格子上の Fractional Schrödinger 作用素 のスペクトル

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格子上の Schrödinger 作用素

$$L\psi(x) = \frac{1}{2d} \sum_{|x=y|=1} (\psi(y) - \psi(x))$$

$$F : l^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d) \Longrightarrow F\psi(\theta) = \sum_{n\in\mathbb{Z}^d} \psi(n) e^{-in\cdot\theta}$$

$$FLF^{-1} = \frac{1}{d} \sum_{j=1}^{d} (\cos\theta_j + 1)$$

Definition

Let $\Psi \in C^1((0,\infty))$ and $\Psi'(x) > 0$ for all $x \in (0,\infty)$. Then

$$\Psi(L) = F^{-1}\Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)F.$$

Potential

$$\sigma(L) = [0,2] \Longrightarrow \sigma(\Psi(L)) = [\Psi(0), \Psi(2)]$$

$V = v\delta_{x,0}$ i.e, $V\psi(x) = \begin{cases} 0 & x \neq 0 \\ v\psi(x) & x = 0 \end{cases}$.

$h = \Psi(L) + V.$

$$H_v = FhF^{-1} = \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right) + v(f, \cdot)_{L^2(\mathbb{T}^d)} f, \qquad (1.1)$$

where $f = (2\pi)^{-d/2}\mathbb{1} \in L^2(\mathbb{T}^d)$.

$L+V$ のスペクトル

[H.,Sasaki,Shirai,Suzuki, J. Math. Industry 2012]
[Bellissard and Schulz-Baldes] preprint 2012

- ○ $d=1,2$, no 0-mode, 2-mode, 0-resonance, and 2-resonance
- ○ $d=3,4$, 0-resonance and 2-resonance
- ○ $d\geq 5$ 0-mode and 2-mode.
- ○ Then e.v.-behavior on both edges, 0 and 2, are the same.

	2-mode	2-resonace	0-mode	0-resonance
$d=1$	no	no	no	no
$d=2$	no	no	no	no
$d=3$	no	yes	no	yes
$d=4$	no	yes	no	yes
$d=5$	yes	no	yes	no

H_v のスペクトル

$H_v\Phi = E\Phi$ I.e., $E\Phi - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)\Phi = v(f,\Phi)f.$

Lemma

E is an e.v. of H_v with some v iff

$$\int_{\mathbb{T}^d} \frac{1}{|E - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)|^2} d\theta < \infty$$

$$\int_{\mathbb{T}^d} \frac{1}{E - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)} d\theta \neq 0.$$

Furthermore if E is e.v. of H_v then v is given by

$$v = (2\pi)^d \left(\int_{\mathbb{T}^d} \frac{1}{E - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)} d\theta\right)^{-1}. \qquad (1.2)$$

$\Psi(*)$-resonance and $\Psi(*)$-mode, $* = 0, 2$

$$I(E) = \int_{\mathbb{T}^d} \frac{1}{\left|E - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)\right|^2} d\theta, \tag{1.3}$$

$$J(E) = \int_{\mathbb{T}^d} \frac{1}{E - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)} d\theta \tag{1.4}$$

スペクトル

Lemma

Let $E \in \mathbb{R} \setminus [\Psi(0), \Psi(2)]$*. Then* E *is e.v. of* H_v *with some* v*.*

証明: $I(E) < \infty$ and $J(E) \neq 0$. Then E is e.v. □

Lemma

$\sigma(H_v) \cap (\Psi(0), \Psi(2)) = \emptyset$ *for any* v*.*

証明: There exists an unique $x \in (0,2)$ s.t. $\Psi(E) = \Psi(x)$. Then $|E - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)| \leq C|\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j - x)|$ Thus

$$I(E) \geq \int_{\mathbb{T}^d} \frac{1}{|C\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j - x)|^2} d\theta = \infty.$$

□

Resonances

Definition

We call Ψ is (a,b)-type if

$$\exists \lim_{x \to 0+} \frac{\Psi(x) - \Psi(0)}{x^a} \neq 0, \tag{2.1}$$

$$\exists \lim_{x \to 0} \frac{\Psi(2) - \Psi(2-x)}{x^b} \neq 0. \tag{2.2}$$

Lemma

Let Ψ be (a,b)-type. Then
(1)$J(E)\neq 0$ for $E=\Psi(0)$ and $\Psi(2)$.
(2) For $E=\Psi(2)$ (resp. $E=\Psi(0)$) $I(E)<\infty$ if and only if $d\geq 1+4a$ (resp $d\geq 1+4b$)
(3) $J(E)<\infty$ if and only if $d\geq 1+2a$ (resp. $d\geq 1+2b$).

証明: At $\theta\approx(0,\cdots,0)$,
$\Psi(2)-\Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j+1)\right)\approx\left(\frac{1}{2d}\sum_j\theta_j^2\right)^a$. Hence
$I(\Psi(2))=\int_{\mathbb{T}^d}\frac{1}{|\Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j+1)\right)-\Psi(2)|^2}\theta\approx\int_0^1\frac{r^{d-1}}{r^{4a}}dr$ □

Theorem

Suppose that Ψ *is* (a,b)*-type. Let*

$$v_2 = (2\pi)^d \left(\int_{\mathbb{T}^d} \frac{1}{\Psi(2) - \Psi\left(\frac{1}{d}\sum_{j=1}^d (\cos\theta_j + 1)\right)} d\theta \right)^{-1} > 0. \qquad (2.3)$$

Then $\mathbf{[}v > 0\mathbf{]}$

$(d < 1+2b)$ *For all* $v > 0$ *there exists e.v.* $E > \Psi(2)$*.*

$(1+2b \le d < 1+4b)$ *For* $v > v_2$ *there exists e.v.* $E > \Psi(2)$ *and* $v \le v_2$ *there is no eigenvalue.*

$(1+4b \le d)$ *For* $v > v_2$ *there exists e.v.* $E > \Psi(2)$*, and for* $v = v_2$ $E = \Psi(2)$ *is e.v. and* $v < v_2$ *there is no eigenvalue.*

$v > 0$	$d < 1 + 2b$	$1 + 2b \leq d < 1 + 4b$	$1 + 4b \leq d$
$\Psi(2)$-mode	no	no	yes
$\Psi(2)$-resonance	no	yes	no

$v < 0$	$d < 1 + 2a$	$1 + 2a \leq d < 1 + 4a$	$1 + 4a \leq d$
$\Psi(0)$-mode	no	no	yes
$\Psi(0)$-resonance	no	yes	no

Figure: $\Psi(0)$ and $\Psi(2)$-mode and resonance

Examples

Example

○ $\Psi(u) = u^{\alpha/2}$ for $0 < \alpha < 2$. Then $(\alpha/2, 1)$-type.
○ $\Psi(u) = (u + m^{2/\alpha})^{\alpha/2} - m$ for $0 < \alpha < 2$. Then $(1, 1)$-type.
○ Let $\Psi(u) = \log(1 + u^{\alpha/2})$ for $0 < \alpha < 2$. Then $(\alpha/2, 1)$-type.
○ (Bernstein function) $\Psi(u) = bu + \int_0^\infty (1 - e^{-uy})\nu(dy)$ Then $(\alpha/2, 1)$-type with some $0 \leq \alpha \leq 2$.

Definition

○ If Ψ is $(\alpha/2, 1)$-type with $0 < \alpha < 2$ we call it fractional type.
○ If Ψ is $(1, 1)$-type we call it normal type.

Relativistic case

$\sqrt{L}+V$

	$\sqrt{2}$-mode	$\sqrt{2}$-resonace	0-mode	0-resonance
$d=1$	no	no	no	no
$d=2$	no	no	no	yes
$d=3$	no	yes	yes	no
$d=4$	no	yes	yes	no
$d\geq 5$	yes	no	yes	no

○ Note that $\sigma(\sqrt{L})=[0,\sqrt{2}]$.
○ For dimension $d=2,3,4$, eigenvalue-behaviors on 0 and $\sqrt{2}$ are different.

$\oplus$ denotes a resonance, $\bullet$ e.v. , and $\times$ denotes no eigenvalue.

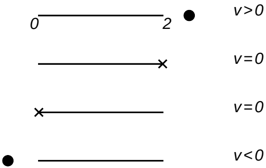

Table: $\alpha = 1$ and $d = 1$

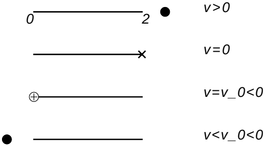

Table: $\alpha = 1$ and $d = 2$

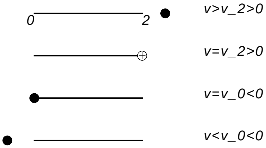

Table: $\alpha = 1$ and $d = 3,4$

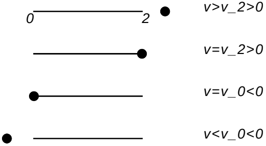

Table: $\alpha = 1$ and $d \geq 5$

Massive VS massless

Let $\psi(u) = \sqrt{u+m^2} - m$ for $m \geq 0$. Then $\sqrt{L+m^2} - m + V$.
$\Psi(u)$ is $(1,1)$-type for $m > 0$, and $(1/2,1)$-type for $m = 0$.
Then the eigenvalue-behavior of $\sqrt{L} + V$ and $\sqrt{L+m^2} - m + V$ are different.