Feynman-Kac type formula with Cadlag path and generalized Schroedinger operator with Spin

Fumio Hiroshima

Faculty of Mathematics, Kyushu University, Japan

Nov. 6, 2010, Kyoto

Outline

Relativistic Schrödinger Operator with spin 1/2

$$h = \sqrt{(\sigma \cdot (p-a))^2 + m^2} - m + V$$

- Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Feynman-Kac formula of $(f, e^{-th}g)$?
- $\psi(u) = \sqrt{2u + m^2} - m \Longrightarrow h = \psi\left(\frac{1}{2}(\sigma \cdot (p-a))^2\right) + V$

Bernstein functions and subordinators

- Bernstein functions $\mathscr{B}$

$$\psi \in \mathscr{B} \overset{def}{\Longleftrightarrow} \begin{cases} \psi(0) = 0 \\ \psi \in C^\infty(R_+) \\ (-1)^n \psi^{(n)}(u) \leq 0 \quad n = 1, 2, 3, \ldots \end{cases}$$

Examples $\psi(u)$

- Fractional Schrödinger $u^{\alpha/2} (0 < \alpha \leq 2)$
- Relativistic Schrödinger $\sqrt{2u + m^2} - m$

$\mathscr{S}$ = the set of subordinators (1-dim Lévy process with increasing paths)

$(T_t)_{t \geq 0} \in \mathscr{S} \Longleftrightarrow \psi \in \mathscr{B}$

$$e^{-t\psi(u)} = \mathbb{E}[e^{-uT_t}] \Longrightarrow e^{-t\psi(H)} = \mathbb{E}[e^{-T_t H}]$$

Bernstein functions and subordinators

- Bernstein functions $\mathscr{B}$

$$\psi \in \mathscr{B} \overset{def}{\Longleftrightarrow} \begin{cases} \psi(0) = 0 \\ \psi \in C^\infty(R_+) \\ (-1)^n \psi^{(n)}(u) \leq 0 \quad n = 1, 2, 3, \ldots \end{cases}$$

Examples $\psi(u)$

- Fractional Schrödinger $u^{\alpha/2} (0 < \alpha \leq 2)$
- Relativistic Schrödinger $\sqrt{2u + m^2} - m$

$\mathscr{S}$ = the set of subordinators (1-dim Lévy process with increasing paths)

$(T_t)_{t \geq 0} \in \mathscr{S} \Longleftrightarrow \psi \in \mathscr{B}$

$$e^{-t\psi(u)} = \mathbb{E}[e^{-uT_t}] \Longrightarrow e^{-t\psi(H)} = \mathbb{E}[e^{-T_t H}]$$

Feynman-Kac Formula for $\psi\left(\frac{1}{2}(p-a)^2\right)+V$

Generalized Schrödinger operator

$$\psi\left(\frac{1}{2}(p-a)^2\right)+V$$

- Feynman-Kac-Itô formula:
  $$(f, e^{-t(\frac{1}{2}(p-a)^2+V)}g) = \int dx \mathbb{E}^x\left[\overline{f(B_0)}g(B_t)e^{-i\int_0^t a(B_s)\circ dB_s - \int_0^t V(B_s)ds}\right]$$

De Angelis - Jona Lasinio - Sirgue83, H.-Ichinose-Lőrinczi08

$$a \in (L^2_{loc})^d,\ \nabla\cdot a \in L^1_{loc},\ V_+ \in L^1_{loc},\ V_- \prec \psi(-\Delta) \Longrightarrow$$

$$(f, e^{-t(\psi(\frac{1}{2}(p-a)^2)+V)}g) = \int dx \mathbb{E}\mathbb{E}^x\left[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s - \int_0^t V(B_{T_s})ds}\right]$$

- $(f, e^{-t\psi(H)}g) = \mathbb{E}\left[(f, e^{-T_tH}g)\right]$ + Trotter prod. formula

Schrödinger operator with spin

Schrödinger operator with spin

$$H_S(a) = \frac{1}{2}(p-a)^2 - \frac{1}{2}\sigma\cdot b = H(a) - \frac{1}{2}\sigma\cdot b$$

- $H_S(a)\begin{bmatrix} f_+ \\ f_- \end{bmatrix} = H(a)\begin{bmatrix} f_+ \\ f_- \end{bmatrix} - \frac{1}{2}\begin{bmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & b_3 \end{bmatrix}\begin{bmatrix} f_+ \\ f_- \end{bmatrix}$
- $\mathbb{Z}_2 = \{-1, +1\}$.

$(x,\theta) \in \mathbb{R}^d \times \mathbb{Z}_2$

$$H_S(a)f(x,\theta) = \left(H(a) - \frac{1}{2}\theta b_3\right)f(x,\theta) - \frac{1}{2}(b_1 - i\theta b_2)f(x,-\theta)$$

- $L^2(\mathbb{R}^d;\mathbb{C}^2) \to L^2(\mathbb{R}^d\times\mathbb{Z}_2;\mathbb{C})$, $\mathbb{C}^2$-valued$\Longrightarrow$$\mathbb{C}$-valued

Jump processes

- Poisson process $(N_t)_{t\geq 0}$: $\mathbb{E}[N_t = n] = e^{-t}t^n/n!$
- $$\int_0^{t+} f(s,N_s)dN_s = \sum_{0<r\leq t, N_r\neq N_{r+}} f(r,N_r)$$

Semimartingale Ito formula

- Potential V:
  $de^{-\int_0^t V(B_s)ds} = -V(B_t)e^{-\int_0^t V(B_s)ds}dt \Longrightarrow V(x) \leftarrow$ Potential
- $\mathbb{Z}_2$-valued stochastic proc. $\theta_s = (-1)^{N_s}$

$$de^{\int_0^{t+} W(B_s,-\theta_s)dN_s} = -\left(1-e^{W(B_t,-\theta_t)}\right)e^{\int_0^{t+} W(B_s,-\theta_s)dN_s}dN_t$$

$\Longrightarrow 1-e^{W(x,-\theta)} \leftarrow$ potential for spin!

- Off diagonal part$\Longrightarrow$

$$-\frac{1}{2}(b_1-\theta b_2) = 1-\exp\left(\log\left(\frac{1}{2}(b_1-\theta b_2)\right)\right)-1$$

Path integration

De Angelis-J.Lasinio-Sirugue (J.Phys.83), H.-Lörinczi (JFA08)

Suppose that $a \in (L^2_{loc})^3$, $b_3 \in L^1_{loc}$, $\nabla \cdot a \in L^1_{loc}$, $0 \le V \in L^1_{loc}$, and

$$\int_0^t ds \int p_s(x-y) \left| \log \frac{1}{2}\sqrt{b_1(y)^2 + b_2(y)^2} \right| dy < \infty.$$

$\Longrightarrow$

$$(f, e^{-tH_S(a)} g) = e^{+t} \sum_{\alpha=\pm 1} \int dx \mathbb{E}^x \mathbb{E}^\alpha \left[\overline{f(B_0, \theta_0)} g(B_t, \theta_t) e^{S_p + S_\sigma} \right]$$

$$S_p = -\int_0^t V(B_s) ds - i \int_0^t a(B_s) \circ dB_s$$

$$S_\sigma = \frac{1}{2}\int_0^t \theta_s b_3(B_s) ds + \int_0^{t+} \log \frac{1}{2}(b_1(B_s) - i\theta_{s-} b_2(B_s)) dN_s$$

Pauli-Fierz model

- **Fock space** $L^2(\mathscr{Q},\mu)$
  - Total Hilbert space $L^2(\mathbb{R}^d)\otimes L^2(\mathscr{Q},\mu)$
- **Quantized Radiation Field**: Gaussian r.v. $\mathscr{A}(f), f\in L^2$

$$\mathbb{E}_\mu[\mathscr{A}_\alpha(f)\cdot\mathscr{A}_\beta(g)]=\int\delta^\perp_{\alpha\beta}(k)\frac{1}{2|k|}\overline{\hat{f}(k)}\hat{g}(k)dk$$

- $\delta^\perp_{\alpha\beta}(k)=\delta_{\alpha\beta}-k_\alpha k_\beta/|k|^2$
- $LH\{:\prod_{j=1}^n\mathscr{A}_\sharp(f_j):\}$ is dense.

Free field Hamiltonian

$$d\Gamma(\sqrt{-\Delta})1\!\!1=0$$

$$d\Gamma(\sqrt{-\Delta}):\prod_{j=1}^n\mathscr{A}_\sharp(f_j):=\sum_{j=1}^n:\mathscr{A}_\sharp(f_1)\cdots\mathscr{A}_\sharp(\sqrt{-\Delta}f_j)\cdots\mathscr{A}_\sharp(f_n):$$

Pauli-Fierz model

UV cutoff

$$\hat{\varphi}(k) = \begin{cases} (2\pi)^{-d/2} & |k| \le \Lambda \\ 0 & |k| \ge \Lambda \end{cases} \Longrightarrow \mathscr{A}(x) = \mathscr{A}(\varphi(\cdot - x))$$

Coulomb gauge $\nabla \cdot \mathscr{A}(x) = 0$

Pauli-Fierz model

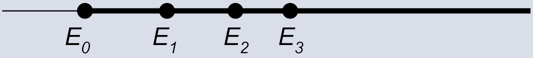
$$H = \frac{1}{2m}(p \otimes \mathbb{1} - \mathscr{A}(x))^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\sqrt{-\Delta})$$

$$\Longrightarrow \frac{1}{2m}(p - \mathscr{A}(x))^2 + V + \mathrm{d}\Gamma(\sqrt{-\Delta})$$

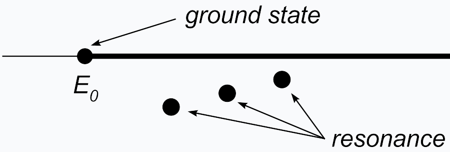
Spectrum of $h \otimes 1\!\!1 + 1\!\!1 \otimes d\Gamma(\sqrt{-\Delta}) + \alpha H_{int}$

$$\mathrm{Sp}(h) = \{E_j\},\ \mathrm{Sp}(d\Gamma(\sqrt{-\Delta})) = [0, \infty)$$

$(\alpha = 0)$ Embedded eigenvalues



$(\alpha \neq 0)$ Resonances and ground state



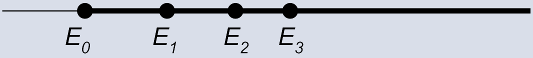
Nelson model

- flat space $\Longrightarrow$ Spohn (99), Betz-H.-Lőrinczi-Minlos-Spohn(01), LMS(02)
- manifold $\Longrightarrow$ Gérard-H.-Panatti-Suzuki (09,10a,10b)

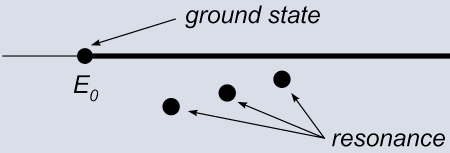
Spectrum of $h \otimes 1\!\!1 + 1\!\!1 \otimes d\Gamma(\sqrt{-\Delta}) + \alpha H_{int}$

$$\mathrm{Sp}(h) = \{E_j\},\ \mathrm{Sp}(d\Gamma(\sqrt{-\Delta})) = [0, \infty)$$

$(\alpha = 0)$ Embedded eigenvalues



$(\alpha \neq 0)$ Resonances and ground state



Nelson model

- flat space $\Longrightarrow$ Spohn (99), Betz-H.-Lőrinczi-Minlos-Spohn(01), LMS(02)
- manifold $\Longrightarrow$ Gérard-H.-Panatti-Suzuki (09,10a,10b)

Feynman-Kac formula for H

- **Euclidean Gaussian r.p.** on $(\mathscr{Q}_E, \mu_E)$: $(\mathscr{A}_t(f))_{t\in\mathbb{R}}$,
$$\mathbb{E}\left[\mathscr{A}_{\alpha,s}(f)\cdot\mathscr{A}_{\beta,t}(g)\right] = \int \delta^{\perp}_{\alpha\beta}(k)\overline{\hat{f}(k)}\hat{g}(k)\frac{e^{-|s-t||k|}}{2|k|}dk$$
- **Isometries** $\exists J_t : L^2(\mathscr{Q}) \to L^2(\mathscr{Q}_E)$ s.t. $J_t^* J_s = e^{-|t-s|\mathrm{d}\Gamma(\sqrt{-\Delta})}$
- Set $F_0 = J_0 F$ and $G_t = J_t G$.

Fröhlich-Park 80, Fefferman-Fröhlich-Graf 97, H.97

$$(F, e^{-tH}G) = \int dx \mathbb{E}^x \mathbb{E}_{\mu_E}\left[\overline{F_0(B_0)} G_t(B_t) e^S\right]$$

$$S = -\int_0^t V(B_s)ds - i\int_0^t \mathscr{A}_s(\varphi(\cdot - B_s))\cdot dB_s$$

Relativistic PF model

Relativistic PF model

$$H_R = \sqrt{(p \otimes 1\!\!1 - \mathscr{A}(x))^2 + m^2} - m + V \otimes 1\!\!1 + 1\!\!1 \otimes d\Gamma(\sqrt{-\Delta})$$

Theorem 1. Feynman-Kac formula for H_R [H.09]

$$(F, e^{-tH_R} G) = \int dx \mathbb{E}\mathbb{E}^x \mathbb{E}_{\mu_E} \left[\overline{F_0(B_0)} G_t(B_{T_t}) e^S \right]$$

$$S = -\int_0^t V(B_{T_s}) ds - i \int_0^{T_t} \mathscr{A}_{T^{-1}s}(\varphi(\cdot - B_s)) \cdot dB_s$$

Similar form to Pauli-Fierz model!

Main theorem

- Existence of ground state is known [Konenberg, Matte, Stockmeyer,09] [H.-Sasaki,09].

Theorem 2.

- (Ess.self-adjointness) H_R is ess. self-adjoint on $\cap_\mu D(p_\mu \otimes 1\!\!1) \cap D(1\!\!1 \otimes d\Gamma(\sqrt{-\Delta}))$.
- (Exp.decay of GS) $\|\varphi_g(x)\|_{L^2(\mathscr{Q})} \leq C_1 e^{-C_2|x|}$ $(m > 0)$
- (Energy comparison ineq.) $\inf \mathrm{Sp}(\sqrt{p^2+m^2} - m + V) \leq \inf \mathrm{Sp}(H_R)$
- (Uniqueness of GS) GS of H_R is unique

Outline of proof

- **(Ess.s.a.)** For $Q = p_\mu$ or $\mathrm{d}\Gamma(\sqrt{-\Delta})$,

$$|(QF, e^{-tH_R}G)| \le \|F\|\|QG\| \Longrightarrow e^{-tH_R}D(Q) \subset D(Q)$$

- **(Exp.decay)** $\varphi_{\mathrm{g}}(x) = e^{tE}e^{-tH_R}\varphi_{\mathrm{g}}(x) = e^{tE}\mathbb{E}^x\left[J_0^* e^S J_t \varphi_{\mathrm{g}}\right]$
- **(Energy ineq.)** Diamagnetic inequality.

$$|(F, e^{-tH_R}G)| \le (|F|, e^{-t(\sqrt{p^2+m^2}-m+V+\mathrm{d}\Gamma(\sqrt{-\Delta}))}|G|)$$

- **(Uniqueness)** It follows from Perron-Frobenius Thm. by

$$(F, e^{-i\frac{\pi}{2}N}e^{-tH_R}e^{i\frac{\pi}{2}N}G) = \int dx \mathbb{E}\mathbb{E}^x\mathbb{E}_{\mu_E}\left[\overline{F_0(B_0)}G_t(B_{T_t})e^{\hat{S}}\right],$$

where $\hat{S} = -\int_0^t V(B_{T_s})ds - i\int_0^{T_t}\Pi_{T^{-1}s}(\varphi(\cdot - B_s))\cdot dB_s$. Π_s is the canonical conjugate of $\mathscr{A}_s$ and $e^{-i\Pi_s(F)}$ is shift.

Translation invariant PF model $(V=0)$

Translation invariant PF model

$$H_R(P)=\sqrt{(P-\mathrm{d}\Gamma(-i\nabla)-\mathscr{A}(0))^2+m^2}-m+\mathrm{d}\Gamma(\sqrt{-\Delta}),\quad P\in\mathbb{R}^d$$

- $[H_R, p_\mu\otimes 1\!\!1+1\!\!1\otimes \mathrm{d}\Gamma(-i\nabla)_\mu]=0$

$$\Longrightarrow\begin{cases} H_R=\int_{\mathbb{R}^d}^{\oplus}H_R(P)dP \\ L^2(\mathbb{R}^3)\otimes L^2(\mathscr{Q})=\int_{\mathbb{R}^d}^{\oplus}L^2(\mathscr{Q})dx\end{cases}$$

- $H_R(P)$ is defined on $L^2(\mathscr{Q})$.
- $P=0\Longrightarrow \exists GS$, $P\neq 0\Longrightarrow \nexists GS$ (Open problem)

Main themreom

Theorem 3. Feynman-Kac formula for $H_R(P)$, [H.JFA07,09]

$$(\Phi, e^{-tH_R(P)}\Psi) = \mathbb{E}^0\mathbb{E}\mathbb{E}_{\mu_E}\left[e^{iP\cdot B_{T_t}}\overline{\Phi_0}e^S e^{-iP\cdot \mathrm{d}\Gamma(-i\nabla)}\Psi_t\right]$$

- $E(P) = \inf \mathrm{Sp}(H_R(P))$

Theorem 4.

- $E(0) \leq E(P)$
- $e^{-i(\pi/2)N}e^{-tH_R(0)}e^{i(\pi/2)N}$ is positivity improving, and GS of $H_R(0)$ is unique.

Main themreom

Theorem 3. Feynman-Kac formula for $H_R(P)$, [H.JFA07,09]

$$(\Phi, e^{-tH_R(P)}\Psi) = \mathbb{E}^0\mathbb{E}\mathbb{E}_{\mu_E}\left[e^{iP\cdot B_{T_t}}\overline{\Phi_0}e^S e^{-iP\cdot \mathrm{d}\Gamma(-i\nabla)}\Psi_t\right]$$

- $E(P) = \inf \mathrm{Sp}(H_R(P))$

Theorem 4.

- $E(0) \leq E(P)$
- $e^{-i(\pi/2)N}e^{-tH_R(0)}e^{i(\pi/2)N}$ is positivity improving, and GS of $H_R(0)$ is unique.

References

Pauli-Fierz model H

- **Self-adjointness** [Arai,JMP81,83], [H. CMP 00, AHP 02], [Haslar-Herbst RMP08],
- **Existence of GS** [Bach-Fröhlich-Sigal, Adv Math97, CMP99], [Arai-Hirokawa,JFA97], [Spohn,LMP99], [Gérard,AHP00], [H.Trans AMS00], [Chen, 00], [Griesemer-Lieb-Loss, Inv.Math.01], [Hirokawa-H.-Spohn,Adv.Math02]
- **Absence of GS** [Lőrinczi-Minlos-Spohn,AHP01], [Hirokawa, Publ.RIMS.06], [Gérard-H.-Suzuki-Panatti I,III,09],
- **Enhanced binding** [H.-Spohn,AHP02], [Catto-Hainzl,JFA02] [Arai-Kawano,RMP02], [Chen,Vogalter,Vugalter,JMP03], [H.-Sasaki,Math.Z09] [H.-Spohn-Suzuki]

References

- **Multiplicity of GS** [Bach-Fröhlich-Sigal,Adv.Math.97], [H. JMP00,JFA 05], [H.-Spohn, ATMP 01], [Bach-Fröhlich-Pizzo CMP 05]
- **Feynman-Kac type formula** [Fefferman-Fröhlich-Graf, CMP97], [H. RMP97, JFA06], [Haba, JMP98], [H.-Lőrinczi, JFA08] [Lőrinczi-H.-Betz, de Gryter Math serise 34, to appear]
- **QFT on a curved space-time** [Gérard-H.-Suzuki-Panatti I,II,III,09]

Relativistic PF model H_R

- **Existence of GS** [Miyao-Spohn,JFA08] [Konenberg, Matte, Stockmeyer,09] [H.-Sasaki,09]

Thank you!