Functional integral approach to mathematically rigorous quantum field theory

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Quantum Mechanics

- Schrödinger operators ($d \geq 3$):

$$h = -\Delta + gV \quad (V \leq 0) \quad on\ L^2(\mathbb{R}^d)$$

- $\sigma(h) = \{E_j\}_{j=0}^{\infty} \cup [0, \infty)$, $E_0 \leq E_1 \leq \cdots \leq 0$, $E_0 =$ ground state energy
- $h\varphi_g = E_0 \varphi_g$, φ_g **is called the ground state**
- Ex(1) Lieb-Thirring bound:

$$\#\{\text{eigenvalues} \leq 0\} \leq a \int |gV(x)|^{d/2} dx$$

Then **0<**$|g| \ll 1 \Longrightarrow h$ has no ground state.

- Ex(2) Birmann-Schwinger principle: $g > \exists g_0 \Longrightarrow h$ has a ground state.
- Ex(3) Small perturbation W of $h \Longrightarrow$ the discrete spectrum of h moves on the real line.

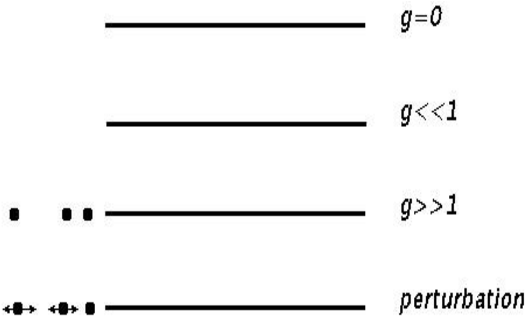

Figure: Spectrum of $-\Delta + gV$ and pertuerbation

Nelson model

- Boson Fock space $\mathscr{F} = \bigoplus_{n=0}^{\infty} \left(\otimes_{\mathrm{sym}}^n L^2(\mathbb{R}^d) \right)$
- $\Omega = \{1, 0, 0, ...\} \in \mathscr{F}$
- Annihilation and creation operators:
  $\Phi = \{\Phi^{(n)}\} \in \mathscr{F} \Longrightarrow (a^\dagger(f)\Phi)^{(n)} = \sqrt{n} S_n(f \otimes \Phi^{(n-1)})$,
  $a(f) = (a^\dagger(\bar{f}))^*$
- **CCR** $[a(f), a^\dagger(g)] = (\bar{f}, g)\mathbb{1}$, $[a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]$.
- 2_{nd} quantization of s.a. T, $d\Gamma(T) : \mathscr{F} \to \mathscr{F}$ defined by

$$d\Gamma(T) \prod_j a^\dagger(f_j)\Omega = \sum_{j=1}^{n} a^\dagger(f_1) \cdots a^\dagger(T f_j) \cdots a^\dagger(f_n)\Omega$$

$$d\Gamma(T)\Omega = 0$$

Nelson model = Schrödinger operator vs bosons

$$H = h \otimes 1\!\!1 + 1\!\!1 \otimes H_{\rm f} + g\phi(x) \quad on\ L^2(\mathbb{R}^d) \otimes \mathscr{F}$$

- $H_{\rm f} = d\Gamma(\omega)$
- $\phi(x) = \dfrac{1}{\sqrt{2}}(a^\dagger(e^{-ikx}h) + a(e^{ikx}h))$
- $\omega = \omega(k) = |k|$
- $g \in \mathbb{R}$ is a coupling constant
- $\sigma(H_{\rm f}) = [0, \infty)$
- $\sigma(h) = \{E_j\}_j \cup [0, \infty) \Longrightarrow \sigma(h \otimes 1\!\!1 + 1\!\!1 \otimes H_{\rm f}) = [E_0, \infty)$
- $E_j \in [E_0, \infty)$ are embedded!

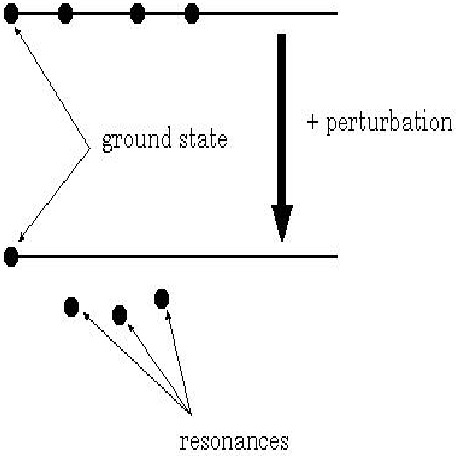

Figure: Spectrum of H and $h \otimes 1\!\!1 + 1\!\!1 \otimes H_{\mathrm{f}}$

Pair potential

Lemma

Wiener space: $(\mathscr{X}, \mathscr{G}, \mathscr{W})$, *BM:* B_t, $B_t(w) = w(t)$ *for* $w \in \mathscr{X}$.

$$(F \otimes \Omega, e^{-tH} G \otimes \Omega) = \int dx \mathbb{E}^x_{\mathscr{W}} \left[F(B_0) G(B_t) e^{S[0,T]} \right]$$

$$S[0,T] = \int_0^T dr \int_0^T ds W(B_r - B_s, r-s) - \int_0^T V(B_s) ds$$

$$W(X,T) = \frac{g^2}{4} \int |h(k)|^2 e^{-ikX} e^{-|T|\omega(k)} dk$$

See **Lőrinczi-Hiroshima-Betz, De Gruyter Study in Math 34**

Ground state of Nelson model

- $F \otimes \Omega = 1\!\!1_F$, $0 < F \in L^2(\mathbb{R}^d)$
- $\Phi_T = e^{-T(H-E)} 1\!\!1_F / \| e^{-T(H-E)} 1\!\!1_F \|, \quad T \geq 0,$
- $\gamma(T) = (1\!\!1_F, \Phi_T)^2 = \dfrac{\left(\int dx \mathbb{E}^x \left[F(B_0) F(B_T) e^{S[0,T]} \right] \right)^2}{\int dx \mathbb{E}^x \left[F(B_{-T}) F(B_T) e^{S[-T,T]} \right]}$

Lemma

A ground state of H exists if and only if $\lim_{T \to \infty} \gamma(T) > 0$.

Theorem

$\hat{h}/\omega \in L^2(\mathbb{R}^d) \Longleftrightarrow H$ has a ground state.

Rem.1 $\hat{h}/\omega \notin L^2(\mathbb{R}^d) \Longrightarrow H$ has no ground state.
Rem.2 $\hat{h}/\omega \in L^2(\mathbb{R}^d) \Longrightarrow (\varphi_g, N\varphi_g) \leq \|h/\omega\|^2$

Gibbs measures

Finite volume Gibbs meas. on $(\mathscr{X}, \sigma(\mathscr{F}))$

$$\mu_T(A) = \frac{1}{Z_T} \int dx \mathbb{E}^x_{\mathscr{W}} \left[1\!\!1_A F(B_{-T}) F(B_T) e^{S[-T,T]} \right]$$

- $\mathscr{F} = \cup_{T>0} \sigma(B_t, t \in [-T, T]) \subset \mathscr{G}$
- Observable $(\varphi_{\rm g}, O\varphi_{\rm g}) = \lim_T (\Phi_T, O\Phi_T) = \lim_T \mathbb{E}_{\mu_T}[f_{O,T}]$

$$\mu_T \to \exists \mu_\infty (T \to \infty)?$$

Theorem

Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then $\exists \mu_\infty$ on $(\mathscr{X}, \sigma(\mathscr{F}))$ st $\mu_T \to \mu_\infty$ in the sense of ***local weak****, i.e., $\mu_T(A) \to \mu_\infty(A)$ for $A \in \mathscr{F}$.*

- $\mathbb{E}_{\mu_T}[f] \to \mathbb{E}_{\mu_\infty}[f]$ for $\mathscr{F}_t$ measurable f

Remarks

- Finite dim. dis. is given by

$$E_{\mu_\infty}[\prod 1\!\!1_{A_j}(B_{t_j})] = (1\!\!1_{A_0}\varphi_{\rm g}, e^{-(t_1-t_0)(H-E)} 1\!\!1_{A_1} \cdots e^{-(t_n-t_{n-1})(H-E)} 1\!\!1_{A_n}\varphi_{\rm g})$$

- For $A \in \mathscr{F}_t$,

$$\mu_\infty(A) = e^{2Et} \int dx \mathbb{E}^x_{\mathscr{W}}[(\varphi_{\rm g}(B_{-t}), Q_{[-t,t]}\varphi_{\rm g}(B_t))_{\mathscr{H}} 1\!\!1_A]$$

- $Q_{[-t,t]} = J^*_{-t} e^{-\phi_E(\int_{-t}^t j_s h(\cdot - B_s)ds) + \int_{-t}^t V(B_s)ds} J_t : \mathscr{F} \to \mathscr{F}$

Gaussian decay of the field operator

$$K(f)=-\frac{g}{2}\int_{-\infty}^{\infty}(e^{-r\omega}e^{ikB_r}\hat{h},\hat{f})_{L^2(\mathbb{R}^d)}dr$$

Theorem

- $(\varphi_{\rm g}, e^{-\beta\phi(f)^2}\varphi_{\rm g})=\frac{1}{\sqrt{1+\beta\|f\|^2}}\mathbb{E}_{\mu_\infty}\left[e^{-\frac{\beta K^2(f)}{1+\beta\|f\|^2}}\right]$ *for* $\beta>0$.
- *Let* $-\infty<\beta<1/\|f\|^2$.
  $$\|e^{(\beta/2)\phi(f)^2}\varphi_{\rm g}\|^2=\frac{1}{\sqrt{1-\beta\|f\|^2}}\mathbb{E}_{\mu_\infty}\left[e^{\frac{\beta K^2(f)}{1-\beta\|f\|^2}}\right].$$
- $\lim_{\beta\uparrow 1/\|f\|^2}\|e^{(\beta/2)\phi(f)^2}\varphi_{\rm g}\|=\infty.$

Expectations of number of bosons

Let N be the number operator in $\mathscr{F}$.

$(\Phi_T, e^{-\beta N}\Phi_T) = \mathbb{E}_{\mu_T}\left[e^{-g^2(1-e^{-\beta})\int_{-T}^0 dt \int_0^T W(B_t - B_s, t-s)ds}\right]$

Theorem

- $(\varphi_g, e^{-\beta N}\varphi_g) = \mathbb{E}_{\mu_\infty}\left[e^{-g^2(1-e^{-\beta})W_\infty}\right]$,
  $$W_\infty = \int_{-\infty}^0 dt \int_0^\infty ds W(B_t - B_s, t-s).$$
- $(\varphi_g, e^{\beta N}\varphi_g) = \mathbb{E}_{\mu_\infty}\left[e^{-g^2(1-e^{\beta})W_\infty}\right]$ ***for all*** $\beta \in \mathbb{C}$

The number of boson in φ_g is very few!

$\varphi_g = \oplus_{n=0}^\infty \varphi_g^{(n)} \Longrightarrow \sum e^{2\beta n}\|\varphi_g^{(n)}\|^2 < \infty$ for $\forall \beta > 0$.

Fall-off of ground state

- Schrödinger operator h, ground state of Φ, $h\Phi = E\Phi$
  $X_t(x) = e^{tE} e^{-\int_0^t V(x+B_s)ds} \Phi(x+B_t)$ is martingale wrt
  $\mathscr{G}_t = \sigma(B_s, 0 \le s \le t)$
- Nelson model $H\varphi_{\rm g} = E\varphi_{\rm g}$,

$$X_t(x) = e^{tE} e^{-g\phi_E(\int_0^t j_s h(\cdot - x - B_s)ds)} e^{-\int_0^t V(x+B_s)ds} J_t \varphi_{\rm g}(x+B_t)$$

  is martingale wrt $\exists \mathscr{H}_t$.
- $\varphi_{\rm g}(x) = \mathbb{E}_{\mathscr{W}}[J_0^* X_t(x)]$
- Let τ be a stopping time. $X_{t\wedge\tau}$ is also martingale and
  $\|\varphi_{\rm g}(x)\|_{\mathscr{F}} \le \|\varphi_{\rm g}\| \mathbb{E}_{\mathscr{W}}[e^{-\int_0^{t\wedge\tau} V(B_s)ds}]$

Theorem

$\lim_{|x|\to\infty} V_-(x) - E < 0$ or $\lim_{|x|\to\infty} V(x) = \infty \Longrightarrow \|\varphi_{\rm g}(x)\|_{\mathscr{F}} \le ce^{-C|x|}$

UV renormalization

E. Nelson proved UV renormalization in 1964 by functional analysis. We prove this by functional integration.

- $H_\varepsilon = -\Delta + H_{\rm f} + \phi_\varepsilon$
- $h = e^{-|k|^2\varepsilon/2}/\sqrt{\omega(k)}$
- $e^{-|k|^2\varepsilon/2} \to 1\!\!1$ **as** $\varepsilon \to 0$
- $(1\!\!1_f, e^{-2TH_\varepsilon} 1\!\!1_g) = \int dx \mathbb{E}^x [f(B_{-T}) g(B_T) e^{S_\varepsilon[-T,T]}]$
- $S_\varepsilon[-T,T] = \int_{-T}^T ds \int_{-T}^T dt \int \frac{e^{-\varepsilon|k|^2}}{2\omega(k)} e^{-ik(B_t - B_s)} e^{-|t-s|\omega(k)} dk$

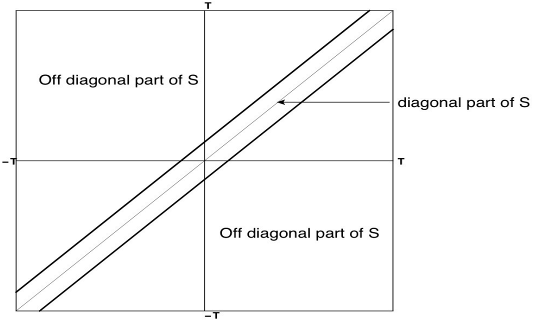

Figure: Renormalization

- $\varphi_\varepsilon(X,T) = \int \frac{e^{-\varepsilon|k|^2} e^{-ik\cdot x - \omega(k)|t|}}{2\omega(k)} \frac{1}{|k|^2/2+\omega(k)} dk, \quad \varepsilon \geq 0$
- Ito formula

$$\int_s^T W(B_t - B_s, t-s)dt - \varphi_\varepsilon(0,0)$$
$$= \int_s^T \nabla\varphi_\varepsilon(B_t - B_s, t-s)\cdot dB_t - \varphi_\varepsilon(B_T - B_s, T-s)$$

- **Girsanov theorem, interpolation, etc.**
  $(1\!\!1_f, e^{-2T(H_\varepsilon - \varphi_\varepsilon(0,0))} 1\!\!1_g) \to \int dx \mathbb{E}^x[f(B_{-T})g(B_T)e^{S_{ren}[-T,T]}]$
- $H_\varepsilon - \varphi_\varepsilon(0,0) > -C$ **(ε independent)**

Theorem

$\exists H_\infty$ *st* $(F, e^{-2T(H_\varepsilon - \varphi_\varepsilon(0,0))}G) \to (F, e^{-2TH_\infty}G)$

This is proven in Gubinelli-FH-Lőrinczi.

- Nelson model

$$(-\frac{1}{2}\Delta_x + V)\otimes 1\!\!1 + g\phi(\hat{\varphi}e^{-ikx}/\sqrt{\omega}) + 1\!\!1 \otimes H_{\rm f}$$

- Semi-relativistic Nelson model

$$(\sqrt{-\Delta_x + M^2} + V)\otimes 1\!\!1 + g\phi(\hat{\varphi}e^{-ikx}/\sqrt{\omega}) + 1\!\!1 \otimes H_{\rm f}$$

- Pauli-Fierz model

$$\frac{1}{2}[\sigma\cdot(p\otimes 1\!\!1 - A(x))]^2 + V\otimes 1\!\!1 + 1\!\!1\otimes H_{\rm f}$$

- Semi-relativistic Pauli-Fierz model

$$\sqrt{[\sigma\cdot(p\otimes 1\!\!1 - A(x))]^2 + M^2} + V\otimes 1\!\!1 + 1\!\!1\otimes H_{\rm f}\ (M>0)$$

$$|\sigma\cdot(p\otimes 1\!\!1 - A(x))| + V\otimes 1\!\!1 + 1\!\!1\otimes H_{\rm f}\ (M=0)$$

- Nelson (JMP 1964) UV ren.
- Bach-Fröhlich-Sigal (Adv Math 97,CMP98) GS. for $|g| \ll 1$
- Gérard (AHP 00), Spohn(LMP 00) GS. $\forall g$
- Fefferman (Adv Math 00) Stability of matter
- FH (JMP99, CMP 01,JFA05) uniqueness and s.a. $\forall g$
- Betz-FH-Lőrinczi-Minlos-Spohn (RMP01) Gibbs meas.
- Griesmer-Lieb-Loss (Inv. Math 01) GS ∀g.
- FH-Spohn(AHP 01) enhanced binding
- FH-Spohn (JMP04), Seiringer-Hainzl (ATMP05) effective mass
- Hirokawa-FH-Spohn(Adv Math 05) GS with UV ren.
- FH-Sasaki (Math Z 08) enhanced binding
- Gérard-FH-Panati-Suzuki (CMP12,JFA12) model on mfd.
- Gubinelli-FH-Lorinczi (preprint 13) UV ren.

Thank you !