Stochastic UV renormalization

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2014/10/15
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Gaussian random variables

- ►Gaussian prob. sp. (Q, Σ, μ), where $Q = \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^3)$
- ►Canonical pair $\langle \phi, f \rangle = \phi(f)$, $\phi \in Q, f \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)$
- ►Gaussian random variables

$$\mathbb{E}_\mu[\phi(f)] = 0$$

$$\mathbb{E}_\mu[\phi(f)\phi(g)] = \frac{1}{2}(f, g)_{L^2(\mathbb{R}^3)}$$

- ►$\phi(f)$ can be extended from $f \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)$ to $f \in L^2(\mathbb{R}^3)$
- ►$f \mapsto \phi(f)$ is linear over $\mathbb{C}$
- ►$(\phi(f), f \in L^2(\mathbb{R}^3))$ **Gaussian r.v.**

Free field Hamiltonian

►Boson Fock space $L^2(Q,\mu) = \mathscr{F}$

►$\left(:\prod_j^m \phi(f_j):, :\prod_j^n \phi(g_j):\right)_{\mathscr{F}} = 0$ if $n \neq m \Longrightarrow \mathscr{F} = \oplus_{n=0}^{\infty} \mathscr{F}^{(n)}$,
where $\mathscr{F}^{(n)} = \mathrm{span}\{:\prod_j^n \phi(g_j):\}$ is n-particle subspace

►$A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \stackrel{Fanctor}{\Longrightarrow} \Gamma(A) : \mathscr{F} \to \mathscr{F}$ by

$$\Gamma(A) : \prod_j \phi(f_j) :=: \prod_j \phi(Af_j) :$$

►$\|A\| \leq 1 \Longrightarrow \|\Gamma(A)\| \leq 1$

►Ex. $\omega = \sqrt{-\Delta} \Longrightarrow U_t = \Gamma(e^{-it\omega})$ is a one-para. unitary group

►$\exists H_{\mathrm{f}}$ **such that** $\Gamma(e^{-it\omega}) = e^{-itH_{\mathrm{f}}}$ **for all** $t \in \mathbb{R}$

►H_{f} free field Hamiltonian, formally "$H_{\mathrm{f}} = \int \omega(k) a^{\dagger}(k) a(k) dk$"

Nelson model

►Total Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathscr{F}$

►UV cutoff function h_ε with UV cutoff parameter $\varepsilon > 0$.

$$\hat{h}_\varepsilon(k) = \frac{e^{-\varepsilon|k|^2/2}}{\sqrt{|k|}} 1\!\!1_{|k|>\lambda} \in L^2(\mathbb{R}^3) \quad \varepsilon > 0$$

►Nelson Hamiltonian with UV parameter $\varepsilon > 0$:

$$H_\varepsilon = H_p \otimes 1\!\!1 + 1\!\!1 \otimes H_{\mathrm{f}} + g\phi(h_\varepsilon(\cdot - x))$$

►$H_p = -\Delta_x/2 + V(x)$ is Schrödinger op. on $L^2(\mathbb{R}^3)$.

Nelson' renormalization

- ►UV renormalization $H_\varepsilon \to \exists H_0?\ \varepsilon \downarrow 0$
- ►Removal of UV cutoff:

$$\lim_{\varepsilon\downarrow 0} \hat{h}_\varepsilon(k) = \frac{1}{\sqrt{|k|}} 1\!\!1_{|k|>\lambda} \notin L^2(\mathbb{R}^3)$$

Thm. E. Nelson, J.Math.Phys.5 (1964)

Let

$$E_\varepsilon = -\frac{g^2}{2}\int_{|k|>\lambda} \frac{e^{-\varepsilon|k|^2}}{|k|}\beta(k)dk, \quad \beta(k) = \frac{1}{|k|+|k|^2/2},$$

where $E_\varepsilon \to -\infty$ as $\varepsilon \downarrow 0$. Then

$$\lim_{\varepsilon\downarrow 0}(H_\varepsilon - E_\varepsilon - z)^{-1} = U_G^{-1}(H_0 - z)^{-1}U_G$$

where U_G is a unitary operator called Gross transform.

E. Nelson tried to have an alternative proof in [Proc. Conference on Analysis in Function Space, W. T. Martin and I. Segal (eds.), p. 87, MIT Press, 1964]. **But he failed !**

Thm. Gubinelli+FH+Lőrinczi (GHL) JFA 2014

$$\lim_{\varepsilon\downarrow 0} e^{-T(H_\varepsilon - E_\varepsilon)} = e^{-TH_0}$$

STEP (1) path int rep and diagonal part

▶Fock vacuum $\mathbb{1} \in \mathscr{F}$, & $f, h \in L^2(\mathbb{R}^3)$ is fixed. Set $V = 0$ for simplicity.

▶By 

(Lőrinczi-FH-Betz),

$$(f \otimes \mathbb{1}, e^{-2TH_\varepsilon} h \otimes \mathbb{1})_{\mathscr{H}} = \int dx \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) e^{\frac{g^2}{2} S_\varepsilon} \right]$$

▶Pair interaction

$$S_\varepsilon = \int_{-T}^{T} ds \int_{-T}^{T} dt W_\varepsilon(B_t - B_{,} t - s)$$

▶Pair potential

$$W_\varepsilon(X, t) = \frac{1}{2} \int \frac{e^{-\varepsilon |k|^2}}{|k|} \mathbb{1}_{|k| > \lambda} e^{-iX \cdot x} e^{-|k||t|} dk.$$

▶The diagonal part of S_ε is singular $\Longrightarrow E_\varepsilon$

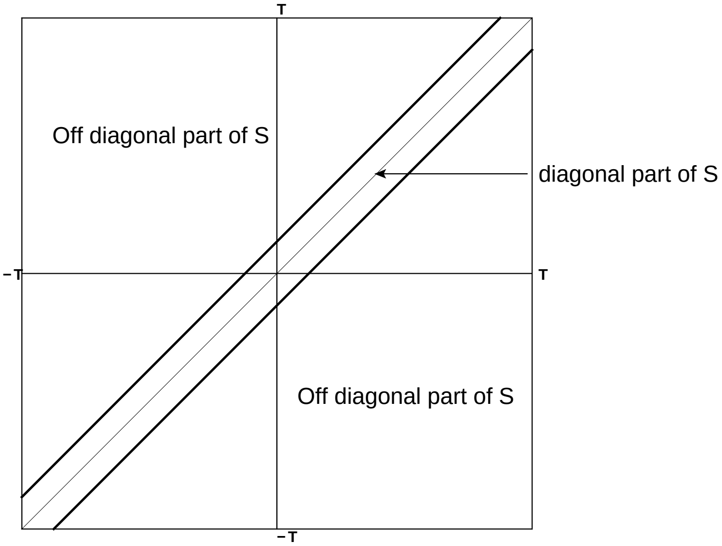

Figure: S_ε

► Let $\varphi_\varepsilon(x,t) = \frac{1}{2}\int 1\!\!1_{|k|>\lambda} \frac{e^{-\varepsilon|k|^2} e^{-ik\cdot x-|k||t|}}{|k|}\beta(k)dk.$

► Ito formula:

$$\begin{aligned} &\varphi_\varepsilon(B_T - B_s, T-s) - \varphi_\varepsilon(0,0) \\ &= \int_s^T \nabla\varphi_\varepsilon(B_t - B_s, t-s)dB_t + \int_s^T (\frac{1}{2}\Delta_x + \frac{\partial}{\partial t})\varphi_\varepsilon(B_t - B_s, t-s)dt \\ &= \int_s^T \nabla\varphi_\varepsilon(B_t - B_s, t-s)dB_t - \int_s^T W_\varepsilon(B_t - B_s, t-s)dt \end{aligned}$$

► $S_\varepsilon = 2\int_{-T}^T ds \int_s^T W_\varepsilon(B_t - B_s, t-s)dt$

► $S_\varepsilon = Y_\varepsilon + Z_\varepsilon + 4T\varphi_\varepsilon(0,0)$ and

$$Y_\varepsilon = 2\int_{-T}^T ds \int_s^T \nabla\varphi_\varepsilon(B_t - B_s, t-s)\cdot dB_t,$$

$$Z_\varepsilon = -2\int_{-T}^T \varphi_\varepsilon(B_T - B_s, T-s)ds.$$

► $S_\varepsilon^{ren} = S_\varepsilon - 4T\varphi_\varepsilon(0,0)$

STEP (2) vacuum expectation

Let

$$a(\varepsilon) = (f \otimes 1\!\!1, e^{-2T(H_\varepsilon + g^2 \varphi_\varepsilon(0,0))} h \otimes 1\!\!1) = \int_{\mathbb{R}^3} \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) e^{\frac{g^2}{2} S_\varepsilon^{ren}} \right] dx$$

Thm. (Vacuum exp.) GHL, JFA 14

- ►(0) $|a(\varepsilon)| < \infty$ ($\forall \varepsilon > 0$).
- ►(1) $\exists S_0^{ren}$ st $\lim_{\varepsilon \downarrow 0} a(\varepsilon) = \int_{\mathbb{R}^3} \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) e^{\frac{g^2}{2} S_0^{ren}} \right] dx$
- ►(2) $\exists C$ st $|a(\varepsilon)| \leq \|f\| \|h\| e^{CT} \quad \varepsilon \geq 0.$

STEP(3) extension to a dense subspace

▶Dense subspace

$$\mathscr{D} = span\{f \otimes F(\phi(f_1), \cdots, \phi(f_n)); f \in L^2(\mathbb{R}^3), F \in \mathscr{S}(\mathbb{R}^n), n \geq 1\} \subset \mathscr{H}$$

▶For $F, G \in \mathscr{D}$ we can easily show the existence of the limit:

$$\exists \lim_{\varepsilon \downarrow 0} (F, e^{-2T(H_\varepsilon + g^2 \varphi_\varepsilon(0,0))} G)$$

▶Furthermore we can see the path int rep of above limit.

STEP(4) The most difficult step

▶ $\exists \lim_{\varepsilon \downarrow 0} (F, e^{-2T(H_\varepsilon + g^2 \varphi_\varepsilon(0,0))} G)$ for $F, G \in \mathscr{H}$?

▶ It is enough to show the existence of a lower bound C:

$$H_\varepsilon + g^2 \varphi_\varepsilon(0,0) > C, \quad \varepsilon > 0$$

※ Note that $\varphi_\varepsilon(0,0) \to +\infty$ but $\inf \sigma(H_\varepsilon) \to -\infty$.

▶ Nelson showed it by functional analysis.

▶ **How can we prove by path int?**

Outline of the idea

▶Adding a dummy potential $\delta|x|^2$:

$$H_\varepsilon(\delta) = H_\varepsilon + \delta|x|^2 + g^2\varphi_\varepsilon(0,0)$$
$$H_\varepsilon(0) = H_\varepsilon + g^2\varphi_\varepsilon(0,0)$$

▶Let $\varphi_{\rm g}^T = e^{-TH_\varepsilon(\delta)} f\otimes 1\!\!1 / \| e^{-TH_\varepsilon(\delta)} f\otimes 1\!\!1\| \sim$ gs.

▶We can show that by path int rep. $\lim_{T\to\infty}(f\otimes 1\!\!1, \varphi_{\rm g}^T) > 0$.

$$\Downarrow$$

$H_\varepsilon(\delta)$ has a ground state $\varphi_{\rm g}(\delta) > 0$ a.e. [H. Spohn, 1999]

▶From

(1) $(\varphi_{\rm g}(\delta), f\otimes 1\!\!1)\neq 0$ for $0\leq f\in L^2(\mathbb{R}^3)$,

(2) $(f\otimes 1\!\!1, e^{-2TH_\varepsilon(\delta)}h\otimes 1\!\!1)\leq \|f\|\|h\|e^{CT}$,

it follows that

$$\inf\sigma(H_\varepsilon(\delta))\stackrel{(1)}{=}-\frac{1}{2T}\lim_{T\to\infty}\log(f\otimes 1\!\!1, e^{-2TH_\varepsilon(\delta)}f\otimes 1\!\!1)\stackrel{(2)}{\geq}-C,$$

▶C is independent of $\delta>0$.

► $(F, e^{-2TH_\varepsilon(\delta)}F) \geq e^{2TC}\|F\|^2$, $\forall F \in \mathscr{H}$

► $(F, e^{-2TH_\varepsilon(0)}F) \geq e^{2TC}\|F\|^2$ by Lebesgue dom.conv.

Thm. (uniform lower bound) GHH JFA14

$\inf \sigma(H_\varepsilon + g^2 \varphi_\varepsilon(0,0)) \geq C$.

STEP(5)

▶Hence $\exists \lim_{\varepsilon\downarrow 0}(F, e^{-2T(H_\varepsilon+g^2\varphi_\varepsilon(0,0))}G)$ for $\forall F, G \in \mathscr{H}$.

▶By Riesz rep. theorem, $\exists T_t$ st

$$\lim_{\varepsilon\downarrow 0}(F, e^{-t(H_\varepsilon+g^2\varphi_\varepsilon(0,0))}G) = (F, T_t G)$$

▶T_t ic C_0 semigroup, and $T_t = e^{-t\exists H_0}$.

▶Then we obtain that

$$\lim_{\varepsilon\downarrow 0} e^{-t(H_\varepsilon+g^2\varphi_\varepsilon(0,0))} = e^{-tH_0}$$

Translation invariant model

▶(Fiber Hamiltonian) $V=0 \Longrightarrow H=\int_{\mathbb{R}^3}^{\oplus} H(P)dP$

▶Nelson model with total momentum P

$$H(P)=\frac{1}{2}(P-P_f)^2+\phi(\varphi)+H_{\rm f}, \quad P\in\mathbb{R}^3$$

acting on $\mathscr{F}$

▶$e^{-itP_f}=\Gamma(e^{-it(-i\nabla)})$

▶$H_\varepsilon(P)=\frac{1}{2}(P-P_f)^2+\phi(\varphi_\varepsilon)+H_{\rm f}$

▶UV renormalization for each P?

►In the same was as H_ε we can show that

$$\exists \lim_{\varepsilon \downarrow 0} (\mathbb{1}, e^{-T(H_\varepsilon(P)-E_\varepsilon)} \mathbb{1})$$

►But $|P| < p_* \Longrightarrow \exists$ ground state of $H_\varepsilon(P)$

Thm. (UV ren. for fiber Hamiltonian)

$\exists H_0(P)$ such that $\lim_{\varepsilon \downarrow 0} e^{-TH_\varepsilon(P)} = e^{-TH_0(P)}$ for $\forall P \in \mathbb{R}^3$.

Proof

►By path int rep of $e^{-TH_\varepsilon(P)}$ we can see that

(1) $\inf \sigma(H_\varepsilon(0)) \le \inf \sigma(H_\varepsilon(P))$ diamagnetic inequality,

(2) $H_\varepsilon(0)$ has a ground state $\varphi_{\rm g}(0)$ and $\varphi_{\rm g}(0) > 0$,

(3) $(\mathbb{1}, \varphi_{\rm g}(0)) \neq 0$

►Hence the uniform lower bound, $H_\varepsilon(0) - E_\varepsilon > -\exists C$, exists

►$H_\varepsilon(P) - E_\varepsilon > -C$ follows!

Weak coupling limit

►Many body Nelson model

$$H_\varepsilon = h_p \otimes 1\!\!1 + 1\!\!1 \otimes H_{\rm f} + \sum_{j=1}^N \phi(h_\varepsilon(\cdot - x_j)),$$

where $h_p = \sum_{j=1}^N (-\frac{1}{2}\Delta_j) + V(x_1, ..., x_N)$

►$H_\varepsilon(\kappa) = h_p \otimes 1\!\!1 + \kappa^2 1\!\!1 \otimes H_{\rm f} + \kappa \sum_{j=1}^N \phi(h_\varepsilon(\cdot - x_j))$, $\kappa > 0$

►$E_\varepsilon(\kappa) = -\frac{g^2 N}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{-\varepsilon|k|^2}}{|k|} \frac{\kappa^2}{\kappa^2|k| + |k|^2/2} 1\!\!1_{|k|>\lambda} dk.$

Effective potentials

Thm. (Effective potential)

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{\kappa\to\infty}(f\otimes 1\!\!1, e^{-t(H_\varepsilon(\kappa)-E_\varepsilon(\kappa))}h\otimes 1\!\!1)(\text{H.JMP98})\\
=&\lim_{\kappa\to\infty}\lim_{\varepsilon\downarrow 0}(f\otimes 1\!\!1, e^{-t(H_\varepsilon(\kappa)-E_\varepsilon(\kappa))}h\otimes 1\!\!1)(\text{GHL.JFA14})\\
=&(f, e^{-th_{\text{eff}}}h),
\end{aligned}$$

where

$$h_{\text{eff}} = -\frac{1}{2}\sum_{j=1}^{N}\Delta_j + V(x^1,\ldots,x^N) - \frac{g^2}{4\pi}\sum_{i<j}\frac{1}{|x_i - x_j|}.$$

Summary

►E. Nelson: E_ε is derived from a commutator, and the Gross trans U_G is needed

►GHL: E_ε is a diagonal part of pair int. and the Gross trans is not needed

►Applications

1. model on mfd: BM$\to$ diffusion proc.
2. Non-local Nelson model: BM$\to$ Lévy proc.
   $\sqrt{-\Delta+m^2}\otimes 1\!\!1 + 1\!\!1\otimes H_{\rm f} + \phi(h_\varepsilon(\cdot - x))$,
3. Translation invariant case

Referemces

- Bach-Fröhlich-Sigal (Adv Math 97,CMP98) GS. for $|g| \ll 1$
- Gérard (AHP 00), Spohn(LMP 00) GS. $\forall g$
- Fefferman (Adv Math 00) Stability of matter
- FH (JMP99, CMP 01,JFA05) uniqueness and s.a. $\forall g$
- Betz-FH-Lőrinczi-Minlos-Spohn (RMP01) Gibbs meas.
- Griesmer-Lieb-Loss (Inv. Math 01) GS ∀g.
- FH-Spohn(AHP 01,JMP05) enhanced binding and eff mass
- Hirokawa-FH-Spohn(Adv Math 05) GS with UV ren.
- FH-Lőrinczi (JFA 08) model with spin
- Gérard-FH-Panati-Suzuki (CMP12,JFA12,LMP12) model on mfd.
- FH (Adv Math 14) Non-local model