Analysis of ground state of quantum field theory by Gibbs measures

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2. Questions of ground states of SRPF model
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- The Pauli-Fierz (PF) model is introduced by W. Pauli and M. Fierz at 1938, which describes an interaction between a **nonrelativistic particle** and a quantized radiation field.
- The Hamiltonian can be realized as a **self-adjoint operator** H on a Hilbert space $\mathscr{H}$. Spectrum of H has been studied from mathematical point of view.
- In this talk a nonrelativistic particle is replaced by a **semi-relativistic particle**.

Particle part and field part

Particle part is governed by h_{sr}.

- $h_{sr} = \sqrt{-\Delta + m^2} + V$ on $L^2(\mathbb{R}^3)$

  (1) $\lim_{|x|\to\infty} V(x) = \infty$ (confining pot.)

  (2) $\|Vf\| \leq a\|\sqrt{-\Delta+m^2}f\| + b\|f\|$, $a < 1$ (relative pot.)
- Particle mass $m \geq 0$.
- **Massless case** $m = 0 \Longrightarrow h = \sqrt{-\Delta} + V$

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Field part is governed by a boson free field Hamiltonian $\mathrm{H_f}(\nu)$.

- Boson Fock space $\mathscr{F} = \bigoplus_{n=0}^{\infty} \otimes_s^n [L^2(\mathbb{R}^3 \times \{1,2\})]$
- CCR $[a(k,j), a^\dagger(k',j')] = \delta(k-k')\delta_{jj'}$
- Dispersion relation $\omega(k) = \sqrt{|k|^2+\nu^2}$, $\nu \ge 0$
- Free field Hamiltonian

$$\mathrm{H_f}(\nu) = \sum_{j=1}^{2} \int \omega(k) a^\dagger(k,j) a(k,j) dk$$

Quantized radiation field and total system

Quantized radiation field We introduce a quantized radiation field $A(x)$.

- $A_\mu(x) = \sum_{j=1}^{2} \int a^\dagger(k,j) e_\mu(k,j) \frac{e^{-ikx}\hat{\varphi}(k)}{\sqrt{\omega(k)}} + h.c. dk$
- $\hat{\varphi}$ is UV cutoff and $\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$
- $e(k,1), e(k,2)$ polarization st $e(k,1) \times e(k,2) = \frac{k}{|k|}$
- $\nabla A(x) = 0$ (Coulomb gauge)

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Total Hamiltonian

- State space (Hilbert sp) $\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathscr{F}$
- Decoupled operator $H(0) = h_{sr} \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}(\nu)$
- minimal coupling $-i\nabla_x \otimes 1\!\!1 \Longrightarrow T_A = -i\nabla_x \otimes 1\!\!1 - \alpha A(x)$, $\alpha \in \mathbb{R}$.

SRPF Hamiltonian is defined by

$$\mathbf{H} = \mathbf{H}(\alpha, \mathbf{m}, \nu) = \sqrt{\mathbf{T_A^2} + \mathbf{m^2}} + \mathbf{V} \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}(\nu)$$

Questions of ground states of SRPF model

Particle mass m and boson mass ν

particle mass ＼ boson mass	$\nu > 0$	$\nu = 0$
$m > 0$	OK	OK
$m = 0$	OK	**my Talk**

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Questions

- Self-adjointness of H?
- Existence of ground state φ_{g}?
- Properties of ground state?
  - Spatial decay $\|\varphi_{\mathrm{g}}(x)\| \sim e^{-|x|}, 1/|x|^a$
  - Gaussian domination $\|e^{+\beta A^2(f)}\varphi_{\mathrm{g}}\| < \infty$?
- Gibbs measures μ_{Gibbs} on **càdlàg** paths?
  - $(\varphi_{\mathrm{g}}, O\varphi_{\mathrm{g}}) = \mathbb{E}_{\mu_{\mathrm{Gibbs}}}[f_O]$

Existence of the ground state and resonances

Figure: $\mathrm{Spec}(H(0))$

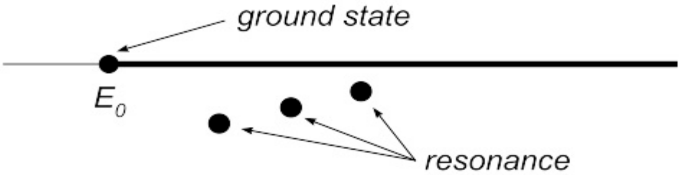

Figure: $\mathrm{Spec}(H)$

Related models

- Nelson model **(particle+scalar field)**

$$(-\frac{1}{2m}\Delta_x + V)\otimes \mathbb{1} + \mathbb{1}\otimes \mathrm{H_f}(\nu) + \alpha\phi(x)$$

- Spin-boson model **(spin+scalar field)**

$$\sigma_z \otimes \mathbb{1} + \mathbb{1}\otimes \mathrm{H_f}(\nu) + \alpha\sigma_x \otimes \phi(f)$$

- PF model **(particle+radiation field)**

$$\frac{1}{2m}T_A^2 + V\otimes \mathbb{1} + \mathbb{1}\otimes \mathrm{H_f}(\nu)$$

- SRPF model **(my talk)**

$$\sqrt{T_A^2 + m^2} + V\otimes \mathbb{1} + \mathbb{1}\otimes \mathrm{H_f}(\nu) \quad m \geq 0$$

- **Self-adjointness** Arai (81,83), H. (00,02), Loss-Miyao-Spohn (07), Haslar-Herbst (08)
- **Existence of GS** Bach-Fröhlich-Sigal (97,99), Arai-Hirokawa (97), Spohn (99), Gérard (00), Griesemer-Lieb-Loss (01), Lieb-Loss (02), Arai(01), Bruneau-Dereziński (04), Hirokawa-H.-Spohn (05), Miyao-Spohn (08), Gérard-H-Suzuki-Panatti (09), Könenberg-Matte-Stockmeyer (11)+many papers,
- **Enhanced binding** H.-Spohn (02), Catto-Hainzl (02), Chen-Vogalter-Vugalter (03), H-Sasaki (08,14)
- **Absence of GS** Arai-Hirokawa-H (99), Chen(01), Derezinski-Gérard(04), Lorinczi-Minlos-Spohn(02), Hirokawa (06), Haslar-Herbst (06) Gérard-H-Suzuki-Panatti (10)
- **Multiplicity of GS** Bach-Fröhlich-Sigal (97), Arai-Hirokawa (97), H. (00,02), H.-Spohn (01), Bach-Fröhlich-Pizzo (05)
- **Asymptotic field** Spohn(97), Derezinski-Gérard(99,04), Gérard (02), Fröhlich-Griesemer-Schlein (01, 02)

Gibbs measures

Example of Nelson model H_N

$$(f \otimes 1\!\!1_{\mathscr{F}}, e^{-2TH_N} g \otimes 1\!\!1_{\mathscr{F}}) = \mathbb{E}[e^{\alpha^2 \int_{-T}^{T} ds \int_{-T}^{T} dt W}]$$

- Vacuum $1\!\!1_{\mathscr{F}} \in \mathscr{F}$
- $\mathbb{E}[\cdots] = \int_{\mathbb{R}^3} dx \mathbb{E}^x [\bar{f}(B_{-T}) g(B_T) e^{-\int_{-T}^{T} V(B_s) ds} \cdots]$
- Pair interaction $W = W(B_t - B_s, t-s)$,

$$W(X,t) = \frac{1}{2} \int_{\mathbb{R}^3} \frac{\hat{\varphi}(k)^2}{\omega(k)} e^{-ikX} e^{-|T|\omega(k)} dk$$

Ground state and Gibbs measures

- Finite vol.Gibbs meas.(meas.on cont. paths)

$$A \mapsto \mu_T(A) = \frac{1}{Z_T}\mathbb{E}[\mathbb{1}_A e^{\alpha^2 \int_{-T}^{T} ds \int_{-T}^{T} dtW}]$$

Thm. Betz-H-Lorinczi-Minlos-Spohn (01)
$\exists \mu_{\mathrm{Gibbs}} = \lim_T \mu_T$ in *local weak* for Nelson model.

- $\exists \varphi_{\mathrm{g}} \Longrightarrow \varphi_{\mathrm{g}}^T = \frac{e^{-TH} f \otimes \mathbb{1}_{\mathscr{F}}}{\|e^{-TH} f \otimes \mathbb{1}_{\mathscr{F}}\|} \to \varphi_{\mathrm{g}}$

$$(\varphi_{\mathrm{g}}, O\varphi_{\mathrm{g}}) = \lim_T (\varphi_{\mathrm{g}}^T, O\varphi_{\mathrm{g}}^T) = \lim_T \mathbb{E}_{\mu_T}[f_0^T] = \mathbb{E}_{\mu_{\mathrm{Gibbs}}}[f_0^\infty]$$

Main results

- **Thm1** Under some condition on $\hat{\varphi}$ and V, H is self-adjoint on $Dom(-\Delta) \cap Dom(\mathrm{H_f}(\nu))$.
- **Thm2** Let V be confining. Then $\exists$ground state φ_{g} and unique for $\forall \alpha \in \mathbb{R}, \forall m, \nu \geq 0$
- **Thm3** $\exists \mu_{\mathrm{Gibbs}}$ on cadlag path space.

- **Thm4** $\|e^{+(\beta/2)A(f)^2}\varphi_{\rm g}\| < \infty$ for $\beta < \exists c$
- **Thm5** Let $V = V_+ - V_-$.
  - $m \geq 0$ and $\lim_{|x|\to\infty} V(x) = \infty$
    $\Longrightarrow \|\varphi_{\rm g}(x)\|_{\mathscr{F}} \leq e^{-c|x|}$ exp decay
  - $m > 0$ $V_+ = 0$ and $\lim_{|x|\to\infty} V_-(x) + E < 0$
    $\Longrightarrow \|\varphi_{\rm g}(x)\|_{\mathscr{F}} \leq e^{-c|x|}$ exp decay
  - $m = 0$ $V_+ = 0$ and $\lim_{|x|\to\infty} V_-(x) + E < 0$
    $\Longrightarrow \|\varphi_{\rm g}(x)\|_{\mathscr{F}} \leq \frac{c}{1+|x|^4}$ poly. decay

Idea of proofs

- Application of a functional integral representation: $(F, e^{-TH}G) = \int dx \mathbb{E}[\cdots]$ in terms of a combination of **∞-dim.OU, Brownian Motion, Poisson process and subordinator**.

- **gs for** $m = \nu = 0$
  Cf. Gérard (00)+Griesemer-Lieb-Loss (01)
  - For $\nu > 0$, H has a spectral gap (HVZ Thm) and $\exists \varphi_g^\nu$.
  - Pull through formula
    $a(k, j)\varphi_g = (H - E - |k|)^{-1} \left[|-i\nabla_x - \alpha A(x)|, a(k, j) \right] \varphi_g$
  - $|u| = \sqrt{u^2} = \int_0^\infty (1 - e^{-\beta u^2}) d\lambda(\beta)$+ Hardy inequality +FKF
  - By this we have $\|N^{1/2}\varphi_g^\nu\| \le c \| |x| \varphi_g^\nu \|$
  - Spatial decay $\|\varphi_g^\nu(x)\|_{\mathscr{F}} \le c e^{-|x|}$ or $\le c(1+|x|^4)^{-1}$ implies

$$\varphi_g^\nu \to \exists \varphi_g \neq 0 \ (\nu \to 0).$$

Conclusion and remarks

- We show **(1)** self-adjointness of H **(2)** $\exists\varphi_{\rm g}$ **(3)** spatial decay and Gaussian domination of $\varphi_{\rm g}$ **(4)** $\exists\mu_{\rm Gibbs}$ on cadlag path, for $m,\nu\geq 0$ and $\forall\alpha\in\mathbb{R}$.
- Our results are valid for the case of $m=0$ and $\nu=0$

$$|-i\nabla_x\otimes 1\!\!1-\alpha A(x)|+V\otimes 1\!\!1+1\!\!1\otimes \mathrm{H_f}(0)$$

- do **not** need $\int\frac{|\hat\varphi|^2}{\omega^3}dk<\infty$. Bach-Fröhlich-Sigal (99)
- Extension to H_S with spin 1/2, and to translation invariant case H_P. Functional integral representation of e^{-TH_S} and e^{-TH_P} can be also constructed.
- UV renormalization?
  Cf Nelson model $\Longrightarrow$ Nelson(64),
  Gérard-H-Panatti-Suzuki(10), Gubinelli-H-Lorinczi(14)