The Relativistic Pauli-Fierz model

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Dedicated to Izumi Ojima and Kei-ichi Ito
on the occasion of their sixtieth birthdays

Abstract

The relativistic Pauli-Fierz model is discussed. The Feynman-Kac type formula with cádlág path is shown and its applications are given.

In Sections 1 and 2 we review the results on the Pauli-Fierz model and in Section 3 we are concerned with the relativistic Pauli-Fierz model.

1 The Pauli-Fierz model

1.1 Definition

We begin with reviewing results on the Pauli-Fierz model. The Pauli-Fierz model describes the minimal interaction between electrons and a quantized radiation field, but electrons are assumed to be low energy and to be governed by a Schrödinger equation.

Let $L^2(\mathbb{R}^3)\otimes\mathcal{F}$ be the total Hilbert space describing the joint electron-photon state vectors. Here $\mathcal{F}=\mathcal{F}(\mathcal{H})$, $\mathcal{H}=L^2(\mathbb{R}^3\times\{\pm\})$, denotes the Boson Fock space over the one-photon Hilbert space $\mathcal{H}$. The elements of the set $\{\pm\}$ account for the fact that a photon is a transversal wave perpendicular to the direction of its propagation, thus it has two components. The Fock vacuum in $\mathcal{F}$ is denoted by Ω. Let $a(f)$ and $a^*(f)$, $f\in\mathcal{H}$, be the annihilation operator and the creation operator, respectively. We also use the identification: $\mathcal{H}\cong L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$ and set $a^\sharp(f,+)=a^\sharp(f\oplus 0)$ and $a^\sharp(f,-)=a^\sharp(0\oplus f)$ for $f\in L^2(\mathbb{R}^3)$. The annihilation operator and the creation operator satisfy the canonical commutation relations:

$$[a(f,i),a^*(g,j)]=\delta_{ij}(\bar{f},g),\quad [a^\sharp(f,i),a^\sharp(g,j)]=0.$$

Let T be a contraction operator on $\mathcal{H}$. Then

$$\Gamma(T) = \bigoplus_{n=0}^{\infty} \underbrace{T \otimes \cdots \otimes T}_{n}$$

is also contraction on $\mathcal{F}$. The second quantization of a self-adjoint operator h on $\mathcal{H}$ is defined by

$$\mathrm{d}\Gamma(h) = \bigoplus_{n=0}^{\infty} \sum_{j=1}^{n} \underbrace{1\!\!1 \otimes \cdots \overset{j}{h} \cdots \otimes 1\!\!1}_{n}.$$

The quantized radiation field with a given cutoff function $\hat{\varphi}$ is defined by

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{j=\pm} \int dk \frac{e_\mu^j(k)}{\sqrt{\omega(k)}} \left(a^*(k,j)\hat{\varphi}(-k)e^{-ik\cdot x} + a(k,j)\hat{\varphi}(k)e^{ik\cdot x} \right) \quad (1.1)$$

for $x \in \mathbb{R}^3$, where $\omega(k) = |k|$. The vectors $e^+(k)$ and $e^-(k)$ are polarization vectors.

The Hamiltonian of one electron is given by the Schrödinger operators with external potential V: $-\frac{1}{2}\Delta + V$, where we assume that the mass of electron is one. On the other hand the free Hamiltonian of the field is defined by $\mathrm{d}\Gamma(\omega)$. Then the decoupled Hamiltonian is

$$(-\frac{1}{2}\Delta + V) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega).$$

Let $\mathrm{D} = -i\nabla_x$. The Pauli-Fierz Hamiltonian is defined by the minimal coupling of the decoupled Hamiltonian with the quantized radiation field:

$$H = \frac{1}{2}(\mathrm{D} \otimes 1\!\!1 - eA)^2 + V \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega). \quad (1.2)$$

Here e denotes the coupling constant. Throughout we use the following assumptions (1) $\hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$, (2) $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$, (3) there exists $0 \le a < 1$ and $0 \le b$ such that

$$\|Vf\| \le a\| - \frac{1}{2}\Delta f\| + b\|f\|$$

for $f \in D(-\frac{1}{2}\Delta)$. We put $D_{\mathrm{PF}} = D\left(-\frac{1}{2}\Delta \otimes 1\!\!1\right) \cap D(1\!\!1 \otimes \mathrm{d}\Gamma(\omega))$. Then H is self-adjoint on D_{PF} and essentially self-adjoint on any core of the decoupled Hamiltonian.

Remark 1.1 *We notice that Pauli-Fierz Hamiltonians with different polarization vectors are isomorphic with each other. Then we fix polarization vectors throughout.*

1.2 Function space

We introduce a function space $(\mathscr{Q}, \mu)$ associated with the quantized radiation field and reformulate the Pauli-Fierz Hamiltonian on $L^2(\mathbb{R}^3) \otimes L^2(\mathscr{Q}, d\mu)$ instead of $L^2(\mathbb{R}^3) \otimes \mathcal{F}$. In order to have a functional integral representation of $(F, e^{-tH}G)$, $F, G \in L^2(\mathbb{R}^3) \otimes \mathcal{F}$, we construct probability spaces $(\mathscr{Q}_\beta, \Sigma_\beta, \mu_\beta)$, $\beta = 0, 1$, and the Gaussian random variables $\mathscr{A}_\beta(\mathbf{f})$ indexed by $\mathbf{f} = (f_1, f_2, f_3) \in \oplus^3 L^2_{\mathbb{R}}(\mathbb{R}^{3+\beta})$ of mean zero and covariance given by

$$\mathrm{q}_\beta(\mathbf{f}, \mathbf{g}) = \begin{cases} \frac{1}{2}(\hat{\mathbf{f}}, \delta^\perp \hat{\mathbf{g}})_{L^2(\mathbb{R}^3)}, & \beta = 0, \\ \frac{1}{2}(\hat{\mathbf{f}}, \mathbb{1} \otimes \delta^\perp \hat{\mathbf{g}})_{L^2(\mathbb{R}^4)}, & \beta = 1. \end{cases}$$

Note that transversal delta function $\delta^\perp(k) = (\delta_{\mu\nu} - k_\mu k_\nu / |k|^2)_{1 \le \mu, \nu \le 1}$ depends only on $k \in \mathbb{R}^3$. In what follows we denote

$$\begin{array}{lll} \text{(Minkowskian)} & \mathscr{A} = \mathscr{A}_0, & \mathscr{Q} = \mathscr{Q}_0, \\ \text{(Euclidean)} & \mathscr{A}_{\mathrm{E}} = \mathscr{A}_1, & \mathscr{Q}_{\mathrm{E}} = \mathscr{Q}_1 \end{array} \tag{1.3}$$

using the subscript E for Euclidean objects. Let $\mathscr{A}(x) = \mathscr{A}(\oplus^3 \tilde{\varphi}(\cdot - x))$, where $\tilde{\varphi}$ is the inverse Fourier transformation of $\hat{\varphi}/\sqrt{\omega}$. The Pauli-Fierz Hamiltonian in function space is defined by

$$H = \frac{1}{2}(\mathrm{D} \otimes \mathbb{1} - e\mathscr{A}(x))^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(\mathrm{D})). \tag{1.4}$$

Let $\mathrm{j}_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^4)$, $t \in \mathbb{R}$, be the family of isometries such that $j_t^* j_s = e^{-|t-s|\omega(\mathrm{D})}$ and we define $\mathrm{J}_t : L^2(\mathscr{Q}) \to L^2(\mathscr{Q}_{\mathrm{E}})$ by $\mathrm{J}_t = \Gamma(\mathrm{j}_t)$, and $\mathrm{J}_t^* \mathrm{J}_s = e^{-|t-s|\mathrm{d}\Gamma(\mathrm{D})}$ follows.

Let $(B_t)_{t \ge 0}$ denote the three dimensional Brownian motion on the probability space $(\mathscr{X}, B(\mathscr{X}), \mathscr{W}^x)$, where $\mathscr{X} = C([0, \infty); \mathbb{R}^3)$ endowed with the locally uniform topology, $B(\mathscr{X})$ is the Borel σ-field on $\mathscr{X}$, and $\mathscr{W}^x$ the Wiener measure. Write $\mathbb{E}^x[\cdots] = \int_{\mathscr{X}} \cdots d\mathscr{W}^x$.

Theorem 1.2 *Let $F, G \in L^2(\mathbb{R}^3) \otimes L^2(\mathscr{Q}_{\mathrm{E}})$. Then*

$$(F, e^{-tH}G) = \int dx \mathbb{E}^x \left[e^{-\int_0^t V(B_s) ds} \Big(\mathrm{J}_0 F(B_0), e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)} \mathrm{J}_t G(B_t) \Big)_{L^2(\mathscr{Q}_{\mathrm{E}})} \right]. \tag{1.5}$$

Here $K_t = \bigoplus_{\mu=1}^3 \int_0^t \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) dB_s^\mu$ denotes the $\bigoplus^3 L^2(\mathbb{R}^4)$-valued stochastic integral.

From this functional integral representation a lot of properties of ground state of H can be derived in the non-perturbative way.

1.3 Translation invariant Pauli-Fierz model

We consider the translation invariant Pauli-Fierz Hamiltonian. This is obtained by setting the external potential V identically zero. Put ${\rm P_f}_\mu = {\rm d}\Gamma(k_\mu)$, which describes the field momentum. The total momentum operator P on $L^2(\mathbb{R}^3) \otimes \mathcal{F}$ is defined by

$$
{\rm P}_\mu = {\rm D}_\mu \otimes 1\!\!1 + 1\!\!1 \otimes {\rm P_f}_\mu, \quad \mu = 1, 2, 3. \tag{1.6}
$$

We can see that $[H, {\rm P}_\mu] = 0$. This leads us to decompose H on the spectrum of the total momentum operator P. The Pauli-Fierz Hamiltonian with a fixed total momentum $H(p)$ is defined by

$$
H(p) = \frac{1}{2}(p - {\rm P_f} - eA(0))^2 + {\rm d}\Gamma(\omega), \quad p \in \mathbb{R}^3, \tag{1.7}
$$

with domain $D(H(p)) = D({\rm d}\Gamma(\omega)) \cap D({\rm P_f^2})$. Here $p \in \mathbb{R}^3$ is called the total momentum. Define the unitary operator $\mathscr{T} : L^2(\mathbb{R}^3_x) \otimes \mathcal{F} \to L^2(\mathbb{R}^3_p) \otimes \mathcal{F}$ by

$$
(\mathscr{T}\Psi)(p) = \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{-ix\cdot p} e^{ix\cdot {\rm P_f}} \Psi(x) dx.
$$

Then $H(p)$ is a nonnegative self-adjoint operator and

$$
\mathscr{T} \left(\int_{\mathbb{R}^3}^{\oplus} H(p) dp \right) \mathscr{T}^{-1} = H \tag{1.8}
$$

holds. As in the previous section, we move to Schrödinger representation from Fock representation to construct a functional integral representation. In that picture $H(p)$ becomes

$$
H(p) = \frac{1}{2}(p - {\rm d}\Gamma({\rm D}) - e\mathscr{A}(0))^2 + {\rm d}\Gamma(\omega({\rm D})), \quad p \in \mathbb{R}^3, \tag{1.9}
$$

on $L^2(\mathscr{Q})$. The functional integral representation of $e^{-tH(p)}$ can be also constructed as an application of that of e^{-tH}.

Theorem 1.3 *Let $\Psi, \Phi \in L^2(\mathscr{Q})$. Then*

$$
(\Psi, e^{-tH(p)}\Phi) = \mathbb{E}^0 \left[e^{ip\cdot B_t} \left({\rm J}_0 \Psi, e^{-ie\mathscr{A}_{\rm E}(K_t)} {\rm J}_t e^{-i{\rm d}\Gamma(\omega({\rm D}))\cdot B_t} \Phi \right)_{L^2(\mathscr{Q}_{\rm E})} \right]. \tag{1.10}
$$

1.4 Effective mass

Let $E(p) = \inf \mathrm{Spec}(H(p))$. Introducing a cutoff function with infrared cutoff $\kappa > 0$:

$$\hat{\varphi}(k) = \begin{cases} 0, & |k| < \kappa, \\ (2\pi)^{-3/2} & \kappa \le |k| \le \Lambda, \\ 0, & |k| > \Lambda, \end{cases}$$

we can see that $E(p)$ is analytic in p_μ for sufficiently small e. The effective mass m_{eff} is defined by

$$\frac{1}{m_{\text{eff}}} = \frac{1}{3}\Delta_p E(p)\Big\lceil_{p=0} \tag{1.11}$$

and we have expansion with respect to $\alpha = e^2/4\pi$:

$$m_{\text{eff}} = 1 + \frac{8}{3\pi}\left(\int_\kappa^\Lambda \frac{1}{r+2}dr\right)\alpha + a_2\alpha^2 + \cdots .$$

Then $a_1 \sim \log\Lambda$. The conventional claim is $a_n \sim (\log\Lambda)^n$ but our model does not satisfies this. In particular $a_2 \sim \sqrt{\Lambda}$ as $\Lambda \to \infty$ is shown in [HS05].

When the Hamiltonian includes spin 1/2, then

$$H(p) = \frac{1}{2}(p - \mathrm{P_f} - eA(0))^2 + \mathrm{d}\Gamma(\omega) - \frac{1}{2}\sigma \cdot B(0),$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the 2×2 Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $B(0) = \nabla \times A(0)$ denotes the quantized magnetic field. In this case the effective mass is also computed: $m_{\text{eff}} = 1 + a_1\alpha + a_2\alpha^2 + \cdots$, where

$$a_1 = \frac{8}{3\pi}\left(\int_\kappa^\Lambda \frac{1}{r+2}dr + \int_\kappa^\Lambda \frac{r^2}{(r+2)^3}dr\right) \tag{1.12}$$

and the behavior of a_2 is [HI05, HI07]

$$-C_1 < \lim_{\Lambda\to\infty} a_2/\Lambda^2 < C_2.$$

2 The dipole approximation

2.1 Symplectic structure

We first of all consider the perturbation of the annihilation operator and the creation operator by c-number. Then CCR leaves invariant.

Let $c(f) = a(f) + (g, f)$ and $c^*(f) = a^*(f) + (\bar{g}, f)$. Then $c(f)$ and $c^*(f)$ satisfy CCR and adjoint relation: $c(f)^* = c^*(\bar{f})$. Thus $c(f)$ and $c^*(f)$ satisfy the same CCR and adjoint relation as those of $a(f)$ and $a^*(f)$. Moreover the unitary operator $U = e^{-a^*(\bar{g})+a(g)}$ induces the unitary equivalence:

$$Ua^\sharp(f)U^{-1} = c^\sharp(f)$$

and also transforms the free Hamiltonian $\mathrm{d}\Gamma(\omega)$ to

$$U\mathrm{d}\Gamma(\omega)U^{-1} = \mathrm{d}\Gamma(\omega) + a^*(\omega\bar{g}) + a(\omega g) + (\omega g, g).$$

This can be extended to more complicated transformation $a(f) \mapsto b(f)$ and $a^*(f) \mapsto b^*(f)$ such that $b(f)$ and $b^*(f)$ satisfy the same CCR and adjoint relation as those of $a(f)$ and $a^*(f)$.

Let $B(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. Let

$$J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

where $\mathbb{1}$ denotes the identity operator on $\mathcal{H}$. For $S \in B(\mathcal{H})$ we define $\bar{S}f = \overline{S\bar{f}}$. Define

$$\mathrm{Sp}_\infty = \left\{ A = \begin{pmatrix} S & \bar{T} \\ T & \bar{S} \end{pmatrix} \in B(\mathcal{H}) \oplus B(\mathcal{H}) \,\middle|\, AJA^* = A^*JA = J \right\}.$$

Sp_∞ is called the infinite dimensional symplectic group. Let $A = \begin{pmatrix} S & \bar{T} \\ T & \bar{S} \end{pmatrix} \in \mathrm{Sp}_\infty$ and we set

$$b(f) = a(Sf) + a(Tf),$$
$$b^*(f) = a^*(\bar{S}f) + a(\bar{T}f).$$

Since $A \in \mathrm{Sp}_\infty$, $\{b(f), b^*(g)\}$ satisfies CCR and $b(f)^* = b^*(\bar{f})$. We furthermore define the subgroup of Sp_∞ by

$$\Sigma_2 = \left\{ A = \begin{pmatrix} S & \bar{T} \\ T & \bar{S} \end{pmatrix} \in \mathrm{Sp}_\infty \,\middle|\, T \text{ is a Hilbert Schmidt class} \right\}.$$

It is known ([Ber66, HI03]) that there exists a projective unitary representation[1] $U : \Sigma_2 \mapsto \{\text{unitary on } \mathcal{F}\}$ such that[2]

$$U(A)a^\sharp(f)U(A)^{-1} = b^\sharp(f) \tag{2.1}$$

for all $f \in \mathcal{H}$. Conversely if a unitary operator U satisfies (2.1), then $A \in \Sigma_2$. Using this fact, one can diagonalize quadratic Hamiltonians as

$$U\left(\mathrm{d}\Gamma(\omega) + (a^*(\bar{f}) + a(f))^2\right)U^{-1} = \mathrm{d}\Gamma(\omega) + C$$

with some constant C under some conditions. Furthermore we can see that there exists a unitary operator $\mathscr{U}_p$ such that

$$\mathscr{U}_p\left(\mathrm{d}\Gamma(\omega) + (p + a^*(\bar{f}) + a(f))^2\right)\mathscr{U}_p^{-1} = \mathrm{d}\Gamma(\omega) + C_p,$$

where $p \in \mathbb{R}$ is a parameter. See [Ara90].

2.2 Dipole approximation

Let us now consider the Pauli-Fierz Hamiltonian. We replace $A(x)$ in H with $1\!\!1 \otimes A(0)$, and the mass of electron is assumed to be m. Then H turns to be

$$H_{\mathrm{dip}} = \frac{1}{2m}(\mathrm{D} \otimes 1\!\!1 - e 1\!\!1 \otimes A(0))^2 + V \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega). \tag{2.2}$$

This is called the dipole approximation. Let $V = 0$. In the dipole approximation the Hamiltonian without external potential is *not* translation invariant but it commutes with the momentum operator of particle. Define $H_{\mathrm{dip}}(p)$ by

$$H_{\mathrm{dip}}(p) = \frac{1}{2m}(p - eA(0))^2 + \mathrm{d}\Gamma(\omega), \quad p \in \mathbb{R}^3,$$

acting on $\mathcal{F}$. Note that

$$\int_{\mathbb{R}^3}^{\oplus} H_{\mathrm{dip}}(p)dp = \frac{1}{2m}(\mathrm{D} \otimes 1\!\!1 - e 1\!\!1 \otimes A(0))^2 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega).$$

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[1] $U(A)U(B) = \omega(A,B)U(AB)$ with some phase $\omega(A,B)$.

[2] $U(A)$ is of the form

$$U(A) = \det(1\!\!1 - K_1^* K_1)^{1/4} e^{-\frac{1}{2}\langle a^* | K_1 | a^* \rangle} : e^{-\frac{1}{2}\langle a^* | K_2 | a \rangle} : e^{-\frac{1}{2}\langle a | K_3 | a \rangle},$$

where $K_1 = TS^{-1}$, $K_2 = 2(1\!\!1 - (S^{-1})^T)$ and $K_3 = -S^{-1}\bar{T}$. See [HI07].

Taking the dipole approximation makes the model drastically simpler. It is a quadratic operator as mentioned in the previous section. For each $p \in \mathbb{R}^3$ it can be indeed constructed the family of operators

$$\{b^*(f,p), b(f,p), f \in \mathcal{H}\}$$

such that [Ara83]

(1) $b^*(f,p)$ and $b(g,p)$ satisfy CCR;

(2) $b(g,p)^* = b^*(\bar{g},p)$;

(3) $[H_{\rm dip}(p), b(f,p)] = -b(\omega f,p)$ and $[H_{\rm dip}(p), b^*(f,p)] = b^*(\omega f,p)$.

We can also see that there exists a bounded operator S, a Hilbert-Schmidt operator T and a function L_p such that

$$\begin{aligned} b(f,p) &= a(Sf) + a^*(Tf) + (L_p, f), \\ b^*(f,p) &= a(\bar{T}f) + a^*(\bar{S}f) + (\bar{L}_p, f). \end{aligned}$$

Then $A = \begin{pmatrix} S & \bar{T} \\ T & \bar{S} \end{pmatrix} \in \Sigma_2$. There exists a unitary operator $S_p = e^{iep\cdot\phi}$ such that

(1) ϕ is of the form

$$\phi = i \sum_{j=\pm} (a^*(\bar{F}_j, j) - a(F_j, j))$$

with some function F_j,

(2) $U_p = S_p U(A)$ satisfies that

$$U_p a^\sharp(f) U_p^{-1} = b^\sharp(f), \quad U_p H_{\rm dip}(p) U_p^{-1} = \mathrm{d}\Gamma(\omega) + \frac{1}{2m_{\rm eff}} p^2 + g,$$

(3) constants $m_{\rm eff}$ and g are given by

$$m_{\rm eff} = m + \frac{2}{3} e^2 \|\hat{\varphi}/\omega\|^2, \quad g = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^2 t^2 \|\hat{\varphi}/(t^2+\omega^2)\|^2}{m + e^2 \frac{2}{3} \|\hat{\varphi}/\sqrt{t^2+\omega^2}\|^2} dt. \quad (2.3)$$

Let $U = e^{ie\mathrm{D}\otimes\phi}(\mathbb{1} \otimes U(A))$. Then

$$UH_{\rm dip}U^{-1} = -\frac{1}{2m_{\rm eff}} \Delta \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + g + V(\cdot - e\phi), \quad (2.4)$$

In particular $\inf \mathrm{Spec}(H_{\mathrm{dip}}) = g$ follows when $V = 0$. Let us take a special cutoff function

$$\hat{\varphi}(k) = \begin{cases} (2\pi)^{-3/2} & |k| \le \Lambda, \\ 0, & |k| > \Lambda. \end{cases}$$

Then $g \to \infty$ as $\Lambda \to \infty$. Indeed we can directly see that g has the bound:

$$e^2 \frac{8}{3} \left(\frac{3}{8\pi} \frac{1}{m} \right)^{1/2} \frac{\pi}{2} \le \lim_{\Lambda \to \infty} \frac{g}{\Lambda^{3/2}} \le e^2 \frac{8}{3} \left(\frac{9}{8\pi} \frac{1}{m} \right)^{1/2} \frac{\pi}{2}.$$

From (2.4) it follows that H_{dip} is unitary equivalent to

$$\left(-\frac{1}{2m_{\mathrm{eff}}} \Delta + V \right) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + g + e(V(\cdot - e\phi) - V). \tag{2.5}$$

It is seen that $m_{\mathrm{eff}} \sim e^2$ and $e(V(\cdot - e\phi) - V) \sim e\phi \cdot \nabla V \sim e$ when $\nabla \cdot V \in L^\infty$. Hence heuristically enhanced binding may occur under some conditions, i.e., the existence of ground state of H_{dip} can be shown for sufficiently large $|e|$ even when we do not assume the existence of ground state of $-\frac{1}{2m}\Delta + V$. The enhanced binding arising in H_{dip} is shown in [HS01].

2.3 Lorentz covariant Pauli-Fierz model

Quantization of the electromagnetic field does not cohere with normal postulates such as Lorentz covariance and existence of a positive definite metric. Then we chose to quantize in a manner sacrificing manifest Lorentz covariance; conversely if the electromagnetic field is quantized in a manifestly covariant fashion, the notion of a positive definite metric must be sacrificed and the existence of negative probability arising from the indefinite metric renders invalid a probabilistic interpretation of quantum field theory. One prescription for quantization of the electromagnetic field in a Lorentz covariant manner is the Gupta-Bleuler procedure ([Ble50, Gup50] and [KO79]).

Let us construct $A_\mu(f, x)$, $x \in \mathbb{R}^3$, $\mu = 0, 1, 2, 3$, with test function $f \in L^2(\mathbb{R}^3)$ such that $[A_\mu(f), A_\nu(g)] = -ig_{\mu\nu}(\bar{f}, g)$, where

$$g_{\mu\nu} = \begin{cases} 1 & \mu = \nu = 0, \\ -1, & \mu = \nu = 1, 2, 3, \\ 0, & \mu \ne \nu. \end{cases}$$

Let $\mathcal{F} = \mathcal{F}(\oplus^4 L^2(\mathbb{R}^3))$. The annihilation operator and the creation operator are denoted by $a(f, \mu)$ and $a^*(f, \mu)$, respectively. Define

$$a^\dagger(f, \mu) = \begin{cases} -a^*(f, 0), & \mu = 0, \\ a^*(f, \mu), & \mu = 1, 2, 3. \end{cases}$$

Then it follows that

$$[a(f,\mu), a^\dagger(g,\nu)] = -ig_{\mu\nu}(\bar{f}, g).$$

Let $e^j(k) \in \mathbb{R}^3$, $k \in \mathbb{R}^3$, $j = 1,2,3$, be unit vectors such that $e^3(k) = k/|k|$, and three vectors $e^1(k)$, $e^2(k)$ and $e^3(k)$ form a right-hand system for each $k \in \mathbb{R}^3$. We fix them. The quantized radiation field, smeared by the test function $f \in L^2(\mathbb{R}^3)$ at the time zero is defined by

$$A_\mu(f,x) = \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \int dk \frac{e^j_\mu(k)}{\sqrt{\omega(k)}} \left(a^*(k,j)\hat{f}(k)e^{-ikx} + a(k,j)\hat{f}(-k)e^{ikx} \right),$$

$$A_0(f,x) = \frac{1}{\sqrt{2}} \int dk \frac{1}{\sqrt{\omega(k)}} \left(a^*(k,0)\hat{f}(k)e^{-ikx} + a(k,0)\hat{f}(-k)e^{ikx} \right)$$

and their conjugate momenta by

$$\dot{A}_\mu(g,x) = \frac{i}{\sqrt{2}} \sum_{j=1}^{3} \int dk e^j_\mu(k)\sqrt{\omega(k)} \left(a^*(k,j)\hat{g}(k)e^{-ikx} - a(k,j)\hat{g}(-k)e^{ikx} \right),$$

$$\dot{A}_0(g,x) = \frac{i}{\sqrt{2}} \int dk \sqrt{\omega(k)} \left(a^*(k,0)\hat{g}(k)e^{-ikx} - a(k,0)\hat{g}(-k)e^{ikx} \right).$$

Set $A_\mu(f) = A_\mu(f,0)$. Note that $A_\mu(f)$, $\mu = 1,2,3$, are symmetric but $A_0(f)$ skew symmetric. We then have commutation relations between A_μ and $\dot{A}_\nu$:

$$[A_\mu(f), \dot{A}_\nu(g)] = -ig_{\mu\nu}(\bar{f}, g), \quad \mu,\nu = 0,1,2,3,$$

and $[A_\mu(f), A_\nu(g)] = 0$, $[\dot{A}_\mu(f), \dot{A}_\nu(g)] = 0$. Then the Lorentz covariant Pauli-Fierz Hamiltonian with the dipole approximation is defined by

$$H = \frac{1}{2}(\mathrm{D} \otimes \mathbb{1} - e\mathbb{1} \otimes A(0))^2 + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + e\mathbb{1} \otimes A_0(0).$$

Take the fiber p. Then we define

$$H(p) = \frac{1}{2}(p - eA(0))^2 + \mathrm{d}\Gamma(\omega) + eA_0(0).$$

This Hamiltonian is not self-adjoint on $\mathcal{F}$, since A_0 is skew symmetric. We introduce the indefinite scalar product on $\mathcal{F}$ by $(F|G) = (F, \Gamma[g]G)$, where $[g] = [g_{\mu\nu}] : \oplus^4 L^2(\mathbb{R}^3) \to \oplus^4 L^2(\mathbb{R}^3)$. Then $H(p)$ is symmetric with respect to $(\cdot|\cdot)$.

In [HS09] we prove the asymptotic completeness of $H(p)$ based on the LSZ method, and characterize the physical subspace of $H(p)$.

3 Relativistic Pauli-Fierz model

In quantum mechanics the relativistic Schrödinger operator is defined by

$$H_{\rm R}(a) = \sqrt{(p-a)^2+m^2} - m + V.$$

In this section the analogue version of the Pauli-Fierz model is defined and its functional integral representation is given. We would like to study the spectral property, effective mass and enhanced binding of the relativistic Pauli-Fierz Hamiltonian as well as the standard Pauli-Fierz Hamiltonian mentioned in the previous section. Some spectral property of the relativistic Pauli-Fierz model is studied in e.g., [HS10, KMS09, MS09].

In this section we overview the relativistic Pauli-Fierz Hamiltonian and the detail [Hir10] will be published somewhere.

3.1 Definition

The so-called relativistic Pauli-Fierz Hamiltonian is defined by

$$H_{\rm R} = \sqrt{({\rm D}\otimes 1\!\!1 - eA)^2+m^2} - m + V\otimes 1\!\!1 + 1\!\!1\otimes {\rm d}\Gamma(\omega) \tag{3.1}$$

on $L^2(\mathbb{R}^3)\otimes\mathcal{F}$ as a self-adjoint operator.

First of all we have to define $H_{\rm R}$. It is however not trivial to do it, since $H_{\rm R}$ has non-local operator $\sqrt{({\rm D}\otimes 1\!\!1 - eA)^2+m^2}$. Although one standard way to define $({\rm D}\otimes 1\!\!1 - eA)^2+m^2$ as a self-adjoint operator is to take the self-adjoint operator associated with the quadratic form:

$$F,G\mapsto \frac{1}{2}\sum_{\mu=1}^3(({\rm D}\otimes 1\!\!1 - eA)_\mu F, ({\rm D}\otimes 1\!\!1 - eA)_\mu G) + m^2(F,G),$$

we do not take it. Instead of this we will find a core of $({\rm D}\otimes 1\!\!1 - eA)^2+m^2$ by using a functional integration. Let

$$L_t = \oplus_{\mu=1}^3\int_0^t \tilde\varphi(\cdot - B_s)dB_s^\mu.$$

Then we can see that $\int dx\mathbb{E}^x\left[(F(B_0), e^{-ie\mathscr{A}(L_t)}G(B_t))\right]$ defines the semigroup generated by a self-adjoint operator K such that

$$(F, e^{-tK}G) = \int dx\mathbb{E}^x\left[(F(B_0), e^{-ie\mathscr{A}(L_t)}G(B_t))\right], \tag{3.2}$$

and see that

$$K \supset \frac{1}{2}(\mathrm{D} \otimes \mathbb{1} - e\mathscr{A})^2 \lceil_{D_{\mathrm{PF}}}. \tag{3.3}$$

Let $N = \mathbb{1} \otimes \mathrm{d}\Gamma(\mathbb{1})$ be the number operator and $\mathscr{D} = D(\Delta) \cap \cap_{n=1}^{\infty} D(N^n)$.

Lemma 3.1 *Suppose that* $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$. *Then* $\frac{1}{2}(\mathrm{D} \otimes \mathbb{1} - e\mathscr{A})^2 \lceil_{\mathscr{D}}$ *is essentially self-adjoint.*

Proof: By using (3.2) we will show that e^{-tK} leaves $\mathscr{D}$ invariant. First of all it can be proven that $e^{-tK}\mathscr{D} \subset D(\Delta)$. Next let us see that $e^{-tK}\mathscr{D} \subset \cap_{n=1}^{\infty} D(N^n)$. Let $z \in \mathbb{N}$ and $F, G \in D(N^\alpha)$. We have

$$(N^\alpha F, e^{-tK} G) = \int dx \mathbb{E}^x \left[(N^\alpha F(B_0), e^{-ie\mathscr{A}(L_t)} G(B_t)) \right]. \tag{3.4}$$

Let $\Pi(f) = i[N, A(f)]$. Note that

$$e^{ie\mathscr{A}(L_t)} N e^{-ie\mathscr{A}(L_t)} = N - e\Pi(L_t) + \frac{e^2}{2}\|L_t\|^2 \tag{3.5}$$

and then

$$\begin{aligned} &(N^\alpha F, e^{-tK} G) \\ &= \int dx \mathbb{E}^x \left[(F(B_0), e^{-ie\mathscr{A}(L_t)} \left(N - e\Pi(L_t) + \frac{e^2}{2}\|L_t\|^2 \right)^\alpha G(B_t)) \right]. \end{aligned} \tag{3.6}$$

By the Burkholder-Davis-Gundy type inequality,

$$\mathbb{E}^x \left[\left\| \int_0^t \tilde{\varphi}(\cdot - B_s) dB_s^\mu \right\|^{2z} \right] \le \frac{(2z)!}{2^\alpha} t^\alpha \|\hat{\varphi}\|^{2z}.$$

we can see that

$$\int dx \mathbb{E}^x \left[\left\| \left(N - e\Pi(L_t) + \frac{e^2}{2}\|L_t\|^2 \right)^\alpha F(B_t) \right\|^2 \right] \le C_\alpha^2 \|(N+1)^\alpha F\|^2 \tag{3.7}$$

with some constant C_α. Combining (3.6) and (3.7) we have

$$|(N^\alpha F, e^{-tK} G)| \le C_\alpha \|F\| \|(N+1)F\|. \tag{3.8}$$

This implies $e^{-tK} \cap_{n=1}^{\infty} D(N^n) \subset \cap_{n=1}^{\infty} D(N^n)$ and $e^{-tK}\mathscr{D} \subset \mathscr{D}$ follows. Hence K is essential self-adjoint on $\mathscr{D}$. $\square$

We denote the self-adjoint extension of $K\lceil_{\mathscr{D}}$ by the same symbol K for simplicity, and $\sqrt{2K+m^2}$ by the spectral resolution of K. Let $(T_t)_{t\geq 0}$ be the subordinator on a probability space $(\mathcal{T}, \mathcal{B}_{\mathcal{T}}, \nu)$ such that

$$\mathbb{E}_\nu^0[e^{-uT_t}] = \exp\left(-t(\sqrt{2u+m^2}-m)\right), \quad u \geq 0.$$

Since

$$(F, e^{-t(\sqrt{2K+m^2}-m)}G) = \mathbb{E}_\nu^0[(F, e^{-T_tK}G)],$$

we immediately have

$$(F, e^{-t(\sqrt{2K+m^2}-m)}G) = \int ds \mathbb{E}^{x,0}\left[(F(B_0), e^{-ie\mathscr{A}(L_{T_t})}G(B_{T_t}))\right]. \qquad (3.9)$$

From (3.9) we directly see the diamagnetic inequality:

$$|(F, e^{-t(\sqrt{2K+m^2}-m)}G)| \leq (|F|, e^{-t(\sqrt{-\Delta+m^2}-m)}|G|). \qquad (3.10)$$

From the diamagnetic inequality we have:

(1) Suppose that V is $\sqrt{-\Delta+m^2}-m$ -form bounded with a relative bound a. Then $|V|$ is also K-form bounded with a relative bound smaller than a.

(2) Suppose that V is relatively bounded with respect to $\sqrt{-\Delta+m^2}-m$ with a relative bound a, then V is also relatively bounded with respect to K with a relative bound a.

Let $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$. Suppose that $V = V_+ - V_-$ satisfies that V_- is relatively form bounded with respect to $\sqrt{-\Delta+m^2}-m$ and $D(V_+) \supset D(\Delta)$. Then H_{R} is defined by

$$H_{\mathrm{R}} = \sqrt{2K+m^2} - m \dotplus V_+ \otimes \mathbb{1} \dot{-} V_- \otimes \mathbb{1} \dotplus \mathbb{1} \otimes \mathrm{d}\Gamma(\omega). \qquad (3.11)$$

3.2 Functional integration

Now we will construct the functional integral representation of $e^{-tH_{\mathrm{R}}}$ through the Trotter product formula. We fix $t > 0$. Let $t_j = tj/2^n$, $j = 0, ..., 2^n$. Define $L^2(\mathbb{R}^4)$-valued stochastic process S_n^μ on $\mathscr{X} \times \mathcal{T}$ by

$$S_n^\mu = \sum_{j=1}^{2^n} \int_{T_{t_{j-1}}}^{T_{t_j}} \mathrm{j}_{t_{j-1}} f(\cdot - B_s) dB_s^\mu, \qquad (3.12)$$

where $f \in L^2(\mathbb{R}^3)$ and $\int_{T_{t_{j-1}}}^{T_{t_j}} \cdots dB_s^\mu = \int_T^S \cdots dB_s^\mu$ evaluated at $T = T_{t_{j-1}}$ and $S = T_{t_j}$.

Lemma 3.2 *$\{S_n^\mu\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mathscr{X}\times\mathcal{T};\mathscr{W}^x\otimes\nu)\otimes L^2(\mathbb{R}^4)$.*

Proof: Set S_n for S_n^μ for simplicity. We can directly see that

$$\int dx\mathbb{E}^{x,0}[\|S_{n+1}-S_n\|^2]\le\sum_{j=1}^{2^n}\int_{(2j-1)t/2^n}^{2jt/2^n}2\mathbb{E}^{0,0}[\|f(\cdot-x)\|^2]\frac{t}{2^{n+1}}.$$

Hence we have

$$\left(\int dx\mathbb{E}^{x,0}[\|S_m-S_n\|^2]\|\right)^{1/2}\le\|f\|\sum_{j=n+1}^{m}\frac{t}{2^{(j+1)/2}}$$

and it follows that S_n is a Cauchy sequence. □

We define the $L^2(\mathbb{R}^4)$-valued stochastic process $\int_0^{T_t}\mathrm{j}_{(T^{-1})_s}f(\cdot-B_s)dB_s^\mu$ on the probability space $(\mathscr{X}\times\mathcal{T},B(\mathscr{X})\times B_{\mathcal{T}},\mathscr{W}^x\otimes\nu)$ by the strong limit of S_n^μ:

$$\int_0^{T_t}\mathrm{j}_{(T^{-1})_s}f(\cdot-B_s)dB_s^\mu=\mathrm{s}-\lim_{n\to\infty}S_n^\mu. \tag{3.13}$$

Remark 3.3 *We give a remark with respect to (3.13). The subordinator $[0,\infty)\ni t\mapsto T_t\in[0,\infty)$ is monotonously increasing, but not injective. So the inverse T^{-1} can not be defined. (3.13) is a formal description of the limit of S_n^μ.*

Theorem 3.4 *Let $\omega^{3/2}\hat{\varphi}\in L^2(\mathbb{R}^3)$. Suppose that $V=V_+-V_-$ satisfies that V_- is relatively form bounded with respect to $\sqrt{-\Delta+m^2}-m$ and $D(V_+)\supset D(\Delta)$. Then*

$$(F,e^{-tH_{\mathrm{R}}}G)=\int dx\mathbb{E}^{x,0}\left[e^{-\int_0^t V(B_{T_s})ds}(\overline{\mathrm{J}_0F(B_0)},e^{-ie\mathscr{A}_{\mathrm{E}}(K_t^{\mathrm{rel}})}\mathrm{J}_tG(B_{T_t}))\right], \tag{3.14}$$

where $K_t^{\mathrm{rel}}=\oplus_{\mu=1}^3\int_0^{T_t}\mathrm{j}_{(T^{-1})_s}\tilde{\varphi}(\cdot-B_s)dB_s^\mu$.

Proof: We set $V=0$ for simplicity. By the Trotter product formula we have

$$(F,e^{-tH_{\mathrm{R}}}G)=\lim_{n\to\infty}\left(F,\left(e^{-t/2^nK}e^{-t/2^n\mathrm{d}\Gamma(\omega)}\right)^{2^n}G\right).$$

By the Markov property of $\mathrm{E}_t=\mathrm{J}_t^*\mathrm{J}_t$ the right hand side above is equal to

$$\lim_{n\to\infty}\left(\mathrm{J}_0F,\left(\prod_{j=0}^{2^n}e^{-t/2^n(\sqrt{(\mathrm{D}\otimes\mathbb{1}-e\mathscr{A}_{\mathrm{E}}(\mathrm{j}_{tj/2^n}\tilde{\varphi}))^2+m^2}-m)}\right)\mathrm{J}_tG\right).$$

Thus we have

$$(F, e^{-tH_{\rm R}}G) = \lim_{n\to\infty}\int dx \mathbb{E}^{x,0}\left[(\overline{{\rm J}_0F(B_0)}, e^{-ie\mathscr{A}_{\rm E}(K_t(n))}{\rm J}_tG(B_{T_t}))\right],$$

where

$$K_t(n) = \sum_{j=1}^{2^n}\int_{T_{t(j-1)/2^n}}^{T_{tj/2^n}} {\rm j}_{t(j-1)/2^n}\tilde\varphi(\cdot - B_s)dB_s^\mu.$$

By Lemma 3.2 and a limiting argument we can show the theorem for $V=0$. In the case of $H_{\rm R}$ with a bounded continuous V, we can also prove the theorem by the Trotter product formula. It can be also extended to $V = V_+ - V_-$ such that V_- is relatively form bounded with respect to $\sqrt{-\Delta+m^2}-m$ and $D(V_+)\supset D(\Delta)$ by a limiting argument. □

By using this functional integral representation we can see similar results to those of H.

Corollary 3.5 *Suppose the same assumptions as Theorem 3.4.*

(1) Let $E(e) = \inf {\rm Spec}(H_{\rm R})$. *Then*

$$|(F, e^{-tH_{\rm R}}G)| \le (|F|, e^{-t(\sqrt{-\Delta+m^2}-m+{\rm d}\Gamma(\omega))}|G|). \qquad (3.15)$$

In particular $E(0)\le E(e)$.

(2) Let $\mathfrak{S} = e^{-i(\pi/2)N}$. *Then* $\mathfrak{S}e^{-tH_{\rm R}}\mathfrak{S}^{-1}$ *is positivity improving. In particular the ground state of* $H_{\rm R}$ *is unique.*

3.3 Translation invariant relativistic Pauli-Fierz Hamiltonian

In the case of the relativistic Pauli-Fierz Hamiltonian with $V=0$, we can also show similar results to those of H by using the functional integral representation of $e^{-tH_{\rm R}}$, but we omit the detail. We give only the results. The relativistic Pauli-Fierz Hamiltonian with a fixed total momentum p, $H_{\rm R}(p)$, is defined by

$$H_{\rm R}(p) = \sqrt{(p - {\rm P_f} - eA(0))^2+m^2} - m + {\rm d}\Gamma(\omega), \quad p\in\mathbb{R}^3, \qquad (3.16)$$

with domain $D(H_{\rm R}(p)) = D({\rm d}\Gamma(\omega))\cap D(|{\rm P_f}|)$.

Theorem 3.6 *Suppose* $\omega^{3/2}\hat\varphi\in L^2(\mathbb{R}^3)$.

(1) $H_{\rm R}(p)$ *is essentially self-adjoint and* $H_{\rm R}\cong\int_{\mathbb{R}^3}^{\oplus} H_{\rm R}(p)dp$.

(2) Let $\Psi, \Phi \in \mathscr{Q}$. Then

$$(\Psi, e^{-tH_{\rm R}(p)}\Phi) = \mathbb{E}^{0,0}\left[e^{ip\cdot B_{T_t}}\left(J_0\Psi, e^{-ie\mathscr{A}_{\rm E}(K_t^{\rm rel})}J_t e^{-i{\rm P_f}\cdot B_{T_t}}\Phi\right)\right]. \quad (3.17)$$

From this functional integral representation we immediately have corollaries. Let $E(p) = \inf {\rm Spec}(H_{\rm R}(p))$.

Corollary 3.7 *(1) It follows that*

$$|(\Psi, e^{-tH_{\rm R}(p)}\Phi)| \le (|\Psi|, e^{-t(\sqrt{(p-{\rm P_f})^2+m^2}-m+{\rm d}\Gamma(\omega))}|\Phi|). \quad (3.18)$$

(2) $\mathfrak{S}^{-1}e^{-tH_{\rm R}(0)}\mathfrak{S}$ is positivity improving. In particular

(a) $E(0) \le E(p)$,

(b) the ground state of $H_{\rm R}(0)$ is unique if it exists.

References

[Ara83] A. Arai, Rigorous theory of spectra and radiation for a model in quantum electrodynamics, *J. Math. Phys.* **24** (1983), 1896–1910.

[Ara90] A.Arai, Perturbation of embedded eigenvalues: general class of exactly solvable models in Fock space, *Hokkaido Math.J.* **19** (1990), 1–34.

[Ber66] E. A. Berezin, *The method of second quantization*, Academic press, 1966.

[Ble50] K. Bleuler, Eine neue methode zur Behandlung der longitudinalen und skalaren photonen, *Helv. Phys. Acta.* **23** (1950), 567-586.

[Gup50] S. N. Gupta, Theory of longitudinal photon in quantum electrodynamics, *Proc. Phys. Soc.* **A63** (1950), 681-691.

[Hir10] F. Hiroshima, Non-local Pauli-Fierz model by path measures, in preparation.

[HI03] F. Hiroshima and K.R. Ito, Local exponent and infinitesimal generators of proper canonical transformations on a Boson Fock space, *Inf. Dim. Anal., Quant. Prob. and Rel. Top.* **7** (2004), 547-572.

[HI05] F. Hiroshima and K. R. Ito, Effective mass of nonrelativistic quantum electrodynamics, mp-arc 05-539, preprint 2005.

[HI07] F. Hiroshima and K.R. Ito, Mass renormalization in nonrelativistic quantum electrodynamics with spin 1/2, *Rev. Math. Phys.* **19** (2007), 405–454.

[HS10] F. Hiroshima and I. Sasaki,On the ionization energy of semi-relativistic Pauli-Fierz model for a single particle, to appear in RIMS Kôkyûroku Bessatsu.

[HS01] F. Hiroshima and H. Spohn, Enhanced binding through coupling to quantum field, *Ann. Henri Poincaré* **2** (2001), 1159–1187.

[HS05] F. Hiroshima and H. Spohn, Mass renormalization in nonrelativistic QED, *J. Math. Phys.* **46** (2005), 042302 (27 pages).

[HS09] F. Hiroshima and A. Suzuki, Physical state of nonrelativistic quantum electroduynamics, *Ann. Henri Poincaré* **10** (2009), 913–953.

[KO79] T. Kugo and I. Ojima, Local covariant operator formalism of non-abelian gauge theories and quark confinement problem, *Prog. Theor. Phys. Suppl.* **66** (1979), 1-130.

[KMS09] M. Könenberg, O. Matte and E. Stockmeyer, Existence of ground states of hydrogen likje atom in relativistic QEDI:The semi-relativistic Pauli-Fierz operator, arXiv:math-ph/09124223.v1, preprint 2009.

[MS09] T. Miyao and H. Spohn, Spectral analysis of the semi-relativistic Pauli-Fierz Hamiltonian, *J. Funct. Anal.* **256** (2009), 2123–2156.