $\bar{\delta}_z(\lambda) \neq 0$ for any $z \in (-\infty, 0)$.
Let $\lambda = \lambda_c$. Then $\bar{\delta}_0(\lambda) = 0$, and $z = 0$ has multiplicity $d - 1$.
Let $\lambda > \lambda_c$. Then $\bar{\delta}_\cdot(\lambda)$ has the unique zero in $(-\infty, 0)$ with multiplicity $d - 1$.

Case of H^{o}_{λ}

In the case of H^{o}_{λ} we can proceed in a similar way to the the case of $H^{\mathrm{e}}_{\lambda\mu}$ and rather easier than that of $H^{\mathrm{e}}_{\lambda\mu}$. We can define the matrix $G_o(z)$ and show that

$$\det(G_o(z) - I) = (\lambda s - 1)^d,$$

where $s = s(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\sin^2 \theta_j}{g(\theta) - z} d\theta$. Let $\lambda_s = \frac{1}{s(0)}$.

Lemma

Let $\lambda \le \lambda_s$. Then $\det(G_o(z) - I) \neq 0$ for any $z \in (-\infty, 0)$.
Let $\lambda = \lambda_s$. Then $\det(G_o(0) - I) = 0$ and $z = 0$ has multiplicity d.
Let $\lambda > \lambda_s$. Then $\det(G_o(\cdot) - I)$ has the unique zero in $(-\infty, 0)$ with multiplicity d.

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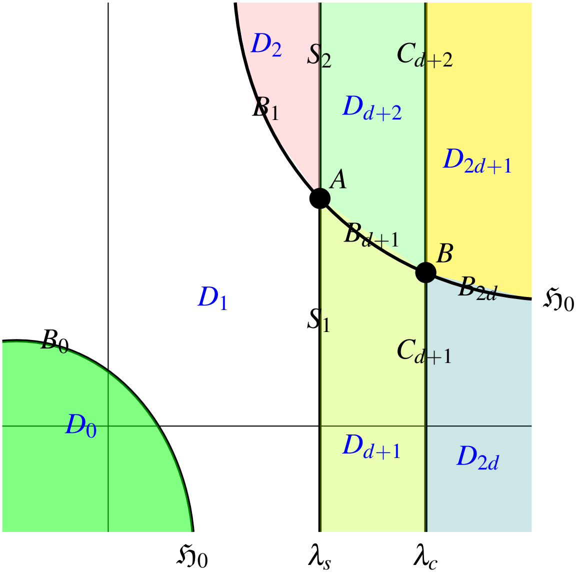


	D_0	D_1	D_2	D_{d+1}	D_{d+2}	D_{2d}	D_{2d+1}
E.v.< 0	0	1	2	$d+1$	$d+2$	$2d$	$2d+1$

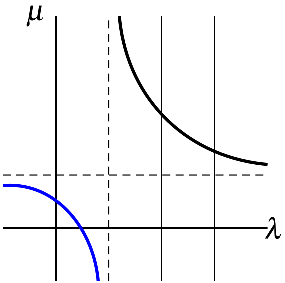
	B_k	S_k	C_k
E.v.< 0	k	k	k
Res.0	$d=2$ — $d=3,4$ 1 $d\geq 5$ —	$d=2$ 2 $d\geq 3$ —	$d\geq 2$ —
Th.0	$d=2$ — $d=3,4$ — $d\geq 5$ 1	$d=2$ — $d\geq 3$ d	$d\geq 2$ $d-1$

Table

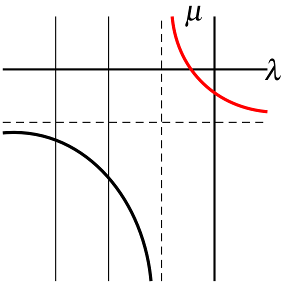
	Point A		Point B	
E.v.< 0	1		$d+1$	
Res.0	$d=2$	$-$	$d=2$	$-$
	$d=3,4$	1	$d=3,4$	1
	$d\geq 5$	$-$	$d\geq 5$	$-$
Th.0	$d=2$	$-$	$d=2$	1
	$d=3,4$	d	$d=3,4$	$d-1$
	$d\geq 5$	$d+1$	$d\geq 5$	d

Two embedded eigenvalues

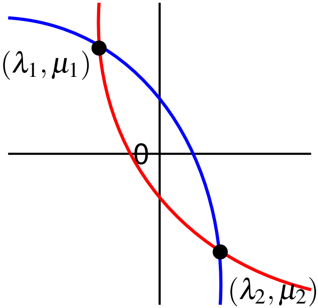
We can construct $H_{\lambda\mu}$ which has two embedded eigenvalues.



0 での resonance と threshold

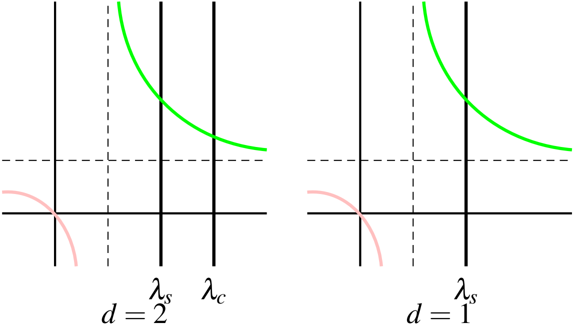


2 での resonance と threshold



Embedded spectra of $H_{\lambda_1\mu_1}$ and $H_{\lambda_2\mu_2}$

(1) In the case of $d=1,2$ point $(0,0)$ is included in hyperbola $\mathfrak{H}_z$.



(2) Fractional Schrödinger operator with multi-delta function.

(3) Resonance and threshold appear for $d=2$.

(4) Let $\lambda=\frac{a(0)}{b(0)}$. Then $H_{\lambda\mu}$ has only one negative ev for $\forall\mu$.

(5) Let $d\geq 5$. There exist (λ,μ) st 0 and 2 simultaneously are threshold.

(6) For λ,μ of A and B, $H_{\lambda\mu}$ has simultaneously both resonance and threshold at 0.