# Gibbs measure approach to spin-boson model

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# Quantum Mechanics

- Schödinger operators ($d \geq 3$):

$$h = -\Delta + gV \quad (V \leq 0) \quad on\ L^2(\mathbb{R}^d)$$

- $\sigma(h) = \{E_j\}_{j=0}^{\infty} \cup [0, \infty)$, $E_0$ is called ground state energy
- $h\varphi_{\rm g} = E_0 \varphi_{\rm g}$, $\varphi_{\rm g}$ **is called the ground state**
- Ex(1) $(g = 0)$ $\sigma(h) = [0, \infty)$ and no ground state!
- Ex(2) Lieb-Thirring bound:

$$\#\{\text{eigenvalues} \leq 0\} \leq a \int |gV(x)|^{d/2} dx$$

  Then $|g| \ll 1 \Longrightarrow h$ has no ground state.
- Small perturbation $W \Longrightarrow$ the discrete spectrum of $h$ moves on the real line.

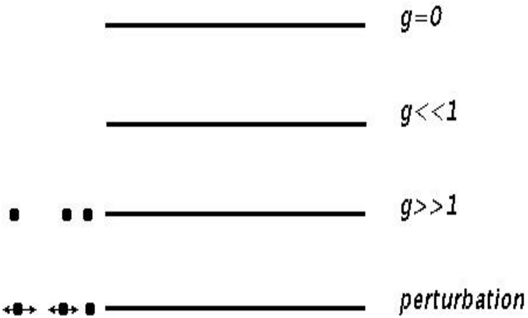


Figure: Spectrum of $-\Delta + gV$ and pertuerbation

# Quantum Field Theory

- - **Spin boson model(2level atom + scalar field)**
  - Nelson model(strong interaction)
  - Polaron model(phonon)
  - Pauli-Fierz model (QED)
- $H = H_0 + gH_\mathrm{I}$ on $\mathscr{K} \otimes \mathscr{F}$
- $H_0 = A \otimes \mathbb{1} + \mathbb{1} \otimes H_\mathrm{f}$
- $\sigma(H_\mathrm{f}) = [0, \infty)$, $\sigma_p(H_\mathrm{f}) = \{0\} \Longrightarrow \sigma(H_0) = \{E_j\}_{j=0}^\infty \cup [E_0, \infty)$
- $\{E_j\}_{j=0}^\infty$ **are embedded in the continuous spectrum**
- We can "not" apply the regular perturbation theory for the discrete spectrum.
- $\sigma(H) = [\inf \sigma(H), \infty) \Longrightarrow \inf \sigma(H)$ is a point spectrum?

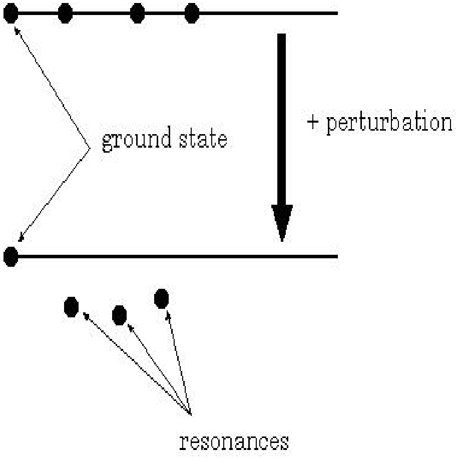


Figure: Spectrum of $H$ and $H_0$

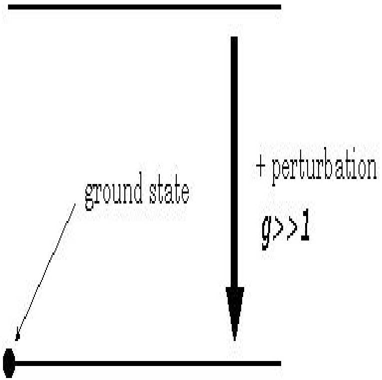


Figure: Enhanced binding

# Spin-boson model

- Boson Fock space $\mathscr{F} = \bigoplus_{n=0}^{\infty} \left( \otimes_{\rm sym}^n L^2(\mathbb{R}^d) \right)$
- Annihilation and creation operators: $\Phi = \{\Phi^{(n)}\} \in \mathscr{F} \Longrightarrow$ $(a^\dagger(f)\Phi)^{(n)} = \sqrt{n} S_n(f \otimes \Phi^{(n-1)})$, $a(f) = (a^\dagger(\bar{f}))^*$
- **CCR** $[a(f), a^\dagger(g)] = (\bar{f}, g)\mathbb{1}$, $[a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]$.
- $2_{nd}$ quantization of s.a. $T$, $d\Gamma(T) : \mathscr{F} \to \mathscr{F}$ defined by $d\Gamma(T) = 0 \oplus_{n=1}^{\infty} \sum_{j=1}^{n} \underbrace{\mathbb{1} \otimes \cdots \overset{j}{T} \cdots \mathbb{1}}_{n}$.

**Spin-boson model** $H_{\rm SB}=\sigma_z\otimes 1\!\!1+1\!\!1\otimes H_{\rm f}+g\sigma_x\otimes\phi(\hat{h})$

- $\mathscr{H}=\mathbb{C}^2\otimes\mathscr{F}\cong L^2(\mathbb{Z}_2\times\mathscr{S}',db\otimes d\mu)$, $\mathbb{Z}_2=\{\pm 1\}$
- $g\in\mathbb{R}$ is a coupling constant
- Pauli matrices $\sigma_x=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $\sigma_y=\begin{bmatrix}0&-i\\i&0\end{bmatrix}$ $\sigma_z=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$
- $H_{\rm f}=d\Gamma(\omega)=\int\omega(k)a^\dagger(k)a(k)dk$, free boson Hamiltonian with $\omega(k)=|k|$. I.e., $(H_{\rm f}\Psi)^{(n)}(k_1,...,k_n)=\left(\sum_{j=1}^n\omega(k_j)\right)\Psi^{(n)}(k_1,...,k_n)$
- $\sigma(H_{\rm f})=[0,\infty)$, $\sigma_p(H_{\rm f})=\{0\}$
- $\phi(\hat{h})=\frac{1}{\sqrt{2}}(a^\dagger(\bar{\hat{h}})+a(\hat{h}))$

$H_{\rm SB}\stackrel{one-mode}{\Longrightarrow} Rabi-model\ \sigma_z+a^\dagger a+g\sigma_x(a^\dagger+a)$
$\Longrightarrow$ **Noble prize for physics 2012, Haroche and Wineland!!**

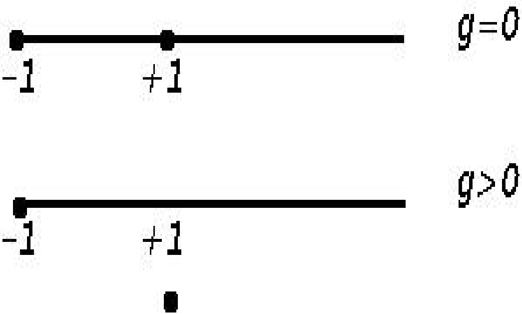


Figure: Spectrum of SB model

# Cadlag path

## Lemma

$(\mathscr{X},\mathscr{G},\exists\mathscr{W})$ *and* $X_t$ *st*

$$(\mathbb{1}, e^{-tH_{\mathrm{SB}}}\mathbb{1}) = e^t \sum_{\sigma\in\mathbb{Z}_2} \mathbb{E}^{\sigma}_{\mathscr{W}}\left[e^{\frac{g^2}{2}\int_0^t dr\int_0^t W(X_r\cdot X_s, r-s)ds}\right],$$

$$W(X,T) = X\int \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-|T|\omega(k)}dk$$

See **Lőrinczi-Hiroshima-Betz, De Gruyter Study in Math 34**

- $\mathscr{X} = D(\mathbb{R};\mathbb{Z}_2)$: the space of càdlàg paths with values in $\mathbb{Z}_2$
- $\mathscr{G}$: the $\sigma$-field generated by cylinder sets
- Coordinate process: $X_t(\omega) = \omega(t)$ for $\omega \in \mathscr{X}$.
- $X_t \stackrel{d}{=} (-1)^{N_t}$

**Nelson model** $(-\Delta_x + V) \otimes 1\!\!1 + g\phi(\hat{\varphi}e^{-ikx}/\sqrt{\omega}) + 1\!\!1 \otimes H_{\mathrm{f}}$

Pair potential

- $\int_0^T dt \int_0^T ds W(X_t - X_s, t-s)$,
  $(X_t)_{t\geq 0}$ **ground state process=** $P(\phi)_1$**-process**
- $W(X,T) = \int \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-ikX} e^{-|T|\omega(k)} dk$

- Nelson (JMP 1964) UV renormalization
- Bach-Fröhlich-Sigal (Adv Math 97) $|g| \ll 1$
- Gérard (AHP 00), Spohn(LMP 00) $\forall g$
- **Betz-FH-Lőrinczi-Minlos-Spohn (RMP01)**
- Hirokawa-FH-Spohn(Adv Math 05) $|g| \ll 1$ + UV ren.
- FH-Sasaki (Math Z 08) enhanced binding
- Gérard-FH-Panati-Suzuki (CMP12,JFA12) model on mfd
- Gubinelli-FH-Lorinczi (preprint 2013) UV ren. by path measure

**PF model** $H_{PF} = \frac{1}{2m}(p \otimes 1\!\!1 + gA)^2 + V \otimes 1\!\!1 + 1\!\!1 \otimes H_{\rm f}$

Pair potential

- $\int_0^T \int_0^T dB_t^\mu W_{\mu\nu}(B_t - B_s, t-s) dB_s^\nu$
- $W_{\mu\nu}(X,T) = \int \frac{|\hat{\varphi}(k)|^2}{2\omega(k)}(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}) e^{-ikX} e^{-|T|\omega(k)} dk$

- Bach-Fröhlich-Sigal (CMP 97) $|g| \ll 1$
- Fefferman (Adv Math 00) Stability of matter
- FH (JMP99, CMP 01) uniqueness and s.a. $\forall g$
- Griesmer-Lieb-Loss (Inv. Math 01), Lieb-Loss (ATMP02) $\forall g$
- FH-Spohn(AHP 01) enhanced binding (dip. app.)
  $SUH_{PF}^{dip}U^{-1}S^{-1} = -\frac{1}{2m_{\rm eff}}\Delta + \tilde{V} + H_{\rm f} \Longrightarrow$ Prof Obata's talk
- FH-Spohn (JMP04), Seiringer-Hainzl (ATMP05) effective mass

**3D Polaron model** $\frac{1}{2}(P-P_{\rm f})^2+g\phi+N$

Pair potential

- $\int_0^T dt \int_0^T ds \frac{e^{-|t-s|}}{|B_t-B_s|}$
- L.Gross (JFA 72) ground state
- Donsker-Varadhan (CPAM 83) ground state energy
- Spohn (Ann Phys 87) effective mass
- Frank-Lieb-Seiringer-Lawrence (preprint 2010) binding energy

# Ground state of SB model

- $\Phi_T = e^{-T(H_{\rm SB}-E)}1\!\!1/\|e^{-T(H_{\rm SB}-E)}1\!\!1\|, \quad T \geq 0,$
- $\gamma(T) = (1\!\!1, \Phi_T)^2 = \dfrac{(1\!\!1, e^{TH_{\rm SB}}1\!\!1)^2}{(1\!\!1, e^{-2TH_{\rm SB}}1\!\!1)} = \dfrac{\left(\sum_\sigma \mathbb{E}^\sigma\left[e^{\frac{g^2}{2}\int_0^T dt \int_0^T W ds}\right]\right)^2}{\sum_\sigma \mathbb{E}^\sigma\left[e^{\frac{g^2}{2}\int_{-T}^T dt \int_{-T}^T W ds}\right]}$

**Lemma**

*A ground state of $H_{\rm SB}$ exists if and only if $\lim_{T\to\infty}\gamma(T) > 0$.*

# Existence and uniqueness of ground state

**Lemma**

*If $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, then $H_{\rm SB}$ has a ground state and it is unique.*

*Proof:*

- $\gamma(T) \geq e^{-g^2\|\hat{h}/\omega\|^2} \Longrightarrow$ a ground state $\varphi_{\rm g}$ of $H_{\rm SB}$ exists.
- $(\Psi, e^{-tH_{\rm SB}}\Phi) > 0$ for $\Psi, \Phi \geq 0$.
- $e^{-tH_{\rm SB}}$ is positivity improving $\Longrightarrow$ Ground state is unique.

$\hat{h}/\omega \in L^2(\mathbb{R}^d)$ is called IR regular condition.

- **Parity** $P = \sigma_x(-1)^N$ $\sigma(P) = \{\pm 1\}$
- $[H_{\rm SB}, P] = 0$

**Theorem**

*$P\varphi_{\rm g} = -\varphi_{\rm g}$, i.e. the parity of $\varphi_{\rm g}$ is -1*

*Proof:*

$\varphi_{\rm g} = \lim_T e^{-TH_{\rm SB}} \mathbb{1} / \|e^{-TH_{\rm SB}} \mathbb{1}\|$ and

$Pe^{-TH_{\rm SB}} \mathbb{1} = e^{-TH_{\rm SB}} P \mathbb{1} = -e^{-TH_{\rm SB}} \mathbb{1}$.

## Gibbs measures

**Finite volume Gibbs meas. on $(\mathscr{X}, \sigma(\mathscr{F}))$**

$$\mu_T(A) = \frac{e^{2T}}{Z_T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}^{\sigma}_{\mathscr{W}} \left[ 1\!\!1_A e^{\frac{g^2}{2} \int_{-T}^{T} dt \int_{-T}^{T} ds W(X_t, X_s, t-s)} \right]$$

- $\mathscr{G}_{[-T,T]} = \sigma(X_t, t \in [-T,T]) \subset \mathscr{G}$
- $\mathscr{F} = \cup_T \mathscr{G}_{[-T,T]}$
- Observable $(\varphi_{\rm g}, O\varphi_{\rm g}) = \lim_T (\Phi_T, O\Phi_T) = \lim_T \mathbb{E}_{\mu_T}[f_{O,T}]$
- $O = e^{+\beta N}, e + \beta\phi(f)^2$ etc.

$$\mu_T \to \exists \mu_\infty (T \to \infty)?$$

**Theorem**

*Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then $\exists \mu_\infty$ on $(\mathscr{X}, \sigma(\mathscr{F}))$ st $\mu_T \to \mu_\infty$ in the sense of* ***local weak****, i.e., $\mu_T(A) \to \mu_\infty(A)$ for $A \in \mathscr{F}$.*

- $\mathbb{E}_{\mu_T}[f] \to \mathbb{E}_{\mu_\infty}[f]$ for $\mathscr{F}_t$ measurable $F$.

## Remarks

- **Finite dim. dis. is given by**

$$E_{\mu_\infty}[\prod 1\!\!1_{A_j}(X_{t_j})] = (\varphi_{\rm g}, 1\!\!1_{A_0} e^{-(t_1-t_0)(H_{\rm SB}-E)} \dots e^{-(t_n-t_{n-1})(H_{\rm SB}-E)} 1\!\!1_{A_n} \varphi_{\rm g})$$

- For $A \in \mathscr{F}_t$,

$$\mu_\infty(A) e^{2t} e^{2Et} \sum_\sigma \mathbb{E}^\sigma_{\mathscr{W}}[(\varphi_{\rm g}(X_{-t}), Q_{[-t,t]} \varphi_{\rm g}(X_t))_{\mathscr{H}} 1\!\!1_A]$$

- Field moment

$$(\varphi_{\rm g}, \phi(f)^n \varphi_{\rm g}) = i^n \mathbb{E}_{\mu_\infty}\left[h_n(\frac{-iK(f)}{\|f\|/\sqrt{2}})\right](\|f\|/\sqrt{2})^n$$

$h_n$ is the nth Hermite polynomial.

- $K(f) = -\frac{g}{2}\int_{-\infty}^{\infty}(e^{-r\omega}\hat{h}, \hat{f})_{L^2(\mathbb{R}^d)} dr$

# Gaussian decay of the field operator

## Theorem

- $(\varphi_{\rm g}, e^{-\beta\phi(f)^2}\varphi_{\rm g}) = \frac{1}{\sqrt{1+\beta\|f\|^2}}\mathbb{E}_{\mu_\infty}\left[e^{-\frac{\beta K^2(f)}{1+\beta\|f\|^2}}\right]$ *for* $\beta > 0$.
- *Let* $-\infty < \beta < 1/\|f\|^2$.

  $$\|e^{(\beta/2)\phi(f)^2}\varphi_{\rm g}\|^2 = \frac{1}{\sqrt{1-\beta\|f\|^2}}\mathbb{E}_{\mu_\infty}\left[e^{\frac{\beta K^2(f)}{1-\beta\|f\|^2}}\right].$$
- $\lim_{\beta\uparrow 1/\|f\|^2}\|e^{(\beta/2)\phi(f)^2}\varphi_{\rm g}\| = \infty.$

# Expectations of number of bosons

$$(\Phi_T, e^{-\beta N}\Phi_T) = \mathbb{E}_{\mu_T}\left[e^{-g^2(1-e^{-\beta})\int_{-T}^0 dt \int_0^T W(X_t,X_s,t-s)ds}\right]$$

## Theorem

- - $(\varphi_g, e^{-\beta N}\varphi_g) = \mathbb{E}_{\mu_\infty}\left[e^{-g^2(1-e^{-\beta})W_\infty}\right]$,
  - $W_\infty = \int_{-\infty}^0 dt \int_0^\infty W(X_t, X_s, t-s)ds.$
- - $\varphi_g \in D(e^{\beta N})$ *for all* $\beta \in \mathbb{C}$
  - $(\varphi_g, e^{\beta N}\varphi_g) = \mathbb{E}_{\mu_\infty}\left[e^{-g^2(1-e^{\beta})W_\infty}\right]$

$\varphi_g = \oplus_{n=0}^\infty \varphi_g^{(n)} \Longrightarrow \sum e^{2\beta n}\|\varphi_g^{(n)}\|^2 < \infty$ for $\forall\beta > 0$.

# UV renormalization

- Nelson model $H = -\Delta + H_{\mathrm{f}} + \phi_\varepsilon$
- $\phi_\varepsilon = \frac{1}{\sqrt{2}}\left(a^\dagger(e^{-ikx}\frac{e^{-|k|^2\varepsilon/2}}{\sqrt{\omega}}) + a(e^{ikx}\frac{e^{-|k|^2\varepsilon/2}}{\sqrt{\omega}})\right)$
- $e^{-|k|^2\varepsilon/2} \to 1\!\!1$ **as** $\varepsilon \to 0$
- $(f\otimes\Omega, e^{-2TH_\varepsilon} g\otimes\Omega) =$
  $\int dx \mathbb{E}^x[f(B_{-T})g(B_T)e^{\int_{-T}^T dt \int_{-T}^T ds W(B_t - B_s, t-s)}]$
- $W(B_t - B_s, t-s) = \int \frac{e^{-\varepsilon|k|^2}}{2\omega(k)} e^{-ik(B_t - B_s)} e^{-|t-s|\omega(k)} dk$

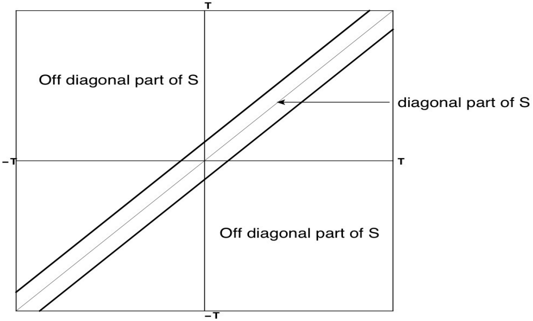


Figure: Renormalization

- Ito formula

$$W(B_t - B_s, t-s) - \varphi_\varepsilon(0,0) = \nabla\varphi_\varepsilon(B_t - B_s, t-s)\cdot dB_t - \varphi_\varepsilon(B_T - B_s, T-s)ds$$

- $\varphi_\varepsilon(X,T) = \int \frac{e^{-\varepsilon|k|^2} e^{-ik\cdot x - \omega(k)|t|}}{2\omega(k)} \frac{1}{|k|^2/2+\omega(k)} dk, \quad \varepsilon \geq 0$
- **Girsanov theorem, interpolation, Kato-class potential...etc** $\Longrightarrow (f\otimes\Omega, e^{-2T(H_\varepsilon-\varphi_\varepsilon)} g\otimes\Omega) \to \int dx \mathbb{E}^x[f(B_{-T})g(B_T)e^{\int_{-T}^T dt \int_{-T}^T ds W_{ren}(B_t - B_s, t-s)}]$
- $H_\varepsilon - \varphi_\varepsilon > -C$ **($\varepsilon$ independent)**
- $\exists H_\infty$ st $(F, e^{-2T(H_\varepsilon-\varphi_\varepsilon)}G) \to (F, e^{-2TH_\infty}G)$

## Summary and concluding remarks

- $\hat{h}/\omega \in L^2(\mathbb{R}^d) \Longrightarrow H_{\mathrm{SB}}$ has a ground state $\varphi_{\mathrm{g}}$
- **Open problem:** $\hat{h}/\omega \notin L^2(\mathbb{R}^d) \Longrightarrow H_{\mathrm{SB}}$ **has no ground state ?**
- $H_{\mathrm{SB}} \to$ Gibbs measure $\mu_\infty$
- $\mu_\infty$=local weak convergence of $\mu_T$.
- $(\varphi_{\mathrm{g}}, O\varphi_{\mathrm{g}}) = \mathbb{E}_{\mu_\infty}[f_O] \Longrightarrow \varphi_{\mathrm{g}} \in D(e^{\beta\phi(f)^2})$, $\varphi_{\mathrm{g}} \in D(e^{\beta N})$
- UV renormalization $\Longrightarrow$ New approaches: white noise analysis, interaction Fock space

# Thank you !