Time operators associated with Schrödinger operators

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Let A and B be linear operators on a complex Hilbert space $\mathscr{H}$,

satisfying the canonical commutation relation (CCR)

$$[A,B] = -i1$$

on a dense subspace

$$\mathscr{D} \subset D(AB) \cap D(BA)$$

● We call $\mathscr{D}$ a CCR-domain for the pair (A,B).
● $(\mathscr{H}, \mathscr{D}, \{A,B\})$ is called a representation of CCR

● Extension to d-degree system:
Let A_j and B_j be linear operators on $\mathscr{H}$ and $\mathscr{D}$ be a dense subspace of $\mathscr{H}$ such that

$$\mathscr{D} \subset \cap_{j,k=1}^{d}[D(A_jB_k)\cap D(B_kA_j)\cap D(A_jA_k)\cap D(B_jB_k)].$$

Then

$$(\mathscr{H},\mathscr{D},\{A_j,B_j|j=1,\ldots,d\})$$

is also called a representation of CCR if

$$[A_j,B_k]=-i\delta_{jk}1, \quad [A_j,A_k]=0, \quad [B_j,B_k]=0$$

hold on $\mathscr{D}$.

Example 1(Heizenberg) $[P_i, Q_j] = -i\delta_{ij}1$
Example 2 $[-\frac{1}{2m}P^2, T_{AB}] = -i$, where

$$T_{AB} = \frac{m}{2}\sum_j (P_j^{-1}Q_j + Q_j P_j^{-1})$$

Example 3(H.+Kuribayashi+Matsuzawa(09)) $[f(P), T_f] = -i1$, where

$$T_f = \frac{1}{2}\sum_j ((\partial_j f(P))^{-1}Q + Q(\partial_j f(P))^{-1})$$

Example 4 $[\sqrt{P^2 + m^2}, T] = -i1$, $T = \cdots$.

We would like to consider CCR representation associated with Schrödinger operator of the form:

$$H = \frac{1}{2m}\sum_{j=1}^{d} P_j^2 + V(Q)$$

on $L^2(\mathbb{R}^d)$. One may infer that a quantum Hamiltonian H may have a symmetric operator T corresponding to time, satisfying CCR

$$[H, T_H] = -i1.$$

Such an operator T_H is called a **time operator** of H. I.e.,

$$T_H = i\frac{d}{dH}$$

Difficulty

Let $H\phi = E\phi$. Then $[H,T]\phi = HT\phi - ET\phi = (H-E)T\phi = -i\phi$ and $T\phi = -i(H-E)^{-1}\phi$? Thus $\phi \notin D(T)$ for any e.v. ϕ of H.

Hierarchy of time operators

[Weyl relation] A pair (A,B) consisting of self-adjoint operators A and B is called a weak Weyl representation if the **Weyl relations** holds:

$$e^{-itA}e^{-isB} = e^{ist}e^{-isB}e^{-itA}$$

[Weak Weyl relation] A pair (A,B) consisting of a self-adjoint operator A and symmetric operator B is called a **weak Weyl representation** if

$$e^{-itA}D(B) \subset D(B)$$
$$Be^{-itA}\psi = e^{-itA}(B+t)\psi$$

holds for all $\psi \in D(B)$ and all $t \in \mathbf{R}$.

Important

Weyl relation $\Longrightarrow$ weak Weyl relation $\Longrightarrow$CCR

Definition (Ultra-strong time operator)

A self-adjoint operator T is called a **ultra-strong time operator** of H if (H,T) is a Weyl representation.

Definition (Strong time operator)

A symmetric operator T is called a **strong time operator** of H on $\mathscr{H}$ if (H,T) is a weak Weyl representation.

[Remark1] Let $H > \infty$ and T be the strong time op. of H. Then T has **no self-adjoint extension**!
[Remark2] Let T be a strong time op. of H. Then $\sigma(H)$ is purely absolutely cont. In particular **if H has an eigenvalue, then H has no strong time op.**

Definition (weak time operator)

A symmetric operator T is called a **weak time operator** of H if a dense subspace $\exists \mathscr{D}_{\mathrm{w}} \subset D(T) \cap D(H)$ s.t.

$$(H\phi, T\psi) - (T\phi, H\psi) = -i(\phi, \psi), \quad \phi, \psi \in \mathscr{D}_{\mathrm{w}}.$$

We call $\mathscr{D}_{\mathrm{w}}$ a weak-CCR domain for the pair (H, T).

Definition (ultra-weak time operator)

Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be dense subspaces of $\mathscr{H}$. A sesquilinear form

$$\mathfrak{t}:\mathscr{D}_1\times\mathscr{D}_2\to\mathbf{C}$$

is called a **ultra-weak time operator** of H if dense subspaces $\exists\mathscr{D}$ and $\exists\mathscr{E}$ of $\mathscr{D}_1\cap\mathscr{D}_2$ such that (i)–(iii) hold:

(i) $\mathscr{E}\subset D(H)\cap\mathscr{D}$.

(ii) $\mathfrak{t}[\phi,\psi]^*=\mathfrak{t}[\psi,\phi]$, $\phi,\psi\in\mathscr{D}$,

(iii) $H\mathscr{E}\subset\mathscr{D}_1$

$$\mathfrak{t}[H\phi,\psi]-\mathfrak{t}[H\psi,\phi]^*=-i(\phi,\psi)\quad \psi,\phi\in\mathscr{E}.$$

We call $\mathscr{E}$ an ultra-weak CCR-domain for $(H,\mathfrak{t})$ and $\mathscr{D}$ a symmetric domain of $\mathfrak{t}$.

Remark

Weak time operator $\subset$ ultra weak time operator

- Let T be a weak time operator of H.
- $\mathfrak{t}_T : \mathscr{H} \times D(T) \to \mathbf{C}$ by

$$\mathfrak{t}_T[\phi, \psi] = (\phi, T\psi), \quad \phi \in \mathscr{H}, \psi \in D(T).$$

- Then $\mathfrak{t}_T$ is an ultra-weak time operator of H.

Hierarchy of time operators

5 classes of time operators

ultra-strong time $\subset$ strong time $\subset$ time $\subset$ weak time $\subset$ **ultra-weak time**

※ The von Neumann uniqueness theorem yields that if (H, T) satisfies Weyl relation (i.e., T is an ultra-strong time op.), then $H \cong \oplus P$ and $T \cong \oplus Q$.
※ The purpose of my talk is to find ultra-weak time op. of H.

Absolutely cont. spectrum and strong time operators

Scattering theory

Let H and H' be s.a. Assume
(A,1)$\exists W_\pm = \text{s-}\lim_{t\to\pm\infty} e^{itH'} J e^{-itH} P_{\text{ac}}(H)$
(A.2) $\lim_{t\to\pm\infty} \|Je^{-itH} P_{\text{ac}}(H)\psi\| = \|P_{\text{ac}}(H)\psi\|$
(A.3)$\text{Ran}(W_\pm) = \mathscr{H}_{\text{ac}}(H')$. Let $U_\pm = W_\pm \lceil \mathscr{H}_{\text{ac}}(H)$.
Then $U_\pm : \mathscr{H}_{\text{ac}}(H) \to \mathscr{H}_{\text{ac}}(H')$ a unitary s.t. $H'_{\text{ac}} = U_\pm H_{\text{ac}} U_\pm^{-1}$.

Theorem (Arai (06), Strong time operators)

Assume (A.1)–(A.3). Suppose that H_{ac} has a strong time op. T. Then $T'_\pm = U_\pm T U_\pm^{-1}$ are strong time op. of H'_{ac}. I.e.,

$$[H,T] = -i1 \Longrightarrow [H', T'_\pm] = -i1.$$

[Example] $H = \frac{1}{2}\sum_j P_j^2$ and $H' = H + V$. Hence $T_{H'} = U_\pm T_{AB} U_\pm^{-1}$ is the strong time op. of H_{ac}.

Discrete spectrum and ultra-weak time operators

[Question] Let $H = H_{\rm p} \oplus H_{\rm ac}$.
We know that $T_{H_{\rm ac}}$ can be derived by scattering theory. How can we construct $T_{H_{\rm p}}$?

Galapon(02), Arai-Matsuzawa(08)

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal bases. Suppose that $\sigma(H) = \{E_j\}_{j=1}^{\infty}$, every E_j is simple, and $\sum_{j=1}^{\infty} \frac{1}{E_j^2} < \infty$. Then

$$T\phi = i \sum_{n=1}^{\infty} \left(\sum_{m \neq n} \frac{(e_m, \phi)}{E_n - E_m} \right) e_n$$

is a time operator of H. I.e.,

$$[H, T]\phi = -i\phi.$$

Ultra-weak time op. of the case $\sigma(H)=\sigma_{disc}(H)=\{E_n\}_{n=1}^{\infty}$

[$E_n \to \infty$] Let $\lim_{n\to\infty} E_n = \infty$. Then **time op.** $\exists T$ of H.

[$E_n \to 0$] Let $E_n < 0$, $\lim_{n\to\infty} E_n = 0$ and $0 \notin \sigma_{\rm p}(H)$. Then **ultra weak time op.** $\exists T$ of H.

Remark 1: In the case of $E_n \to \infty$, T is usually of the form

$$T = \oplus_{j=1}^{\infty} T_j$$

and each T_j is a weak time operator of some simple self-adjoint operator H_j and $H = \oplus_j H_j$ (Sasaki+Wada(14)).

[Example 1: Two photon Rabi Hamiltonian]

Let $\omega = 1$ and $g \geq 0$.

$$H_R = \Delta\sigma_z + g({a^\dagger}^2 + a^2) + a^\dagger a.$$

($g < 1/2$**)**
$\sigma(H_R) = \{E_n\} \Longrightarrow \exists$ **Time op.**

($g = 1/2$**)**
$\Longrightarrow H_R \cong \begin{pmatrix} P^2 - \frac{1}{2} + \Delta & 0 \\ 0 & Q^2 - \frac{1}{2} - \Delta \end{pmatrix} \Longrightarrow \sigma(H_R) = [-\frac{1}{2} - \Delta, 0)$
$\Longrightarrow \exists$ **Strong time op.**

[Example 2: NcHO]

Let A and J be 2×2 matrices defined by

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha, \beta \geq 0, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $\alpha\beta > 1$. Then the **non-commutative harmonic oscillator** is defined by

$$H(\alpha, \beta) = A \otimes (\frac{1}{2}P^2 + \frac{1}{2}Q^2) + J \otimes (QP + \frac{1}{2})$$

$\Longrightarrow \exists$ **Time operator**

Definition (class $S(\mathscr{H})$)

A self-adjoint operator H on $\mathscr{H}$ is said to be in the class $S(\mathscr{H})$ if it has the following properties (H.1)–(H.4):

(H.1) $\sigma_{\rm sc}(H) = \emptyset$.

(H.2) $\sigma_{\rm ac}(H) = [0, \infty)$.

(H.3) $\sigma_{\rm disc}(H) = \sigma_{\rm p}(H) = \{E_n\}_{n=1}^{\infty}$, $E_1 < E_2 < \cdots < 0$, $\lim_{n\to\infty} E_n = 0$ (hence $0 \notin \sigma_{\rm p}(H)$).

(H.4) There exists a strong time operator $T_{\rm ac}$ of $H_{\rm ac}$ in $\mathscr{H}_{\rm ac}(H)$.

Theorem (Arai-H.(16) Ultra-weak time op. of H)

Let $H \in S(\mathscr{H})$. Then $\exists \mathfrak{t}_H$ ultra-weak time op.

Proof:

$$H = H_{\rm p} \oplus H_{\rm ac},$$

where $H_{\rm p} = H\lceil_{\mathscr{H}_{\rm p}(H)}$ $H_{\rm ac} = H\lceil_{\mathscr{H}_{\rm ac}(H)}$.

- By (H.3), $H_{\rm p}$ has an **ultra-weak time op.** $\mathfrak{t}_{\rm p}$ such that

$$\mathfrak{t}_{\rm p}[H_{\rm p}\phi, \psi] - \mathfrak{t}_{\rm p}[\phi, H_{\rm p}\psi] = -i(\phi, \psi), \quad \phi, \psi \in \exists \mathscr{E}_{\rm p}.$$

- By (H.4), $H_{\rm ac}$ has a **strong time op.** $T_{\rm ac}$ such that

$$[H_{\rm ac}, T_{\rm ac}] = -i1$$

- $\mathfrak{t}_H : (\mathscr{H}_{\rm ac}(H) \oplus \mathscr{D}_{\rm p}) \times (D(T_{\rm ac}) \oplus \mathscr{D}_{\rm p}) \to \mathbf{C}$ by

$$\mathfrak{t}_H[\phi_1 \oplus \phi_2, \psi_1 \oplus \psi_2] = (\phi_1, T_{\rm ac}\psi_1) + \mathfrak{t}_{\rm p}[\phi_2, \psi_2].$$

Example (Agmon potential)

Let $d \geq 3$. Suppose that $U \in L^\infty(\mathbf{R}^3)$. Then

$$V(x) = \frac{U(x)}{(1+|x|^2)^{1/2+\varepsilon}}$$

is an Agmon potential for all $\varepsilon > 0$. Suppose that U is negative, continuous, spherically symmetric and satisfies that $U(x) = -1/|x|^\alpha$ for $|x| > R$ with $0 < \alpha < 1$ and $R > 0$. Let $2\varepsilon + \alpha < 2$. Then H has an ultra-weak time operator.

Example (hydrogen atom)

The hydrogen Schrödinger operator $H_{\mathrm{hyd}} = -\Delta - \gamma/|x|$ is self-adjoint with $D(H_{\mathrm{hyd}}) = D(H_0)$. The Coulomb potential $-\gamma/|x|$ with $d = 3$ is not an Agmon potential. But we can show that H_{hyd} has an ultra-weak time operator.

Summary

(1) We construct an ultra-weak time op. $\mathfrak{t}$ of $H = \frac{1}{2}P^2 + V$:

$$\mathfrak{t}(H\phi, \psi) - \mathfrak{t}(H\psi, \phi)^* = -i(\phi, \psi)$$

(2) $\mathfrak{t}$ is **densely** defined.
(3-1) We assume $\sigma_{sc}(H) = \emptyset$ and $0 \notin \sigma_p(H)$.
(3-2) We assume $\#\sigma_{disc}(H) = \infty$ or $\#\sigma_{disc}(H) = 0$.
(4) H_{hyd} is included in our results.

Reference: A. Arai and F. Hiroshima, Ultra-Weak Time Operators of Schrödinger Operators, arXiv:1607.04702, 2016