Spin-Boson Model

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1 Spin-boson model

1.1 Definition

Spin-boson model describes an interaction between an ideal two-level atom and a quantum scalar field. Two eigenvalues of the atom are embedded in the continuous spectrum when no perturbation is added. See Figure 1. We are interested in investigating behaviors of embedded eigenvalues after adding a perturbation. In particular we consider properties of the bottom of the spectrum of spin-boson Hamiltonians by functional integrations. In this article we show an outline of [HHL08].

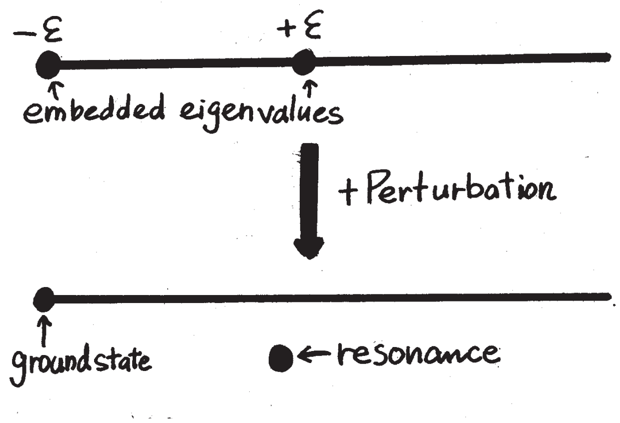

Figure 1: Embedded eigenvalues

Let $\mathscr{F} = \bigoplus_{n=0}^{\infty} \left(\otimes_{\rm sym}^n L^2(\mathbb{R}^d) \right)$ be the boson Fock space over $L^2(\mathbb{R}^d)$, where the subscript means symmetrized tensor product. We denote the boson annihilation and creation operators by $a(f)$ and $a^\dagger(f)$, $f, g \in L^2(\mathbb{R}^d)$, respectively, satisfying the canonical commutation relations

$$[a(f), a^\dagger(g)] = (\bar{f}, g), \quad [a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]. \tag{1.1}$$

We use the informal expression $a^\sharp(f) = \int a^\sharp(k) f(k) dk$ for notational convenience. Consider the Hilbert space $\mathscr{H} = \mathbb{C}^2 \otimes \mathscr{F}$. Denote by $d\Gamma(T)$ be the second quantization of a self-adjoint operator T in $L^2(\mathbb{R}^d)$. The operator on Fock space defined by $H_\mathrm{f} = d\Gamma(\omega)$ is the free boson Hamiltonian with dispersion relation $\omega(k) = |k|$. The operator

$$\phi_\mathrm{b}(\hat{h}) = \frac{1}{\sqrt{2}} \int \left(a^\dagger(k)\hat{h}(-k) + a(k)\hat{h}(k) \right) dk, \tag{1.2}$$

acting on Fock space is the scalar field operator, where $h \in L^2(\mathbb{R}^d)$ is a suitable form factor and $\hat{h}$ is the Fourier transform of h. Denote by σ_x, σ_y and σ_z the 2×2 Pauli matrices given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.3}$$

With these components, the spin-boson Hamiltonian is defined by the linear operator

$$H_\mathrm{SB} = \varepsilon \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes H_\mathrm{f} + \alpha \sigma_x \otimes \phi_\mathrm{b}(\hat{h}) \tag{1.4}$$

on $\mathscr{H}$, where $\alpha \in \mathbb{R}$ is a coupling constant and $\varepsilon \geq 0$ a parameter.

1.2 A Feynman-Kac-type formula

In this section we give a functional integral representation of e^{-tH_SB} by making use of a Poisson point process and an infinite dimensional Ornstein-Uhlenbeck process. First we transform H_SB in a convenient form to study its spectrum in terms of path measures.

Recall that the rotation group in $\mathbb{R}^3$ has an adjoint representation on $SU(2)$. In particular, for $n = (0, 1, 0)$ and $\theta = \pi/2$, we have $e^{(i/2)\theta n \cdot \sigma} \sigma_x e^{-(i/2)\theta n \cdot \sigma} = \sigma_z$ and $e^{(i/2)\theta n \cdot \sigma} \sigma_z e^{-(i/2)\theta n \cdot \sigma} = -\sigma_x$. Let $U = \exp\left(i\frac{\pi}{4}\sigma_y\right) \otimes \mathbb{1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \otimes \mathbb{1}$ be a unitary operator on $\mathscr{H}$. Then H_SB transforms as

$$H = U H_\mathrm{SB} U^* = -\varepsilon \sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes H_\mathrm{f} + \alpha \sigma_z \otimes \phi_\mathrm{b}(\hat{h}). \tag{1.5}$$

If $\hat{h}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$ and h is real-valued, then $\phi_\mathrm{b}(\hat{h})$ is symmetric and infinitesimally small with respect to H_f, hence by the Kato-Rellich theorem it follows that H is a self-adjoint operator on $D(H_\mathrm{f})$ and bounded from below.

To construct the functional integral representation of the semigroup e^{-tH}, it is useful to introduce a spin variable $\sigma \in \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{-1, +1\}$ is the additive group of order 2. For $\Psi = \begin{bmatrix} \Psi(+) \\ \Psi(-) \end{bmatrix} \in \mathscr{H}$, we have $H\Psi = \begin{bmatrix} (H_\mathrm{f} + \alpha\phi_\mathrm{b}(\hat{h}))\Psi(+) - \varepsilon\Psi(-) \\ (H_\mathrm{f} - \alpha\phi_\mathrm{b}(\hat{h}))\Psi(-) - \varepsilon\Psi(+) \end{bmatrix}$. Thus we can transform H on $\mathscr{H}$ to the operator $\tilde{H}$ on $L^2(\mathbb{Z}_2; \mathscr{F})$ by

$$(\tilde{H}\Psi)(\sigma) = \left(H_\mathrm{f} + \alpha\sigma\phi_\mathrm{b}(\hat{h}) \right) \Psi(\sigma) + \varepsilon\Psi(-\sigma), \quad \sigma \in \mathbb{Z}_2. \tag{1.6}$$

In what follows, we identify the Hilbert space $\mathscr{H}$ with $L^2(\mathbb{Z}_2; \mathscr{F})$, and instead of H we consider $\tilde{H}$, and use the notation H for $\tilde{H}$.

Let (Ω, Σ, P) be a probability space, and $(N_t)_{t \in \mathbb{R}}$ be a two-sided Poisson process with unit intensity on this space. We denote by $D = \{t \in \mathbb{R} \,|\, N_{t+} \neq N_{t-}\}$ the set of jump points, and define the integral with respect to this Poisson process by $\int_{(s,t]} f(r, N_r) dN_r = \sum_{\substack{r \in D \\ r \in (s,t]}} f(r, N_r)$

for any predictable function f. In particular, we have for any continuous function g, $\int_{(s,t]} g(r,N_{r-})dN_r = \sum_{\substack{r\in D\\ s<r\le t}} g(r,N_{r-})$. We write $\int_s^{t+}\cdots dN_r$ for $\int_{(s,t]}\cdots dN_r$. Note that $\int_s^{t+} g(r,N_{-r})dN_r$ is right-continuous in t and the integrand $g(r,N_{-r})$ is left-continuous in r and thus a predictable process. Define the random process $\sigma_t = \sigma(-1)^{N_t}$, $\sigma\in\mathbb{Z}_2$. In the Schrödinger representation the boson Fock space $\mathscr{F}$ can be realized as an L^2-space over a probability space (Q,μ), and the field operator $\phi_{\rm b}(\hat f)$ with real-valued function $f\in L^2(\mathbb{R}^d)$ as a multiplication operator, which we will denote by $\phi(f)$. The identity function $\mathbb{1}$ on Q corresponds to the Fock vacuum $\Omega_{\rm b}$ in $\mathscr{F}$.

Let $(Q_{\rm E},\mu_{\rm E})$ be a probability space associated with the Euclidean quantum field. The Hilbert spaces $L^2(Q_{\rm E})$ and $L^2(Q)$ are related through the family of isometries $\{j_s\}_{s\in\mathbb{R}}$ from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{d+1})$ defined by $\widehat{j_s f}(k,k_0) = \frac{e^{-itk_0}}{\sqrt{\pi}}\sqrt{\frac{\omega(k)}{|k_0|^2+\omega(k)^2}}\hat f(k)$. Let $\Phi_{\rm E}(j_s f)$ be a Gaussian random variable on $(Q_{\rm E},\mu_{\rm E})$ indexed by $j_s f\in L^2(\mathbb{R}^{d+1})$ with mean zero and covariance $\mathbb{E}_{\mu_{\rm E}}[\Phi_{\rm E}(j_s f)\Phi_{\rm E}(j_t g)] = \frac12\int_{\mathbb{R}^d} e^{-|s-t|\omega(k)}\overline{\hat f(k)}\hat g(k)dk$. Also, let $\{J_s\}_{s\in\mathbb{R}}$ be the family of isometries from $L^2(Q)$ to $L^2(Q_{\rm E})$ defined by $J_s{:}\phi(f_1)\cdots\phi(f_n){:} = {:}\Phi_{\rm E}(j_s f_1)\cdots\Phi_{\rm E}(j_s f_n){:}$, where ${:}X{:}$ denotes Wick product of X. Then we derive that $(J_s\Phi, J_t\Psi)_{L^2(Q_{\rm E})} = (\Phi, e^{-|t-s|H_{\rm f}}\Psi)_{L^2(Q)}$. We identify $\mathscr{H}$ as $\mathscr{H}\cong L^2(\mathbb{Z}_2;L^2(Q))\cong L^2(\mathbb{Z}_2\times Q)$.

Proposition 1.1 *Let $\Phi,\Psi\in\mathscr{H}$ and $h\in L^2(\mathbb{R}^d)$ be real-valued. Then*

$$(\varepsilon\neq 0)\quad (\Phi, e^{-tH}\Psi)_{\mathscr{H}} = e^t\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}_P\mathbb{E}_{\mu_{\rm E}}\left[\overline{J_0\Phi(\sigma_0)}e^{-\alpha\Phi_{\rm E}\left(\int_0^t\sigma_s j_s h ds\right)}\varepsilon^{N_t}J_t\Psi(\sigma_t)\right] \tag{1.7}$$

$$(\varepsilon = 0)\quad (\Phi, e^{-tH}\Psi)_{\mathscr{H}} = e^t\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}_{\mu_{\rm E}}\left[\overline{J_0\Phi(\sigma)}e^{-\alpha\Phi_{\rm E}\left(\sigma\int_0^t j_s h ds\right)}J_t\Psi(\sigma)\right]. \tag{1.8}$$

Denote $\mathbb{1}_{\mathscr{H}} = \mathbb{1}_{L^2(\mathbb{Z}_2)}\otimes\mathbb{1}_{L^2(Q)}$. Using the above proposition we can compute the vacuum expectation of the semigroup e^{-tH}.

Corollary 1.2 *Let $h\in L^2(\mathbb{R}^d)$ be a real-valued function. Then for every $t>0$ it follows that*

$$(\mathbb{1}_{\mathscr{H}}, e^{-tH}\mathbb{1}_{\mathscr{H}}) = e^t\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}_P\left[\varepsilon^{N_t}e^{\frac{\alpha^2}{2}\int_0^t dr\int_0^t W(N_r-N_s,r-s)ds}\right], \tag{1.9}$$

where the pair interaction potential W is given by

$$W(x,s) = \frac{(-1)^x}{2}\int_{\mathbb{R}^d} e^{-|s|\omega(k)}|\hat h(k)|^2 dk. \tag{1.10}$$

2 Ground state of the spin-boson

In the remainder of this paper we assume that $h\in L^2(\mathbb{R}^d)$ is real-valued. Let $E = \inf{\rm Spec}(H)$. Assume that $\varepsilon\neq 0$. Then e^{-tH}, $t>0$, is a positivity improving semigroup on $L^2(\mathbb{Z}_2\times Q)$, i.e., $(\Psi, e^{-tH}\Phi)>0$ for $\Psi,\Phi\geq 0$ such that $\Psi\not\equiv 0\not\equiv\Phi$. By this we can see that ${\rm Ker}(H-E)=1$ for $\varepsilon\neq 0$. We consider the case of $\varepsilon\neq 0$. Write $\Phi_T = e^{-T(H-E)}\mathbb{1}$ and

$$\gamma(T) = \frac{(\mathbb{1}_{\mathscr{H}},\Phi_T)^2}{\|\Phi_T\|^2} = \frac{(\mathbb{1}_{\mathscr{H}}, e^{TH}\mathbb{1}_{\mathscr{H}})^2}{(\mathbb{1}_{\mathscr{H}}, e^{-2TH}\mathbb{1}_{\mathscr{H}})}. \tag{2.1}$$

A known criterion of existence of a ground state is [LHB11, Proposition 6.8].

Proposition 2.1 *A ground state of H exists if and only if* $\lim_{T\to\infty}\gamma(T)>0$.

By Corollary 1.2 we have

$$\|\Phi_T\|^2 = e^{2TE}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}_P\left[\varepsilon^{N_T}e^{\frac{\alpha^2}{2}\int_{-T}^T dt\int_{-T}^T W(N_t-N_s,t-s)ds}\right], \tag{2.2}$$

$$(\mathbb{1}_{\mathscr{H}},\Phi_T) = e^{TE}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}_P\left[\varepsilon^{N_T}e^{\frac{\alpha^2}{2}\int_{-T}^0 dt\int_{-T}^0 W(N_t-N_s,t-s)ds}\right]. \tag{2.3}$$

Note that $\left|\int_{-T}^0 dt\int_0^T W(N_t-N_s,t-s)ds\right| \le \frac{1}{2}\left\|\hat{h}/\omega\right\|^2$ uniformly in T and in the paths.

Theorem 2.2 *If $\hat{h}/\omega\in L^2(\mathbb{R}^d)$, then H has a ground state and it is unique.*

Proof: We can show that $\lim_{T\to\infty}\gamma(T)>0$ by using (2.2) and (2.3). Then the theorem follows from Proposition 2.1.

□

It is a known fact that $H_{\rm SB}$ has a parity symmetry. Let $P=\sigma_z\otimes(-1)^N$, where $N=d\Gamma(\mathbb{1})$ denotes the number operator in $\mathscr{F}$. From $\mathrm{Spec}(\sigma_z)=\{-1,1\}$ and $\mathrm{Spec}(N)=\{0,1,2,...\}$ it follows that $\mathrm{Spec}(P)=\{-1,1\}$. Then $\mathscr{H}$ can be decomposed as $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$ and H can be reduced by $\mathscr{H}_\pm$.

Corollary 2.3 *Let $\varphi_{\rm SB}$ be the ground state of $H_{\rm SB}$. Then $\varphi_{\rm SB}\in\mathscr{H}_-$.*

3 Path measure associated with the ground state

In this section we set $\varepsilon=1$ for simplicity. Let $\mathscr{X}=D(\mathbb{R};\mathbb{Z}_2)$ be the space of càdlàg paths with values in $\mathbb{Z}_2$, and $\mathscr{G}$ the σ-field generated by cylinder sets. Thus $\sigma_\cdot:(\Omega,\Sigma,P)\to(\mathscr{X},\mathscr{G})$ is an $\mathscr{X}$-valued random variable. We denote its image measure by $\mathcal{W}^\sigma$, i.e., $\mathcal{W}^\sigma(A)=\sigma_\cdot^{-1}(A)$ for $A\in\mathscr{G}$, and the coordinate process by $(X_t)_{t\in\mathbb{R}}$, i.e., $X_t(\omega)=\omega(t)$ for $\omega\in\mathscr{X}$. Hence Proposition 1.1 can be reformulated in terms of $(X_t)_{t\in\mathbb{R}}$ as

$$(\Phi,e^{-tH}\Psi)_{\mathscr{H}} = e^t\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^\sigma_{\mathcal{W}}\mathbb{E}_{\mu_{\rm E}}\left[\overline{J_0\Phi(X_0)}e^{-\alpha\Phi_{\rm E}\left(\int_0^t X_s j_s h ds\right)}J_t\Psi(X_t)\right]. \tag{3.1}$$

Here $\mathbb{E}_{\mathcal{W}^\sigma}=\mathbb{E}^\sigma_{\mathcal{W}}$ so that $\mathbb{E}^\sigma_{\mathcal{W}}[X_0=\sigma]=1$. Let $(\mathbb{Z}_2,\mathscr{B})$ be a measurable space with σ-field $\mathscr{B}=\{\emptyset,\{-1\},\{+1\},\mathbb{Z}_2\}$. We see that the operator $Q_{[S,T]}=J_S^*e^{\Phi_{\rm E}(-\alpha\int_S^T X_s j_s h ds)}J_T$: $L^2(Q)\to L^2(Q)$ is bounded.

Corollary 3.1 *Let $-\infty < t_0 \leq \ldots \leq t_n < \infty$ and $A_0, ..., A_n \in \mathscr{B}$. Then*

$$(1)\ (\Phi, 1\!\!1_{A_0} e^{-(t_1-t_0)H} 1\!\!1_{A_1} e^{-(t_2-t_1)H} \cdots e^{-(t_n-t_{n-1})H} 1\!\!1_{A_n} \Psi)$$
$$= e^{t_n - t_0} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathscr{W}}^{\sigma} \mathbb{E}_{\mu_{\mathrm{E}}} \left[\left(\prod_{j=0}^{n} 1\!\!1_{A_j}(X_{t_j}) \right) \overline{\Phi(X_{t_0})} Q_{[t_0,t_n]} \Psi(X_{t_n}) \right],$$
$$(2)\ (1\!\!1_{A_0}, e^{-(t_1-t_0)H} 1\!\!1_{A_1} e^{-(t_2-t_1)H} \cdots e^{-(t_n-t_{n-1})H} 1\!\!1_{A_n})$$
$$= e^{t_n - t_0} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathscr{W}}^{\sigma} \left[e^{\frac{\alpha^2}{2} \int_{t_0}^{t_n} dt \int_{t_0}^{t_n} ds W(X_s, X_t, t-s)} \prod_{j=0}^{n} 1\!\!1_{A_j}(X_{t_j}) \right],$$

where $W(x,y,t) = \dfrac{xy}{2} \displaystyle\int_{\mathbb{R}^d} e^{-|t|\omega(k)} \hat{h}(k)^2 dk.$

Now we make the assumption that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, so that there is a unique ground state $\varphi_{\mathrm{g}} \in \mathscr{H}$. Let $\mathscr{G}_{[-T,T]} = \sigma(X_t, t \in [-T,T])$ be the family of sub-σ-fields of $\mathscr{G}$ and $\mathcal{G} = \bigcup_{T \geq 0} \mathscr{G}_{[-T,T]}$. Let $\overline{\mathcal{G}} = \sigma(\mathcal{G})$. Define the probability measure μ_T on $(\mathscr{X}, \overline{\mathcal{G}})$ by

$$\mu_T(A) = \frac{e^{2T}}{Z_T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathscr{W}}^{\sigma} \left[1\!\!1_A e^{\frac{\alpha^2}{2} \int_{-T}^{T} dt \int_{-T}^{T} ds W(X_t, X_s, t-s)} \right], \quad A \in \overline{\mathcal{G}}, \tag{3.2}$$

where Z_T is the normalizing constant such that $\mu_T(\mathscr{X}) = 1$. This probability measure is a Gibbs measure for the pair interaction potential W, indexed by the bounded intervals $[-T,T]$. Let μ_∞ be a probability measure on $(\mathscr{X}, \overline{\mathcal{G}})$. The sequence of probability measures $(\mu_n)_n$ is said to converge to the probability measure μ_∞ in *local weak topology* whenever $\lim_{n\to\infty} |\mu_n(A) - \mu_\infty(A)| = 0$ for all $A \in \mathscr{G}_{[-t,t]}$ and $t \geq 0$. By the definition it is seen that whenever $\mu_T \to \mu_\infty$ in local weak sense, we have that $\lim_{T\to\infty} \mathbb{E}_{\mu_T}[f] = \mathbb{E}_{\mu_\infty}[f]$ for any bounded $\mathscr{G}_{[-t,t]}$-measurable function f.

We define below a probability measure ρ_T on $(\mathscr{X}, \mathscr{G}_{[-T,T]})$ and an additive set function μ on $(\mathscr{X}, \mathcal{G})$. The unique extension of μ to a probability measure on $(\mathscr{X}, \bar{\mathcal{G}})$ is denoted by μ_∞. We shall prove that $\mu_T(A) = \rho_T(A)$ for all $A \in \mathscr{G}_{[-t,t]}$ with $t \leq T$, and show that $\rho_T(A) \to \mu(A)$ as $T \to \infty$, which implies that μ_T converges to μ_∞ in the sense of local weak.

We define the finite dimensional distributions indexed by $\Lambda = \{t_0, \ldots, t_n\} \subset [-T,T]$ with $t_0 \leq \ldots \leq t_n$. Let

$$\mu_T^{\Lambda}(A_0 \times \cdots \times A_n) = \frac{e^{2T}}{Z_T} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathscr{W}}^{\sigma} \left[\left(\prod_{j=0}^{n} 1\!\!1_{A_j}(X_{t_j}) \right) e^{\frac{\alpha^2}{2} \int_{-T}^{T} dt \int_{-T}^{T} ds W(X_t, X_s, t-s)} \right] \tag{3.3}$$

be a probability measure on $(\mathbb{Z}_2^{\Lambda}, \mathscr{B}^{\Lambda})$, where $\mathbb{Z}_2^{\Lambda} = \times_{j=1}^{n} \mathbb{Z}_2^{t_j}$ and $\mathscr{B}^{\Lambda} = \times_{j=1}^{n} \mathscr{B}^{t_j}$ for $\Lambda = \{t_1, ..., t_n\}$, and $\mathbb{Z}_2^{t_j}$ and $\mathscr{B}^{t_j}$ are copies of $\mathbb{Z}_2$ and $\mathscr{B}$, respectively. Clearly, $\mathcal{G}$ is a finitely additive family of sets. Define an additive set function on $(\mathscr{X}, \mathcal{G})$ by

$$\mu(A) = e^{2Et} e^{2t} \sum_{\sigma \in \mathbb{Z}_2} \mathbb{E}_{\mathscr{W}}^{\sigma} \left[1\!\!1_A (\varphi_{\mathrm{g}}(X_{-t}), Q_{[-t,t]} \varphi_{\mathrm{g}}(X_t))_{\mathscr{H}} \right], \quad A \in \mathscr{G}_{[-t,t]}. \tag{3.4}$$

Note that $\mu(\mathscr{X}) = (\varphi_{\rm g}, e^{-2t(H-E)}\varphi_{\rm g}) = 1$. There exists a unique probability measure μ_∞ on $(\mathscr{X}, \overline{\mathcal{G}})$ such that $\mu_\infty\lceil_{\mathcal{G}} = \mu$. In particular, $\mu_\infty(A) = \mu(A)$, for every $A \in \mathscr{G}_{[-t,t]}$ and $t \in \mathbb{R}$. In order to show that $\mu_T(A) \to \mu_\infty(A)$ for every $A \in \mathscr{G}_{[-t,t]}$, we define the probability measure ρ_T on $(\mathscr{X}, \mathscr{G}_{[-T,T]})$ for $A \in \mathscr{G}_{[-t,t]}$ with $t \le T$ by

$$\rho_T(A) = e^{2Et}e^{2t}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^\sigma_{\mathcal{W}}\left[\mathbb{1}_A\left(\frac{\Phi_{T-t}(X_{-t})}{\|\Phi_T\|}, Q_{[-t,t]}\frac{\Phi_{T-t}(X_t)}{\|\Phi_T\|}\right)\right]. \tag{3.5}$$

Remark 3.2 *Both μ and ρ_T are well defined. I.e., for $A \in \mathscr{G}_{[-s,s]} \subset \mathscr{G}_{[-t,t]}$ with $s \le t \le T$*

$$\begin{aligned}\mu(A) &= e^{2Es}e^{2s}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^\sigma_{\mathcal{W}}\left[\mathbb{1}_A(\varphi_{\rm g}(X_{-s}), Q_{[-s,s]}\varphi_{\rm g}(X_s))\right] \\ &= e^{2Et}e^{2t}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^\sigma_{\mathcal{W}}\left[\mathbb{1}_A(\varphi_{\rm g}(X_{-t}), Q_{[-t,t]}\varphi_{\rm g}(X_t))\right], \\ \rho_T(A) &= e^{2Es}e^{2s}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^\sigma_{\mathcal{W}}\left[\mathbb{1}_A\left(\frac{\Phi_{T-s}(X_{-s})}{\|\Phi_T\|}, Q_{[-s,s]}\frac{\Phi_{T-s}(X_s)}{\|\Phi_T\|}\right)\right] \\ &= e^{2Et}e^{2t}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^\sigma_{\mathcal{W}}\left[\mathbb{1}_A\left(\frac{\Phi_{T-t}(X_{-t})}{\|\Phi_T\|}, Q_{[-t,t]}\frac{\Phi_{T-t}(X_t)}{\|\Phi_T\|}\right)\right].\end{aligned}$$

The family of probability measures ρ_T^Λ on $(\mathbb{Z}_2^\Lambda, \mathscr{B}^\Lambda)$ indexed by $\Lambda = \{t_0, ..., t_n\} \subset [-T, T]$ is defined by

$$\rho_T^\Lambda(A_0\times\cdots\times A_n) = e^{2Et}e^{2t}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^\sigma_{\mathcal{W}}\left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(X_{t_j})\right)\left(\frac{\Phi_{T-t}(X_{-t})}{\|\Phi_T\|}, Q_{[-t,t]}\frac{\Phi_{T-t}(X_t)}{\|\Phi_T\|}\right)\right] \tag{3.6}$$

for arbitrary t such that $-T \le -t \le \ldots \le t_0 \le \ldots \le t_n \le t \le T$. To show that $\mu_T = \rho_T$, we prove that their finite dimensional distributions coincide.

Lemma 3.3 *Let $\Lambda = \{t_0, t_1, ..., t_n\}$ and $A_0 \times \cdots \times A_n \in \mathscr{B}^\Lambda$. Then $\mu_T^\Lambda(A_0 \times \cdots \times A_n) = \rho_T^\Lambda(A_0 \times \cdots \times A_n)$, and $\mu_T(A) = \rho_T(A)$ follows for $A \in \mathscr{G}_{[-t,t]}$ and $t \le T$.*

Proof: The former statement follows from Corollary 3.1 and the later from Kolmogorov consistency theorm. □

Theorem 3.4 *Suppose $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then the probability measure μ_T on $(\mathscr{X}, \overline{\mathcal{G}})$ converges in local weak sense to μ_∞ as $T \to \infty$.*

Proof: Let $A \in \mathscr{G}_{[-T,T]}$. Then $\mu_T(A) = \rho_T(A)$. Since $\frac{\Phi_T}{\|\Phi_T\|} \to \varphi_{\rm g}$ as $T \to \infty$, we can see that $\rho_T(A) \to \mu(A)$ as $T \to \infty$. Since $\mu(A) = \mu_\infty(A)$, the theorem follows. □

In the case when $\varepsilon \ne 1$ a parallel discussion to the previous section can be made. We summarize this in the theorem below.

Theorem 3.5 *Suppose $\hat{h}/\omega \in L^2(\mathbb{R}^d)$. Then the probability measure μ_T^ε on $(\mathscr{X}, \overline{\mathcal{G}})$ converges in local weak sense to μ_∞^ε as $T \to \infty$.*

We also write $\mu_{\rm g}$ for μ_∞^ε for notational convenience.

4 Ground state properties

In this section without proofs we show to be able to express ground state expectations of some observables in terms of the limit measure $\mu_{\rm g}$ discussed in the previous section.

4.1 Expectations of functions of the form $\xi(\sigma)F(\phi(f))$

Theorem 4.1 *Let f be a $\mathscr{G}_{[-\varepsilon t,\varepsilon t]}$-measurable function on $\mathscr{X}$. Then*

$$\mathbb{E}_{\mu_{\rm g}}[f] = e^{2E\varepsilon t}e^{2\varepsilon t}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^{\sigma}_{\mathcal{W}}\left[\left(\varphi_{\rm g}(X_{-\varepsilon t}), Q^{(\varepsilon)}_{[-\varepsilon t,\varepsilon t]}\varphi_{\rm g}(X_{\varepsilon t})\right) f\right]. \tag{4.1}$$

An immediate consequence of Theorem 4.1 is the following.

Corollary 4.2 *Let $f_j:\mathbb{Z}_2\to\mathbb{C}$, $j=0,...,n$, be bounded functions. Then*

$$\mathbb{E}_{\mu_{\rm g}}\left[\prod_{j=0}^n f_j(X_{\varepsilon t_j})\right] = (\varphi_{\rm g}, f_0 e^{-(t_1-t_0)(H-E)}f_1\cdots e^{-(t_n-t_{n-1})(H-E)}f_n\varphi_{\rm g}). \tag{4.2}$$

In particular, we have for all bounded functions ξ, f and g that

$$\mathbb{E}_{\mu_{\rm g}}[\xi(X_0)] = (\varphi_{\rm g}, \xi(\sigma)\varphi_{\rm g}), \tag{4.3}$$

$$\mathbb{E}_{\mu_{\rm g}}[f(X_t)g(X_s)] = (f(\sigma)\varphi_{\rm g}, e^{-|t-s|(H-E)}g(\sigma)\varphi_{\rm g}). \tag{4.4}$$

Theorem 4.3 *Let $\hat{h}/\omega\in L^2(\mathbb{R}^d)$, $f\in L^2(\mathbb{R}^d)$ be real-valued, $\xi:\mathbb{Z}_2\to\mathbb{C}$ be a bounded function, and $\beta\in\mathbb{R}$. Then*

$$(\varphi_{\rm g}, \xi(\sigma)e^{i\beta\phi(f)}\varphi_{\rm g}) = e^{-\frac{\beta^2}{4}\|f\|^2}\mathbb{E}_{\mu_{\rm g}}\left[\xi(X_0)e^{i\beta K(f)}\right], \tag{4.5}$$

where $K(f)$ is a random variable on $(\mathscr{X},\overline{\mathcal{G}})$ given by $K(f)=\frac{\alpha}{2}\int_{-\infty}^{\infty}(e^{-|r|\omega}\hat{h},\hat{f})X_{\varepsilon r}dr$.

By using Theorem 4.3 the functionals $(\varphi_{\rm g},\xi(\sigma)F(\phi(f))\varphi_{\rm g})$ can be represented in terms of averages with respect to the path measure $\mu_{\rm g}$. Consider the case when F is a polynomial or a Schwartz test function. We will show in Corollary 2.2 below that $\varphi_{\rm g}\in D(e^{+\beta N})$ for all $\beta>0$, thus $\varphi_{\rm g}\in D(\phi(f)^n)$ for every $n\in\mathbb{N}$.

Corollary 4.4 *Let $\hat{h}/\omega\in L^2(\mathbb{R}^d)$, $f\in L^2(\mathbb{R}^d)$ be real-valued, and $\xi:\mathbb{Z}_2\to\mathbb{C}$ a bounded function. Also, let $h_n(x)=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$ be the Hermite polynomial of order n. Then*

$$(\varphi_{\rm g}, \xi(\sigma)\phi(f)^n\varphi_{\rm g}) = i^n\mathbb{E}_{\mu_{\rm g}}\left[\xi(X_0)h_n\left(\frac{-iK(f)}{\|f\|2^{-1/2}}\right)\right](\|f\|2^{-1/2})^n, \quad n\in\mathbb{N}. \tag{4.6}$$

In the next corollary we give the path integral representation of $(\varphi_{\rm g},\xi(\sigma)F(\phi(f))\varphi_{\rm g})$ for $F\in\mathscr{S}(\mathbb{R})$, where $\mathscr{S}(\mathbb{R})$ denotes the space of rapidly decreasing, infinitely many times differentiable functions on $\mathbb{R}$.

Corollary 4.5 *Let $\hat{h}/\omega\in L^2(\mathbb{R}^d)$, $f\in L^2(\mathbb{R}^d)$ be real-valued, $F\in\mathscr{S}(\mathbb{R})$, and $\xi:\mathbb{Z}_2\to\mathbb{C}$ a bounded function. Then $(\varphi_{\rm g},\xi(\sigma)F(\phi(f))\varphi_{\rm g}) = \mathbb{E}_{\mu_{\rm g}}[\xi(X_0)G(K(f))]$, where $G=\check{F}*\check{g}$ and $g(\beta)=e^{-\beta^2\|f\|^2/4}$.*

4.2 Exponential moments of the field operator

In this section we show that $(\varphi_{\rm g}, e^{\beta\phi(f)^2}\varphi_{\rm g}) < \infty$ for some $\beta > 0$.

Theorem 4.6 *Let $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$ be a real-valued function. If $-\infty < \beta < 1/\|f\|^2$, then $\varphi_{\rm g} \in D(e^{(\beta/2)\phi(f)^2})$,*

$$\|e^{(\beta/2)\phi(f)^2}\varphi_{\rm g}\|^2 = \frac{1}{\sqrt{1-\beta\|f\|^2}}\,\mathbb{E}_{\mu_{\rm g}}\Big[e^{\frac{\beta K^2(f)}{1-\beta\|f\|^2}}\Big], \tag{4.7}$$

and $\lim_{\beta\uparrow 1/\|f\|^2}\|e^{(\beta/2)\phi(f)^2}\varphi_{\rm g}\| = \infty$.

Theorem 4.6 says that $\|e^{(\beta/2)\phi(f)^2}\varphi_{\rm g}\| < \infty$. Using this fact we can obtain explicit formulae of the exponential moments $(\varphi_{\rm g}, e^{\beta\phi(f)}\varphi_{\rm g})$ of the field.

Corollary 4.7 *If $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$ is a real-valued function, then $\varphi_{\rm g} \in D(e^{\beta\phi(f)})$ and*

$$(\varphi_{\rm g}, e^{\beta\phi(f)}\varphi_{\rm g}) = (\varphi_{\rm g}, \cosh(\beta\phi(f))\varphi_{\rm g}) = e^{\frac{\beta^2}{4}\|f\|^2}\mathbb{E}_{\mu_{\rm g}}\left[e^{\beta K(f)}\right], \tag{4.8}$$

$$(\varphi_{\rm g}, \sigma e^{\beta\phi(f)}\varphi_{\rm g}) = (\varphi_{\rm g}, \sigma\sinh(\beta\phi(f))\varphi_{\rm g}) = e^{\frac{\beta^2}{4}\|f\|^2}\mathbb{E}_{\mu_{\rm g}}\left[X_0 e^{\beta K(f)}\right]. \tag{4.9}$$

4.3 Expectations of second quantized operators

We consider expectations of the form $(\varphi_{\rm g}, e^{-\beta d\Gamma(\rho(-i\nabla))}\varphi_{\rm g})$, where ρ is a real-valued multiplication operator given by the function ρ. An important example is $\rho = \mathbb{1}$ giving the boson number operator $N = d\Gamma(\mathbb{1})$. We obtain the expression

$$\frac{(\Phi_T, \xi(\sigma)e^{-\beta d\Gamma(\rho(-i\nabla))}\Phi_T)}{\|\Phi_T\|^2} = \mathbb{E}_{\mu_T^\varepsilon}\left[\xi(X_0)e^{-\alpha^2\int_{-T}^0 dt\int_0^T W^{\rho,\beta}(X_{\varepsilon t}, X_{\varepsilon s}, t-s)ds}\right], \tag{4.10}$$

where $W^{\rho,\beta}(x,y,T) = \frac{xy}{2}\int_{\mathbb{R}^d}|\hat{h}(k)|^2 e^{-|T|\omega(k)}(1-e^{-\beta\rho(k)})dk$. Denote

$$W_\infty^{\rho,\beta} = \int_{-\infty}^0 dt\int_0^\infty W^{\rho,\beta}(X_{\varepsilon t}, X_{\varepsilon s}, t-s)ds. \tag{4.11}$$

Notice that $|W_\infty^{\rho,\beta}| \le \|\hat{h}/\omega\|^2/2 < \infty$, uniformly in the paths in $\mathscr{X}$.

Theorem 4.8 *Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $\xi : \mathbb{Z}_2 \to \mathbb{C}$ is a bounded function. Then*

$$(\varphi_{\rm g}, \xi(\sigma)e^{-\beta d\Gamma(\rho(-i\nabla))}\varphi_{\rm g}) = \mathbb{E}_{\mu_{\rm g}}\Big[\xi(X_0)e^{-\alpha^2 W_\infty^{\rho,\beta}}\Big], \quad \beta > 0. \tag{4.12}$$

Corollary 4.9 *Suppose that $\hat{h}/\omega \in L^2(\mathbb{R}^d)$ and $\xi : \mathbb{Z}_2 \to \mathbb{C}$ is a bounded function. Then*

$$(\varphi_{\rm g}, \xi(\sigma)e^{-\beta N}\varphi_{\rm g}) = \mathbb{E}_{\mu_{\rm g}}\Big[\xi(X_0)e^{-\alpha^2(1-e^{-\beta})W_\infty}\Big], \tag{4.13}$$

where $W_\infty = \int_{-\infty}^0 dt\int_0^\infty W(X_{\varepsilon t}, X_{\varepsilon s}, t-s)ds$. Furthermore $\varphi_{\rm g} \in D(e^{\beta N})$ for all $\beta \in \mathbb{C}$ and

$$(\varphi_{\rm g}, e^{\beta N}\varphi_{\rm g}) = \mathbb{E}_{\mu_{\rm g}}\Big[e^{-\alpha^2(1-e^{\beta})W_\infty}\Big] \tag{4.14}$$

follows.

5 Van Hove representation

The *van Hove Hamiltonian* is defined by the self-adjoint operator

$$H_{\rm vH}(\hat g) = H_{\rm f} + \phi_{\rm b}(\hat g) \tag{5.1}$$

in Fock space $\mathscr{F}$. Suppose that $\hat g/\omega \in L^2(\mathbb{R}^d)$ and define the conjugate momentum by

$$\pi_{\rm b}(\hat g) = \frac{i}{\sqrt{2}} \int \left(a^\dagger(k) \frac{\hat g(k)}{\omega(k)} - a(k) \frac{\hat g(-k)}{\omega(k)} \right) dk.$$

Then $e^{i\pi_{\rm b}(\hat g)} H_{\rm vH}(\hat g) e^{-i\pi_{\rm b}(\hat g)} = H_{\rm f} - \frac{1}{2}\|\hat g/\omega\|^2$ and the ground state of $H_{\rm vH}(\hat g)$ is given by $\varphi_{\rm vH}(\hat g) = e^{-i\pi_{\rm b}(\hat g)}\Omega_{\rm b}$. On the other hand, clearly the spin-boson Hamiltonian H with $\varepsilon = 0$ is the direct sum of van Hove Hamiltonians since $H = \begin{bmatrix} H_{\rm f} + \alpha\phi_{\rm b}(\hat h) & 0 \\ 0 & H_{\rm f} - \alpha\phi_{\rm b}(\hat h) \end{bmatrix}$ and $H_{\rm f} \pm \alpha\phi_{\rm b}(\hat h)$ are equivalent. Therefore the ground state of H with $\varepsilon = 0$ can be realized as $\varphi_{\rm g} = \begin{bmatrix} \varphi_{\rm vH}(\alpha\hat h) \\ \varphi_{\rm vH}(-\alpha\hat h) \end{bmatrix}$. Thus in this case

$$(\varphi_{\rm g}, e^{i\beta\phi(f)}\varphi_{\rm g})_{\mathscr{H}} = \frac{1}{2} \sum_{\sigma=\pm 1} (\varphi_{\rm vH}(\sigma\alpha\hat h), e^{i\beta\phi_{\rm b}(\hat f)}\varphi_{\rm vH}(\sigma\alpha\hat h))_{\mathscr{F}} \tag{5.2}$$

and the right hand side above equals $(\Omega_{\rm b}, e^{i\beta(\phi_{\rm b}(\hat f) + \alpha(\hat h/\omega, \hat f))}\Omega_{\rm b})_{\mathscr{F}} = e^{-\beta^2\|\hat f\|^2/4 + i\beta\alpha(\hat h/\omega, \hat f)}$. When $\varepsilon \neq 0$ we can derive similar but non-trivial representations. Define the random boson field operator $\Psi(\hat f) = \phi_{\rm b}(\hat f) + K(f)$ on $\mathscr{F}$. Let $\chi = \frac{\alpha}{2}\omega(k)\hat h(k)\int_{-\infty}^{\infty} e^{-|s|\omega(k)} X_{\varepsilon s} ds$. Note that $\chi \in L^2(\mathbb{R}^d)$, $K(f) = (\chi, \hat f)$, moreover, $\chi/\omega \in L^2(\mathbb{R}^d)$, whenever $\hat h/\omega \in L^2(\mathbb{R}^d)$, and $\chi = \sigma\alpha\hat h$ for $\varepsilon = 0$. We define the *random van Hove Hamiltonian* by $H_{\rm vH}(\chi)$.

Theorem 5.1 *If $\hat h/\omega \in L^2(\mathbb{R}^d)$, then*

$$(\varphi_{\rm g}, e^{i\beta\phi(f)}\varphi_{\rm g}) = \mathbb{E}_{\mu_{\rm g}}\Big[(\Omega_{\rm b}, e^{i\beta\Psi(\hat f)}\Omega_{\rm b})\Big] = \mathbb{E}_{\mu_{\rm g}}\Big[(\varphi_{\rm vH}(\chi), e^{i\beta\phi_{\rm b}(\hat f)}\varphi_{\rm vH}(\chi))\Big]. \tag{5.3}$$

Corollary 5.2 *Suppose $\hat h/\omega \in L^2(\mathbb{R}^d)$ and $F \in \mathscr{S}(\mathbb{R})$. Then we have*

$$(\varphi_{\rm g}, F(\phi(f))\varphi_{\rm g}) = \mathbb{E}_{\mu_{\rm g}}\Big[(\Omega_{\rm b}, F(\Psi(\hat f))\Omega_{\rm b})\Big] = \mathbb{E}_{\mu_{\rm g}}\Big[(\varphi_{\rm vH}(\chi), F(\phi(\hat f))\varphi_{\rm vH}(\chi))\Big], \tag{5.4}$$

$$\|e^{\beta\phi(f)^2/2}\varphi_{\rm g}\|^2 = \mathbb{E}_{\mu_{\rm g}}\Big[\|e^{\beta\Psi(\hat f)^2/2}\Omega_{\rm b}\|^2\Big] = \mathbb{E}_{\mu_{\rm g}}\Big[\|e^{\beta\phi_{\rm b}(\hat f)^2/2}\varphi_{\rm vH}(\chi)\|^2\Big]. \tag{5.5}$$

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