Spin-boson model through a Gibbs measure on cádlág paths

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Gibbs measures

- $(\Omega, \mathscr{Y}, Q)$ probability space
- $(Y_t)_{t\in\mathbb{R}}$ Markov process
- $\mathscr{Y}_T = \sigma(Y_r, r \in [-T, T])$ and $\mathscr{T}_T = \sigma(Y_r, r \in [-T, T]^c)$.
- External potential $\mathscr{V} : \mathbb{R}^d \to \mathbb{R}$: admissible

$$0 < \mathbb{E}_Q[e^{-\int_I \mathscr{V}(Y_s)ds}] < \infty$$

for every bounded interval $I \subset \mathbb{R}$

Pair potential $\mathscr{W} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$: admissible

$$\int_{\mathbb{R}} \sup_{x,y\in\mathbb{R}^d} |\mathscr{W}(x, y, s)| ds < \infty.$$

Gibbs Measures

Let $\mathscr{V}$ and $\mathscr{W}$ be admissible potentials.

(1) $dP_T = \dfrac{1}{Z_T} e^{-\mathscr{E}_{T,T}} dQ$ is a finite volume Gibbs measure for $[-T,T]$.

(2) (a) $P_T(A) \to \exists P_\infty(A)$ as $T \to \infty$ for all $A \in \mathscr{Y}_T$(local weak conv.)
(b) $P_\infty \lceil_{\mathscr{Y}_T} \ll Q \lceil_{\mathscr{Y}_T}$ for every T
$\Longrightarrow P_\infty$ is a Gibbs measure.

$$\mathscr{E}_{S,T} = \int_{[-T,T]} \mathscr{V}(Y_t)dt + \int_{[-S,S]\times[-T,T]} \mathscr{W}(Y_t, Y_s, |t-s|)dsdt$$

Spin-boson model

- Boson Fock space $\mathscr{F} = \bigoplus_{n=0}^{\infty} \left(\otimes_{\mathrm{sym}}^n L^2(\mathbb{R}^d) \right)$
- Annihilation and creation operators, $a(f)$ and $a^\dagger(f)$ for $f \in L^2(\mathbb{R}^d)$
- CCR $[a(f), a^\dagger(g)] = (\bar{f}, g)\mathbb{1}$, $[a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]$.
- $a^\sharp(f) = \int a^\sharp(k) f(k) dk$
- 2_{nd} quantization of s.a. T in $L^2(\mathbb{R}^d)$, $d\Gamma(T) : \mathscr{F} \to \mathscr{F}$.

SB model

$$H_{\rm SB} = \varepsilon\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + \alpha\sigma_x \otimes \phi_{\rm b}(\hat{h})$$

- $\mathscr{H} = \mathbb{C}^2 \otimes \mathscr{F}$
- $\alpha \in \mathbb{R}$ is a coupling constant
- $\varepsilon > 0$ (gap of the spectrum of two level atom)
- Pauli matrices $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $H_{\rm f} = d\Gamma(\omega)$, free boson Hamiltonian with $\omega(k) = |k|$.
- $\phi_{\rm b}(\hat{h}) = \frac{1}{\sqrt{2}} \int \left(a^\dagger(k)\hat{h}(-k) + a(k)\hat{h}(k) \right) dk$

Vacuum Expectation

- $(\mathbb{1}_{\mathscr{H}}, e^{-tH}\mathbb{1}_{\mathscr{H}}) = e^t \sum_{\sigma\in\mathbb{Z}_2} \mathbb{E}_P\left[e^{\frac{\alpha^2}{2}\int_0^t dr \int_0^t W(N_r - N_s, r-s)ds}\right],$

  $\mathbb{Z}_2 = \{\pm 1\}$
- $W(N_r - N_s, r-s) = (-1)^{N_r - N_s}\int_{\mathbb{R}^d} e^{-|r-s|\omega(k)}\frac{\hat{h}(k)^2}{2\omega(k)^2}dk.$
- $(N_t)_{t\in\mathbb{R}}$ a Poisson process with intensity 1 on (Ω, Σ, P)

- $\Phi_T = e^{-T(H-E)}\mathbb{1}, \quad T \geq 0,$
- $\gamma(T) = \frac{(\mathbb{1}_{\mathscr{H}}, \Phi_T)^2}{\|\Phi_T\|^2} = \frac{(\mathbb{1}_{\mathscr{H}}, e^{TH}\mathbb{1}_{\mathscr{H}})^2}{(\mathbb{1}_{\mathscr{H}}, e^{-2TH}\mathbb{1}_{\mathscr{H}})} = \frac{\left(\mathbb{E}_P\left[e^{\frac{\alpha^2}{2}\int_0^T dt \int_0^T W ds}\right]\right)^2}{\mathbb{E}_P\left[e^{\frac{\alpha^2}{2}\int_{-T}^T dt \int_{-T}^T W ds}\right]}$

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Criteria for ground state

A ground state of H exists if and only if $\lim_{T\to\infty}\gamma(T) > 0$.

- $\Phi_T = e^{-T(H-E)}\mathbb{1}, \quad T \geq 0,$
- $\gamma(T) = \frac{(\mathbb{1}_{\mathscr{H}}, \Phi_T)^2}{\|\Phi_T\|^2} = \frac{(\mathbb{1}_{\mathscr{H}}, e^{TH}\mathbb{1}_{\mathscr{H}})^2}{(\mathbb{1}_{\mathscr{H}}, e^{-2TH}\mathbb{1}_{\mathscr{H}})} = \frac{\left(\mathbb{E}_P\left[e^{\frac{\alpha^2}{2}\int_0^T dt \int_0^T W ds}\right]\right)^2}{\mathbb{E}_P\left[e^{\frac{\alpha^2}{2}\int_{-T}^T dt \int_{-T}^T W ds}\right]}$

Existence and uniqueness of ground state

Existence and uniqueness

If $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, then H has a ground state and it is unique.

Proof:

- Independence of N_t and N_{-s}, and reflection symmetry $\Longrightarrow$

$$\|\Phi_T\|^2 \leq (\mathbb{1}_{\mathscr{H}}, \Phi_T)^2 e^{\alpha^2 \|\hat{h}/\omega\|^2}.$$

- $\gamma(T) \geq e^{-\alpha^2 \|\hat{h}/\omega\|^2} \Longrightarrow$ a ground state φ_{g} of H exists.
- $(\Psi, e^{-tH}\Phi) > 0$ for $\Psi, \Phi \geq 0$.
- e^{-tH} is positivity improving $\Longrightarrow$ Ground state is unique.

Cadlag path

Path integral

$$(\mathbb{1}_{\mathscr{H}}, e^{-tH}\mathbb{1}_{\mathscr{H}}) = e^t \sum_{\sigma\in\mathbb{Z}_2} \mathbb{E}_{\mathscr{W}}^{\sigma}\left[e^{\frac{\alpha^2}{2}\int_0^t dr\int_0^t W(X_r - X_s, r-s)ds}\right]$$

- $\mathscr{X} = D(\mathbb{R};\mathbb{Z}_2)$: the space of càdlàg paths with values in $\mathbb{Z}_2$
- $\mathscr{G}$: the σ-field generated by cylinder sets
- $(-1)^{N_{\cdot}} : (\Omega,\Sigma,P) \to (\mathscr{X},\mathscr{G})$: $\mathscr{X}$-valued random variable
- Image measure: $\mathscr{W}^{\sigma}(A) = P(\sigma_{\cdot}^{-1}(A))$ for $A \in \mathscr{G}$
- Coordinate process: $X_t(\omega) = \omega(t)$ for $\omega \in \mathscr{X}$.

Gibbs measures

Finite volume Gibbs meas.on $(\mathscr{X}, \sigma(\mathscr{F}))$

$$\mu_T(A) = \frac{e^{2T}}{Z_T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}^{\sigma}_{\mathscr{W}} \left[1\!\!1_A e^{\frac{\alpha^2}{2} \int_{-T}^{T} dt \int_{-T}^{T} ds W(X_t, X_s, t-s)} \right]$$

- $\mathscr{G}_{[-T,T]} = \sigma(X_t, t \in [-T,T]) \subset \mathscr{G}$
- $\mathscr{F} = \cup_T \mathscr{G}_{[-T,T]}$

- $O = \xi(\sigma) f(\phi(f)),\ e^{-d\Gamma(\rho)}$ etc.

$$\mu_T \to \exists \mu_\infty (T \to \infty)?$$

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- $\mathscr{G}_{[-T,T]} = \sigma(X_t, t \in [-T,T]) \subset \mathscr{G}$
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Motivation

$$(\varphi_g, O\varphi_g) = \lim_n \left(\frac{\Phi_T}{\|\Phi_T\|}, O \frac{\Phi_T}{\|\Phi_T\|} \right) = \lim_n \mathbb{E}_{\mu_T}[f_{O,n}]$$

- $O = \xi(\sigma) f(\phi(f)),\ e^{-d\Gamma(\rho)}$ etc.

$$\mu_T \to \exists \mu_\infty (T \to \infty)?$$

- $\mathscr{F} = \bigcup_{t>0} \mathscr{G}_{[-t,t]}$. For $\Lambda = \{t_0, \ldots, t_n\} \subset [-T, T]$

$$\mu_T^{\Lambda}(\prod_{j=0}^{n} A_j) = \frac{e^{2T}}{Z_T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathscr{W}}^{\sigma} \left[\left(\prod_{j=0}^{n} 1\!\!1_{A_j}(X_{t_j}) \right) e^{\frac{\alpha^2}{2} \int_{-T}^{T} dt \int_{-T}^{T} ds W} \right]$$

- $\mathscr{F}_T = \bigcup_{t \leq T} \mathscr{G}_{[-t,t]}$. For $-T \leq -t \leq t_0 < \cdots < t_n \leq t \leq T$

- $Q_t = J_{-t}^* e^{-\alpha \Phi_{\mathrm{E}}(\int_{-t}^{t} X_s j_s h ds)} J_t$

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- $\mathscr{F}_T = \bigcup_{t \le T} \mathscr{G}_{[-t,t]}$. For $-T \le -t \le t_0 < \cdots < t_n \le t \le T$

$$\rho_T^\Lambda(\prod_{j=0}^n A_j) = e^{2Et} e^{2t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathscr{W}}^\sigma \left[\prod_{j=0}^n 1\!\!1_{A_j}(X_{t_j}) \left(\frac{\Phi_{T-t}(X_{-t})}{\|\Phi_T\|}, Q_t \frac{\Phi_{T-t}(X_t)}{\|\Phi_T\|} \right) \right].$$

- $Q_t = J_{-t}^* e^{-\alpha \Phi_{\mathrm{E}}(\int_{-t}^t X_s j_s h ds)} J_t$

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- $Q_t = J_{-t}^* e^{-\alpha \Phi_{\mathrm{E}}(\int_{-t}^t X_s j_s h ds)} J_t$

Identity

$$\mu_T^\Lambda(\prod_j A_j) = \frac{(1\!\!1_{\mathscr{H}}, e^{-(t_0+T)H} 1\!\!1_{A_0} e^{-(t_1-t_0)H} \cdots 1\!\!1_{A_n} e^{-(T-t_n)H} 1\!\!1_{\mathscr{H}})}{\|\Phi_T\|^2} = \rho_T^\Lambda(\prod_j A_j)$$

- Set function ρ_T on $(\mathscr{X}, \mathscr{F}_T)$:
  $$\mathscr{G}_{[-t,t]} \ni A \mapsto \rho_T(A) = e^{2Et} e^{2t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}^{\sigma}_{\mathscr{W}} \left[1\!\!1_A \left(\frac{\Phi_{T-t}(X_{-t})}{\|\Phi_T\|}, Q_t \frac{\Phi_{T-t}(X_t)}{\|\Phi_T\|} \right) \right].$$
- ρ_T is well defined.

Existence of Measure

$\exists \bar{\rho}_T$ on $(\mathscr{X}, \sigma(\mathscr{F}_T))$ such that $\rho_T \lceil_{\mathscr{F}_T} = \bar{\rho}_T$.

- Consistency $\Longrightarrow$ $\begin{array}{l} \{\mu_T^\Lambda\}_\Lambda \Longrightarrow \mu_T \ on \ (\mathscr{X}, \sigma(\mathscr{F})) \\ \{\rho_T^\Lambda\}_\Lambda \Longrightarrow \bar{\rho}_T \ on \ (\mathscr{X}, \sigma(\mathscr{F}_T)) \end{array}$
- $\mu_T(A) = \bar{\rho}_T(A) = \rho_T(A)$ for $A \in \mathscr{G}_{[-t,t]}$ with $t \leq T$.
- $\Phi_{T-t}/\|\Phi_T\| \to \varphi_{\rm g}$
- $\lim_{T\to\infty} \rho_T(A) = e^{2t} e^{2Et} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}^{\sigma}_{\mathscr{W}} \left[\left(\varphi_{\rm g}(X_{-t}), Q_t \varphi_{\rm g}(X_t) \right) 1\!\!1_A \right]$

- Set function ρ_∞ on $(\mathscr{X},\mathscr{F})$: $\mathscr{G}_{[-t,t]} \ni A \mapsto \rho_\infty(A) = e^{2Et}e^{2t}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}_{\mathscr{W}}^{\sigma}\left[1\!\!1_A(\varphi_{\rm g}(X_{-t}),Q_t\varphi_{\rm g}(X_t))_{\mathscr{H}}\right]$.
- ρ_∞ is well defined.

Existence of Measure

$\exists\bar\rho_\infty$ on $(\mathscr{X},\sigma(\mathscr{F}))$ such that $\rho_\infty\lceil_{\mathscr{F}}=\bar\rho_\infty$.

- Set function ρ_∞ on $(\mathscr{X}, \mathscr{F})$: $\mathscr{G}_{[-t,t]} \ni A \mapsto \rho_\infty(A) = e^{2Et} e^{2t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}^{\sigma}_{\mathscr{W}} \left[1\!\!1_A(\varphi_{\mathrm{g}}(X_{-t}), Q_t \varphi_{\mathrm{g}}(X_t))_{\mathscr{H}} \right]$.
- ρ_∞ is well defined.

Existence of Measure

$\exists \bar{\rho}_\infty$ on $(\mathscr{X}, \sigma(\mathscr{F}))$ such that $\rho_\infty \lceil_{\mathscr{F}} = \bar{\rho}_\infty$.

Main Result: Local Weak Convergence

$$\hat{h}/\omega \in L^2(\mathbb{R}^d) \Longrightarrow \lim_{T \to \infty} \mu_T(A) = \bar{\rho}_\infty(A) \text{ for } A \in \mathscr{G}_{[-T,T]}.$$

Gibbs measure μ_∞

Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then $\bar{\rho}_\infty = \mu_\infty$ is a Gibbs measure.

Gibbs measure μ_∞

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Expectation

Let f be a $\mathscr{G}_{[-t,t]}$-measurable function on $\mathscr{X}$. Then

- $\mathbb{E}_{\mu_\infty}\left[\prod_{j=0}^n f_j(X_{t_j})\right] = (\varphi_{\rm g}, f_0 e^{-(t_1-t_0)(H-E)} f_1 \cdots e^{-(t_n-t_{n-1})(H-E)} f_n \varphi_{\rm g})$.
- $\mathbb{E}_{\mu_\infty}[f] = e^{2Et} \sum_{\sigma\in\mathbb{Z}_2} \mathbb{E}_{\mathscr{W}}^{\sigma}\left[\left(\varphi_{\rm g}(X_{-t}), Q_t \varphi_{\rm g}(X_t)\right) f\right]$

Expectation of $\xi(\sigma)F(\phi(f))$

Key Expectation

- $(\varphi_{\rm g}, \xi(\sigma)e^{i\beta\phi(f)}\varphi_{\rm g}) = e^{-\frac{\beta^2}{4}\|f\|^2}\mathbb{E}_{\mu_\infty}\left[\xi(X_0)e^{i\beta K(f)}\right]$
- $K(f) = \dfrac{\alpha}{2}\displaystyle\int_{-\infty}^{\infty}(e^{-|r|\omega}\hat{h},\hat{f})X_r dr$.

Proof: Note that $(\varphi_{\rm g}, \xi(\sigma)e^{i\beta\phi(f)}\varphi_{\rm g}) = \lim_{T\to\infty}(\frac{\Phi_T}{\|\Phi_T\|}, \xi(\sigma)e^{i\beta\phi(f)}\frac{\Phi_T}{\|\Phi_T\|})$.

$$(\varphi_{\rm g}, \xi(\sigma)e^{i\beta\phi(f)}\varphi_{\rm g}) = \lim_{T\to\infty} e^{-\frac{\beta^2}{4}\|f\|^2}\mathbb{E}_{\mu_T}\left[\xi(X_0)e^{i\beta\int_{-T}^{T}(e^{-|s|\omega}\hat{h},\hat{f})X_s ds}\right].$$

Polynomial

- $(\varphi_g, \xi(\sigma)\phi(f)^n\varphi_g) =$
  $i^n \mathbb{E}_{\mu_\infty}\left[\xi(X_0) h_n\left(\frac{-iK(f)}{\|f\| 2^{-1/2}}\right)\right] (\|f\| 2^{-1/2})^n$.
- $h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$: Hermite polynomial of order n.

Proof: Since $F(\phi(f)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{F}(\beta) e^{i\beta\phi(f)} d\beta$, we have

$$(\varphi_g, \xi(\sigma) F(\phi(f))\varphi_g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{F}(\beta) e^{-\frac{\beta^2}{4}\|f\|^2} \mathbb{E}_{\mu_\infty}\left[\xi(X_0) e^{i\beta K(f)}\right] d\beta.$$

Polynomial

- $(\varphi_g, \xi(\sigma)\phi(f)^n\varphi_g) =$
  $i^n \mathbb{E}_{\mu_\infty}\left[\xi(X_0)h_n\left(\frac{-iK(f)}{\|f\|2^{-1/2}}\right)\right](\|f\|2^{-1/2})^n$.
- $h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$: Hermite polynomial of order n.

Schwartz test function

- $(\varphi_g, \xi(\sigma)F(\phi(f))\varphi_g) = \mathbb{E}_{\mu_\infty}[\xi(X_0)G(K(f))]$.
- $F \in \mathscr{S}$, $G = \check{F} * \check{g}$ and $g(\beta) = e^{-\beta^2\|f\|^2/4}$.

Proof: Since $F(\phi(f)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\check{F}(\beta)e^{i\beta\phi(f)}d\beta$, we have

$$(\varphi_g, \xi(\sigma)F(\phi(f))\varphi_g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\check{F}(\beta)e^{-\frac{\beta^2}{4}\|f\|^2}\mathbb{E}_{\mu_\infty}\left[\xi(X_0)e^{i\beta K(f)}\right]d\beta.$$

Exponential moments of the field operator

Exponential moment

- $\varphi_g \in D(e^{\beta\phi(f)})$
- $(\varphi_g, e^{\beta\phi(f)}\varphi_g) = e^{\frac{\beta^2}{4}\|f\|^2}\mathbb{E}_{\mu_\infty}\left[e^{\beta K(f)}\right]$
- $(\varphi_g, \sigma e^{\beta\phi(f)}\varphi_g) = e^{\frac{\beta^2}{4}\|f\|^2}\mathbb{E}_{\mu_\infty}\left[X_0 e^{\beta K(f)}\right].$

Proof: Analytic continuation: $\beta \to i\beta$. □

Gaussian decay of the field operator

Gaussian moment

$$(\varphi_{\rm g}, e^{-\beta\phi(f)^2}\varphi_{\rm g}) = \frac{1}{\sqrt{1+\beta\|f\|^2}}\mathbb{E}_{\mu_\infty}\left[e^{-\frac{\beta K^2(f)}{1+\beta\|f\|^2}}\right]$$ for $\beta > 0$.

Proof:

$$(\varphi_{\rm g}, e^{-(\beta^2/2)\phi(f)^2}\varphi_{\rm g}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-k^2/2}(\varphi_{\rm g}, e^{i\beta\phi(f)k}\varphi_{\rm g})dk$$

$$= \frac{1}{\sqrt{1+\beta^2\|f\|^2/2}}\mathbb{E}_{\mu_\infty}\left[e^{-\frac{\beta^2 K^2(f)/2}{1+\beta^2\|f\|^2/2}}\right].$$

Replacing $\beta^2/2$ by β completes the proof of the lemma. □

Gaussian decay

Let $-\infty < \beta < 1/\|f\|^2$.

- $\varphi_{\rm g} \in D(e^{(\beta/2)\phi(f)^2})$
- $\|e^{(\beta/2)\phi(f)^2}\varphi_{\rm g}\|^2 = \dfrac{1}{\sqrt{1-\beta\|f\|^2}}\mathbb{E}_{\mu_\infty}\left[e^{\frac{\beta K^2(f)}{1-\beta\|f\|^2}}\right].$
- $\displaystyle\lim_{\beta\uparrow 1/\|f\|^2}\|e^{(\beta/2)\phi(f)^2}\varphi_{\rm g}\| = \infty.$

Expectations of number of bosons

$$(\frac{\Phi_T}{\|\Phi_T\|}, \xi(\sigma)e^{-\beta N}\frac{\Phi_T}{\|\Phi_T\|}) = \mathbb{E}_{\mu_T}\left[\xi(X_0)e^{-\alpha^2(1-e^{-\beta})\int_{-T}^0 dt \int_0^T W(X_t,X_s,t-s)ds}\right]$$

Super-exponential decay $\rho = \mathbb{1}$

-
  - $(\varphi_g, \xi(\sigma)e^{-\beta N}\varphi_g) = \mathbb{E}_{\mu_\infty}\left[\xi(X_0)e^{-\alpha^2(1-e^{-\beta})W_\infty}\right]$,
  - $W_\infty = \int_{-\infty}^0 dt \int_0^\infty W(X_t, X_s, t-s)ds.$
-
  - $\varphi_g \in D(e^{\beta N})$ for all $\beta \in \mathbb{C}$
  - $(\varphi_g, e^{\beta N}\varphi_g) = \mathbb{E}_{\mu_\infty}\left[e^{-\alpha^2(1-e^{\beta})W_\infty}\right]$

Van Hove representation

- The *van Hove Hamiltonian* $H_{\rm vH}(\hat{g}) = H_{\rm f} + \phi_{\rm b}(\hat{g})$.
- Ground state $\varphi_{\rm vH}(\hat{g})$
- $H \cong \begin{bmatrix} H_{\rm f} + \alpha\phi_{\rm b}(\hat{h}) & -\varepsilon \\ -\varepsilon & H_{\rm f} - \alpha\phi_{\rm b}(\hat{h}) \end{bmatrix}$
- H with $\varepsilon = 0$ is the direct sum of van Hove Hamiltonians: $H \cong \begin{bmatrix} H_{\rm f} + \alpha\phi_{\rm b}(\hat{h}) & 0 \\ 0 & H_{\rm f} - \alpha\phi_{\rm b}(\hat{h}) \end{bmatrix}$
- $(\varphi_{\rm g}, e^{i\beta\phi(f)}\varphi_{\rm g})_{\mathscr{H}} = \dfrac{1}{2}\sum_{\sigma=\pm}(\varphi_{\rm vH}(\sigma\alpha\hat{h}), e^{i\beta\phi_{\rm b}(\hat{f})}\varphi_{\rm vH}(\sigma\alpha\hat{h}))_{\mathscr{F}}$

Van Hove representation

$$(\varphi_{\rm g}, e^{i\beta\phi(f)}\varphi_{\rm g}) = \mathbb{E}_{\mu_\infty}\Big[(\varphi_{\rm vH}(\chi), e^{i\beta\phi_{\rm b}(\hat{f})}\varphi_{\rm vH}(\chi))\Big].$$

- $\chi = \frac{\alpha}{2}\omega(k)\hat{h}(k)\int_{-\infty}^{\infty} e^{-|s|\omega(k)}X_s ds$.
- $H_{\rm vH}(\chi)$ *random van Hove Hamiltonian*.

Concluding remarks

Summary

- $H_{SB} \to$ Gibbs measure μ_∞
- μ_∞=local weak convergence of μ_T.
- $(\varphi_{\rm g}, O\varphi_{\rm g}) = \mathbb{E}_{\mu_\infty}[f_O] \Longrightarrow \varphi_{\rm g} \in D(e^{\beta\phi(f)^2})$, $\varphi_{\rm g} \in D(e^{\beta N})$
- Van Hove representation
- New application $\to \sqrt{-\Delta + m^2} + V + H_{\rm f} + \alpha\phi_{\rm b}(\hat{h})$

Thank you !