Translation invariant models in nonrelativistic quantum electrodynamics

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1 The Pauli-Fierz Hamiltonian

In this paper we discuss translation invariant nonrelativistic quantum electrodynamics by *functional integrations*. We assume that an electron is in low energy, its density of charge is smoothly localized. In particular, the ultraviolet divergence does not exist. Let us see some classical model. Let $E(t,x)$ and $B(t,x)$, $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, be an electric field and a magnetic field respectively, and $q(t)$ the position of an electron at time $t \in \mathbb{R}$. The Maxwell equation with form factor φ is given by

$$\begin{aligned} \dot{B} &= -\nabla \times E, \\ \nabla \cdot B &= 0, \\ \dot{E} &= \nabla \times B - e\varphi(\cdot - q(t))\dot{q}(t), \\ \nabla \cdot E &= e\varphi(\cdot - q(t)). \end{aligned}$$

Let $(J,\rho) = (e\varphi(x - q(t))\dot{q}(t), e\varphi(x - q(t)))$. Then the Lagrangian density is given by

$$\mathcal{L}(t,x) = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}(E^2 - B^2) + J \cdot A - \rho\phi, \tag{1.1}$$

where A and ϕ are a vector potential and a scalar potential related to E and B such as $E = -\dot{A} - \nabla\phi$ and $B = \nabla \times A$. Let $L = \int_{\mathbb{R}^3} \mathcal{L}(t,x)dx$. Then the conjugate momenta are given by

$$p(t) := \frac{\partial L}{\partial \dot{q}} = m\dot{q}(t) + e\int A(t,x)\varphi(x - q(t))dx, \quad \Pi(t,x) := \frac{\delta L}{\delta \dot{A}} = \dot{A}(t,x).$$

Then the Hamiltonian is given through the Legendre transformation as

$$
\begin{aligned}
H_{cl} &= p\cdot\dot q + \int \dot A \Pi dx - L \\
&= \frac{1}{2m}\left(p - e\int A(t,x)\varphi(x-q(t))dx\right)^2 + \frac{1}{2}\int\left\{\dot A(t,x)^2 + (\nabla\times A(t,x))^2\right\}dx + V_{cl}(q),
\end{aligned}
$$

where V is a smeared external potential given by

$$
V_{cl}(q) := \frac{1}{2}e^2\int \frac{\varphi(q-y)\varphi(q-y')}{4\pi|y-y'|}dydy'.
$$

We quantized H_{cl} to define the Pauli-Fierz Hamiltonian.

Let us assume that the dimension of the state space is d and the photon is polarized to $d-1$ directions. Physically reasonable choice is $d=3$. Let $\mathcal{F}_{\rm b}$ be the Boson Fock space over $h_{\rm b} := \oplus^{d-1}L^2(\mathbb{R}^d)$, i.e., $\mathcal{F}_{\rm b} := \bigoplus_{n=0}^\infty[\otimes_s^n h_{\rm b}]$, where $\otimes_s^n h_{\rm b}$ denotes the n-fold symmetric tensor product of $h_{\rm b}$ with $\otimes_s^0 h_{\rm b} := \mathbb{C}$. $\Omega = \{1,0,0,...\}\in\mathcal{F}_{\rm b}$ is called the Fock vacuum. The annihilation operator and the creation operator on $\mathcal{F}_{\rm b}$ are denoted by $a(f)$ and $a^*(f)$, $f\in W$, respectively, and are defined by

$$
(a^*(f)\Psi)^{(n)} := \sqrt{n}S_n(f\otimes\Psi^{(n-1)}) \tag{1.2}
$$

and $a(f) := (a^*(\bar f))^*$, where S_n denotes the symmetrizer. Let $\mathcal{F}_{\rm b,fin}$ be the finite particle subspace of $\mathcal{F}_{\rm b}$. The annihilation operator and the creation operator leave $\mathcal{F}_{\rm b,fin}$ invariant and satisfy the canonical commutation relations on it:

$$
[a(f),a^*(g)] = (\bar f, g)1,\quad [a(f),a(g)]=0,\quad [a^*(f),a^*(g)]=0.
$$

For $f=(f_1,...,f_{d-1})\in\oplus^{d-1}L^2(\mathbb{R}^d)$, we informally write $a^\sharp(f)$, where $a^\sharp$ stands for a or a^*, as $a^\sharp(f) = \sum_{j=1}^{d-1}\int a^\sharp(k,j)f_j(k)dk$. The quantized radiation field $A_\mu(x)$ with a form factor φ is defined by

$$
A_\mu(x) = \frac{1}{\sqrt{2}}\sum_{j=1}^{d-1}\int e_\mu(k,j)\left(\frac{\hat\varphi(k)}{\sqrt{\omega(k)}}a^*(k,j)e^{-ik\cdot x} + \frac{\hat\varphi(-k)}{\sqrt{\omega(k)}}a(k,j)e^{ik\cdot x}\right)dk.
$$

Here $e(k,1),\cdots,e(k,d-1)$ denote generalized polarization vectors satisfying $k\cdot e(k,j)=0$ and $e(k,i)\cdot e(k,j)=\delta_{ij}1$, $i,j=1,...,d-1$, and $\hat\varphi$ is the Fourier transform of form factor φ. Note that

$$
\sum_{j=1}^{d-1}e_\alpha(k,j)e_\beta(k,j) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2} := \delta^\perp_{\alpha\beta}(k),\quad \alpha,\beta=1,...,d.
$$

Thus

$$
(A_\mu(x)\Omega, A_\nu(x)\Omega)_{\mathcal{F}_{\rm b}} = \frac{1}{2}\int_{\mathbb{R}^d}\frac{|\hat\varphi(k)|^2}{\omega(k)}\delta^\perp_{\mu\nu}(k)dk
$$

holds. Throughout this paper we use Assumption (A) below.

(A) *Form factor* $\hat{\varphi}$ *satisfies* $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^d)$ *and* $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) = \hat{\varphi}(k)$.

$A_\mu(x)$ is essentially self-adjoint on $\mathcal{F}_{\rm b,fin}$, and its unique self-adjoint extension is denoted by the same symbol. Next we define the second quantization. Let $\mathcal{C}(\mathcal{K} \to \mathcal{L})$ be the set of contraction operators from $\mathcal{K}$ to $\mathcal{L}$. The second quantization Γ is the functor:

$$\Gamma : \mathcal{C}(L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)) \to \mathcal{C}(\mathcal{F}_{\rm b} \to \mathcal{F}_{\rm b})$$

given by

$$\Gamma(T) := \bigoplus_{n=0}^{\infty} \otimes^n (\oplus^{d-1} T).$$

For a self-adjoint operator h on $L^2(\mathbb{R}^d)$, $\{\Gamma(e^{ith})\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\mathcal{F}_{\rm b}$. Then there exists a unique self-adjoint operator $d\Gamma(h)$ on $\mathcal{F}_{\rm b}$ such that $\Gamma(e^{ith}) = e^{itd\Gamma(h)}$. The number operator is defined by $N := d\Gamma(1)$. Let $\omega(k) = |k|$ be the multiplication operator on $L^2(\mathbb{R}^d)$. Define the free Hamiltonian $H_{\rm rad}$ on $\mathcal{F}_{\rm b}$ by

$$H_{\rm rad} := d\Gamma(\omega). \tag{1.3}$$

The Hilbert space $\mathcal{H}$ of state vectors for the total system under consideration is given by

$$\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\rm b}. \tag{1.4}$$

Under the identification $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{F}_{\rm b} dx$, we define the self-adjoint operator A on $\mathcal{H}$ by $A_\mu := \int_{\mathbb{R}^d}^{\oplus} A_\mu(x) dx$. The total Hamiltonian H, the so-called Pauli-Fierz Hamiltonian, is described by

$$H := \frac{1}{2}(-i\nabla \otimes 1 - eA)^2 + V \otimes 1 + 1 \otimes H_{\rm rad}, \tag{1.5}$$

where $e \in \mathbb{R}$ is a coupling constant. The proposition below is established in [H00b, H02].

Proposition 1.1 *Assume that* V *is relatively bounded with respect to* $-\Delta$ *with a relative bound strictly smaller than one. Then* H *is self-adjoint on* $D(H_0)$ *and essentially self-adjoint on any core of self-adjoint operator* $-(1/2)\Delta \otimes 1 + 1 \otimes H_{\rm rad}$*, and bounded from below,*

Define the field momentum by $P_{\rm f\mu} := d\Gamma(k_\mu)$ and the total momentum

$$P_\mu^{\rm T} := \overline{-i\nabla_\mu \otimes 1 + 1 \otimes P_{\rm f\mu}}, \tag{1.6}$$

where $\overline{X}$ denotes the closure of closable operator X. Now we set $V = 0$. Then it is seen that H is *translation invariant*;

$$e^{isP_\mu^{\rm T}} H e^{-isP_\mu^{\rm T}} = H, \quad s \in \mathbb{R}, \quad \mu = 1, ..., d.$$

Then we can decompose H on $\sigma(P_\mu^{\rm T}) = \mathbb{R}$. Define

$$H(P) := \frac{1}{2}(P - P_{\rm f} - eA(0))^2 + H_{\rm rad}, \quad P \in \mathbb{R}^d. \tag{1.7}$$

Note that $H(P)$ is a well defined symmetric operator on $D(H_{\rm rad}) \cap D({P_{\rm f}}^2)$ by assumption **(A)**. The next proposition is established in [H06, LMS06].

Proposition 1.2 *$H(P)$ is self-adjoint on $D(H_{\rm rad})\cap(\cap_{\mu=1}^d D(P_{{\rm f}\mu}^2))$ and it follows that*

$$\int_{\mathbb{R}^d}^{\oplus} H(P)dP \cong H. \tag{1.8}$$

So $H(P)$ is our main object and $P\in\mathbb{R}^d$ is called the total momentum. We want to investigate spectral properties of $H(P)$ by making use of functional integrations.

2 Functional integral representations

Let $(b(t))_{t\geq 0}=(b_1(t),\cdots,b_d(t))_{t\geq 0}$ be the d-dimensional Brownian motion starting at 0 on a probability space $(W,\mathcal{B},db)$. Set $X_s:=x+b(s)$, $x\in\mathbb{R}^d$, and $dX:=dx\otimes db$.

2.1 Functional integral representations for e^{-tH}

Let $\mathcal{A}_0(f)$ be a Gaussian random process on a probability space (Q_0,Σ_0,μ_0) indexed by real $f=(f_1,...,f_d)\in\overset{d}{\oplus}L^2(\mathbb{R}^d)$ with mean zero and covariance given by

$$\int_{Q_0}\mathcal{A}_0(f)\mathcal{A}_0(g)d\mu_0=q_0(f,g), \tag{2.1}$$

where

$$q_0(f,g):=\frac{1}{2}\sum_{\alpha,\beta=1}^d\int_{\mathbb{R}^d}\delta_{\alpha\beta}^{\perp}(k)\overline{\hat{f}_\alpha(k)}\hat{g}_\beta(k)dk.$$

The existence of probability space (Q_0,Σ_0,μ_0) and Gaussian random variable $\mathcal{A}_0(f)$ are known by the Minlos theorem. In a similar way, we can construct two other Gaussian random variables. Let $\mathcal{A}_1(f)$ indexed by real $f\in\overset{d}{\oplus}L^2(\mathbb{R}^{d+1})$ and $\mathcal{A}_2(f)$ by real $f\in\overset{d}{\oplus}L^2(\mathbb{R}^{d+2})$ be Gaussian random processes on probability spaces (Q_1,Σ_1,μ_1) and (Q_2,Σ_2,μ_2), respectively, with mean zero and covariances given by

$$\int_{Q_1}\mathcal{A}_1(f)\mathcal{A}_1(g)d\mu_1=q_1(f,g),\quad \int_{Q_2}\mathcal{A}_2(f)\mathcal{A}_2(g)d\mu_2=q_2(f,g), \tag{2.2}$$

where

$$q_1(f,g):=\frac{1}{2}\sum_{\alpha,\beta=1}^d\int_{\mathbb{R}^{d+1}}\delta_{\alpha\beta}^{\perp}(k)\overline{\hat{f}_\alpha(k,k_0)}\hat{g}_\beta(k,k_0)dkdk_0,$$

$$q_2(f,g):=\frac{1}{2}\sum_{\alpha,\beta=1}^d\int_{\mathbb{R}^{d+1+1}}\delta_{\alpha\beta}^{\perp}(k)\overline{\hat{f}_\alpha(k,k_0,k_1)}\hat{g}_\beta(k,k_0,k_1)dkdk_0dk_1.$$

From now on $q=0,1,2$. We extend it for $f=f_{\rm R}+if_{\rm I}$ with $f_{\rm R}=(f+\bar{f})/2$ and $f_{\rm I}=(f-\bar{f})/(2i)$ as $\mathcal{A}_q(f)=\mathcal{A}_q(f_{\rm R})+i\mathcal{A}_q(f_{\rm I})$. The n-particle subspace $L_n^2(Q_q)$ of $L^2(Q_q)$ is defined by

$$L_n^2(Q_q)=\overline{{\rm L.H.}\{:\mathcal{A}_q(f_1)\cdots\mathcal{A}_q(f_n):|f_j\in L^2(\mathbb{R}^{d+q}),j=1,...,n\}}.$$

Here $: X :$ denotes the Wick product of X. The identity $L^2(Q_q) = \oplus_{n=0}^\infty L^2_n(Q_q)$ is known as the Wiener-Itô decomposition. We also define the second quantization on $L^2(Q_q)$. Let $\Gamma_{qq'} : \mathcal{C}(L^2(\mathbb{R}^{d+q}) \to L^2(\mathbb{R}^{d+q'})) \to \mathcal{C}(L^2(Q_q) \to L^2(Q_{q'}))$ be defined by

$$\Gamma_{qq'}T1 = 1, \quad \Gamma_q(T) : \mathcal{A}_q(f_1) \cdots \mathcal{A}_q(f_n) :=: \mathcal{A}_{q'}([T]_d f_1) \cdots \mathcal{A}_{q'}([T]_d f_n) : .$$

Set $\Gamma_{qq} = \Gamma_q$ for simplicity. In particular since $\{\Gamma_q(e^{ith})\}_{t\in\mathbb{R}}$ with a self-adjoint operator h on $L^2(\mathbb{R}^d)$ is a strongly continuous one-parameter unitary group, there exists a self-adjoint operator $d\Gamma_q(h)$ on $L^2(Q_q)$ such that $\Gamma_q(e^{ith}) = e^{itd\Gamma_q(h)}$, $t \in \mathbb{R}$. We set $N_q := d\Gamma_q(1)$. Let h be a multiplication operator in $L^2(\mathbb{R}^d)$. We define the families of isometries,

$$L^2(\mathbb{R}^d) \xrightarrow{j_s} L^2(\mathbb{R}^{d+1}) \xrightarrow{\xi_t = \xi_t(h)} L^2(\mathbb{R}^{d+2}), \quad s, t \in \mathbb{R}, \tag{2.3}$$

by

$$\widehat{j_s f}(k, k_0) := \frac{e^{-isk_0}}{\sqrt{\pi}} \left(\frac{\omega(k)}{\omega(k)^2 + |k_0|^2} \right)^{1/2} \hat{f}(k), \quad (k, k_0) \in \mathbb{R}^d \times \mathbb{R}, \tag{2.4}$$

$$\widehat{\xi_t f}(k, k_0, k_1) := \frac{e^{-itk_1}}{\sqrt{\pi}} \left(\frac{h(k)}{h(k)^2 + |k_1|^2} \right)^{1/2} \hat{f}(k, k_0), \quad (k, k_0, k_1) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}.$$

Next, define the families of operators J_s and $\Xi_t = \Xi_t(h)$, $s, t \in \mathbb{R}$;

$$L^2(Q_0) \xrightarrow{J_s} L^2(Q_1) \xrightarrow{\Xi_t} L^2(Q_2)$$

by

$$J_s = \Gamma_{01}(j_s), \quad \Xi_t = \Gamma_{12}(\xi_t). \tag{2.5}$$

Define $\mathcal{A}_{q,\mu}(f) = \mathcal{A}_q(\oplus_{\ell=1}^d \delta_{\ell\mu} f)$. We see that $d\Gamma_0(-i\nabla) \cong P_{\rm f}$ and $d\Gamma_0(\omega(-i\nabla)) \cong H_{\rm rad}$. We can see that $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} L^2(Q_0) dx$ i.e., $F \in \mathcal{H}$ can be regarded as an $L^2(Q_0)$-valued L^2-function on $\mathbb{R}^d$. Note that in the Fock representation the test function $\hat{f}$ of $A_\mu(\hat{f})$ is taken in the momentum representation, but in the Schrödinger representation, f of $\mathcal{A}_{0,\mu}(f)$ in the position representation. We can see that

$$H \cong \frac{1}{2}(-i\nabla \otimes 1 - e\mathcal{A}_0^{\tilde{\varphi}})^2 + V \otimes 1 + 1 \otimes H_{\rm rad},$$

where $\tilde{\varphi} := (\hat{\varphi}/\sqrt{\omega})^\vee$. By the Feynman-Kac formula and the fact $J_0^* J_t = e^{-tH_{\rm rad}}$ we can see that

$$(F, e^{-t(-(1/2)\Delta + V + H_{\rm rad})} G)_{\mathcal{H}} = \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s) ds} (J_0 F(X_0), J_t G(X_t))_{L^2(Q_1)} dX.$$

Adding the minimal perturbation: $-i\nabla_\mu \otimes 1 \to -i\nabla_\mu \otimes 1 - e\mathcal{A}_0^{\hat{\varphi}}$, we have the functional integral representation below [H97].

$$(F, e^{-tH} G)_{\mathcal{H}} = \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s) ds} (J_0 F(X_0), e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(x))} J_t G(X_t))_{L^2(Q_1)} dX, \tag{2.6}$$

where $\mathcal{K}_1^{[0,t]}(x) := \oplus_{\mu=1}^d \int_0^t j_s \tilde{\varphi}(\cdot - X_s) db_\mu(s) \in \oplus^d L^2(\mathbb{R}^{d+1})$.

2.2 Functional integral representations for $e^{-tH(P)}$

We now construct the functional integral representation of $(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}}$. We use the identification $\mathcal{F}_{\rm b} \cong L^2(Q_0)$ without notices. For $\Psi \in L^2(Q_0)$, we set $\Psi_t := J_t e^{-iP_{\rm f}\cdot b(t)}\Psi$, $t \geq 0$.

Theorem 2.1 *Let* $\Psi, \Phi \in \mathcal{F}_{\rm b}$. *Then*

$$(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}} = \int_W (\Psi_0, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}\Phi_t)_{L^2(Q_1)} e^{iP\cdot b(t)} db, \tag{2.7}$$

where $\mathcal{K}_1^{[0,t]}(0) := \oplus_{\mu=1}^d \int_0^t j_s\tilde{\varphi}(\cdot - b(s))db_\mu(s)$.

Proof: We show an outline of the proof. See [H06] for detail. Set $F_s = \rho_s \otimes \Psi \in L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\rm b,fin}$ and $G_r = \rho_r \otimes \Phi \in L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\rm b,fin}$, where ρ_s is the heat kernel:

$$\rho_s(x) = (2\pi s)^{-d/2} e^{-|x|^2/(2s)}, \quad s > 0. \tag{2.8}$$

By the fact that $H = U^{-1}\left(\int_{\mathbb{R}^d}^{\oplus} H(P)dP\right)U$ and $Ue^{-i\xi\cdot P^{\rm T}}U^{-1} = \int_{\mathbb{R}^d}^{\oplus} e^{-i\xi\cdot P}dP$, we have

$$(F_s, e^{-tH}e^{-i\xi\cdot P^{\rm T}}G_r)_{\mathcal{H}} = \int_{\mathbb{R}^d} dP((UF_s)(P), e^{-tH(P)}e^{-i\xi\cdot P}(UG_r)(P))_{\mathcal{F}_{\rm b}}, \quad \xi \in \mathbb{R}^d.$$

Here $(UF_s)(P) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot P}e^{ix\cdot P_{\rm f}}\rho_s(x)\Psi dx$. Note that

$$\lim_{s\to 0}(UF_s)(P) = \frac{1}{\sqrt{(2\pi)^d}}\Psi \tag{2.9}$$

strongly in $\mathcal{F}_{\rm b}$ for each $P \in \mathbb{R}^d$. Hence we have by the Lebesgue dominated convergence theorem,

$$\lim_{s\to 0}(F_s, e^{-tH}e^{-i\xi\cdot P^{\rm T}}G_r)_{\mathcal{F}_{\rm b}} = \frac{1}{\sqrt{(2\pi)^d}}\int_{\mathbb{R}^d} dP(\Psi, e^{-tH(P)}e^{-i\xi\cdot P}(UG_r)(P))_{\mathcal{F}_{\rm b}}. \tag{2.10}$$

On the other hand we see that by (2.6)

$$\lim_{s\to 0}(F_s, e^{-tH}e^{-i\xi\cdot P^{\rm T}}G_r)_{\mathcal{H}} = \int_W \rho_r(b(t)-\xi)(J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-i\xi\cdot P_{\rm f}}\Phi)_{L^2(Q_1)}db. \tag{2.11}$$

Here we used that $\int_W db\rho_r(b_t + x - \xi)(J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(x))}J_t e^{-i\xi P_{\rm f}}\Phi)$ is continuous at $x = 0$ and $e^{-i\xi\cdot\tilde{P}_T}(\rho(X_t)\otimes\Phi) = \rho(X_t - \xi)\otimes e^{-i\xi\cdot P_{\rm f}}\Phi$. Then we obtained by (2.10) and (2.11) that

$$\begin{aligned} &\frac{1}{\sqrt{(2\pi)^d}}\int_{\mathbb{R}^d} e^{-i\xi\cdot P}(\Psi, e^{-tH(P)}(UG_r)(P))_{\mathcal{F}_{\rm b}}dP \\ &= \int_W \rho_r(b(t)-\xi)(J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-i\xi\cdot P_{\rm f}}\Phi)_{L^2(Q_1)}db. \end{aligned} \tag{2.12}$$

Since

$$\int_{\mathbb{R}^d} \|e^{-tH(P)}UG_r(P)\|^2_{\mathcal{F}_\mathrm{b}} dP \le \int_{\mathbb{R}^d} \|UG_r(P)\|^2_{\mathcal{F}_\mathrm{b}} dP = \|G_r\|^2_{\mathcal{H}} < \infty,$$

we have $(\Psi, e^{-tH(\cdot)}(UG_r)(\cdot))_{\mathcal{F}_\mathrm{b}} \in L^2(\mathbb{R}^d)$ for $r \ne 0$. Then taking the inverse Fourier transform of both sides of (2.12) with respect to P, we have

$$\begin{aligned}&(\Psi, e^{-tH(P)}(UG_r)(P))_{\mathcal{F}_\mathrm{b}}\\ &= \frac{1}{\sqrt{(2\pi)^d}} \int_W db \int_{\mathbb{R}^d} d\xi e^{iP\cdot\xi} \rho_r(b(t)-\xi)(J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-i\xi\cdot P_\mathrm{f}}\Phi)_{L^2(Q_1)} \quad (2.13)\end{aligned}$$

for almost every $P \in \mathbb{R}^d$. Both sides (2.13) are continuous in P, then (2.13) is true for all $P \in \mathbb{R}^d$. Taking $r \to 0$ on both sides of (2.13), we have by the Lebesgue dominated convergence theorem and (2.9) that

$$(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_\mathrm{b}} = \int_W (J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_\mathrm{f}\cdot b(t)}\Phi)_{L^2(Q_1)} e^{iP\cdot b(t)} db = (2.7).$$

Thus the theorem follows for $\Psi, \Phi \in \mathcal{F}_{\mathrm{b,fin}}$. Let $\Psi, \Phi \in \mathcal{F}_\mathrm{b}$, and $\Psi_n, \Phi_n \in \mathcal{F}_{\mathrm{b,fin}}$ such that $\Psi_n \to \Psi$ and $\Phi_n \to \Phi$ strongly as $n \to \infty$. Since

$$|(J_0\Psi_n, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_\mathrm{f}\cdot b(t)}\Phi_n)_{L^2(Q_1)}| \le \|\Psi_n\|_{\mathcal{F}_\mathrm{b}} \|\Phi_n\|_{\mathcal{F}_\mathrm{b}} \le c$$

with some constant c independent of n, we have by the Lebesgue dominated convergence theorem

$$\begin{aligned}&\lim_{n\to\infty} \int_W (J_0\Psi_n, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_\mathrm{f}\cdot b(t)}\Phi_n)_{L^2(Q_1)} e^{iP\cdot b(t)} db\\ &= \int_W (J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_\mathrm{f}\cdot b(t)}\Phi)_{L^2(Q_1)} e^{iP\cdot b(t)} db,\end{aligned}$$

and it is immediate that $\lim_{n\to\infty}(\Psi_n, e^{-tH(P)}\Phi_n)_{\mathcal{F}_\mathrm{b}} = (\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_\mathrm{b}}$. Hence (2.7) is proven. **qed**

2.3 Applications

Let $L^2_{\mathrm{fin}}(Q_q) := \bigcup_{N=0}^{\infty} [\oplus_{n=0}^N L^2_n(Q_q)]$ and T a self-adjoint operator on $L^2(\mathbb{R}^{d+q})$. Let us define the operator $\Pi_{q,\mu}(Tf)$ on $L^2_{\mathrm{fin}}(Q_q)$ by

$$\Pi_{q,\mu}(Tf) := i[d\Gamma_q(T), \mathcal{A}_{q,\mu}(f)]$$

for $f \in D(T)$. In the case f is real-valued, $\Pi_{q,\mu}(Tf)$ is a symmetric operator. The self-adjoint extension of $\Pi_{q,\mu}(f)$ with real f is denoted by the same symbol.

Let $\mathcal{K}_+ := \{\Psi \in L^2(Q_0) | \Psi \ge 0\}$ and $\mathcal{K}_+^0 := \{\Psi \in \mathcal{K}_+ | \Psi > 0\}$. It is well known that $e^{iP_\mathrm{f}\cdot v}\mathcal{K}_+ \subset \mathcal{K}_+$ for $v \in \mathbb{R}^d$. Fundamental fact is that for real $f \in L^2(\mathbb{R}^{d+1})$,

$$J_0^* e^{i\Pi_{1,\mu}(f)} J_t [\mathcal{K}_+ \setminus \{0\}] \subset \mathcal{K}_+^0, \quad (2.14)$$

i.e., $J_0^* e^{i\Pi_{1,\mu}(f)} J_t$ is positivity improving. See [H00a]. We define $\vartheta := \exp\left(i\frac{\pi}{2}N\right)$.

Theorem 2.2 *$\vartheta e^{-tH(0)}\vartheta^{-1}$ is positivity improving.*

Proof: Let $\Psi,\Phi\in\mathcal{K}_+\setminus\{0\}$. It is seen that

$$(\Psi,\vartheta e^{-tH(0)}\vartheta^{-1}\Phi)_{\mathcal{F}_{\rm b}}=\int_W(\Psi,J_0^*e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))}J_te^{-iP_{\rm f}\cdot b(t)}\Phi)_{L^2(Q_0)}db. \qquad (2.15)$$

Here we used the facts that $J_te^{-iP_{\rm f}\cdot b(t)}e^{-i(\pi/2)N}=e^{-i(\pi/2)\tilde{N}}J_te^{-iP_{\rm f}\cdot b(t)}$ and

$$e^{i(\pi/2)\tilde{N}}e^{-ie\mathcal{A}_1(f)}e^{-i(\pi/2)\tilde{N}}=e^{-ie\Pi_1(f)},$$

where $\tilde{N}=d\Gamma_1(1)$. Since $J_0^*e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))}J_te^{-iP_{\rm f}\cdot b(t)}$ is positivity improving for each $b\in W$, specifically the integrand in (2.15) is strictly positive for each $b\in W$. Hence the right-hand side of (2.15) is strictly positive, which implies that $\vartheta e^{-tH(0)}\vartheta^{-1}\mathcal{K}_+\setminus\{0\}\subset\mathcal{K}_+^0$. Thus the theorem follows. **qed**

Immediate corollaries are as follows.

Corollary 2.3 *The ground state $\varphi_{\rm g}(0)$ of $H(0)$ is unique up to multiple constants, if it exists, and it can be taken as $\vartheta\varphi_{\rm g}(0)>0$ in the Schrödinger representation.*

Corollary 2.4 *It follows that*

$$|(\Psi,e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}}|\leq(|\Psi|,e^{-t(\frac{1}{2}{P_{\rm f}}^2+H_{\rm rad})}|\Phi|)_{L^2(Q_0)}, \qquad (2.16)$$
$$|(\Psi,\vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_{\rm b}}|\leq(|\Psi|,\vartheta e^{-tH(0)}\vartheta^{-1}|\Phi|)_{L^2(Q_0)}. \qquad (2.17)$$

Proof: When L is positivity preserving, we have $|L\Psi|\leq L|\Psi|$. Furthermore,

$$|(\Psi,e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}}|\leq\int_W(J_0|\Psi|,J_te^{-iP_{\rm f}\cdot b(t)}|\Phi|)_{L^2(Q_1)}db=(|\Psi|,e^{-t(\frac{1}{2}{P_{\rm f}}^2+H_{\rm rad})}|\Phi|)_{L^2(Q_0)}$$

where we used that $b(t)$ is Gaussian with $\int|b_\mu(t)|^2db=1/2$. Thus (2.16) follows. We have

$$(\Psi,\vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_{\rm b}}=\int_W(\Psi_0,e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))}\Phi_t)_{L^2(Q_1)}e^{iP\cdot b(t)}db. \qquad (2.18)$$

Then

$$|(\Psi,\vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_{\rm b}}|\leq(|\Psi|,\vartheta e^{-tH(0)}\vartheta^{-1}|\Phi|)_{L^2(Q_0)}.$$

Hence (2.17) follows. **qed**

Let $E(P,e^2)=\inf\sigma(H(P))$.

Corollary 2.5 *(1)* $0=E(0,0)\leq E(0,e^2)\leq E(P,e^2)$, *(2) Assume that the ground state $\varphi_{\rm g}(0)$ of $H(0)$ exists for $e\in[0,e_0)$ with some $e_0>0$. Then $E(0,e^2)$ is concave, continuous and monotonously increasing function on e^2, (3) $E(0,e^2)\leq\inf\sigma(H)$.*

Proof: (2.17) implies $|(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Psi)_{\mathcal{F}_{\rm b}}| \le e^{-tE(0,e^2)}\|\Psi\|^2_{\mathcal{F}_{\rm b}}$. Since ϑ is unitary, (1) follows. Let $\varphi_{\rm g}(0)$ be the ground state of $H(0)$. Thus by Corollary 2.3, $(1,\varphi_{\rm g}(0))_{L^2(Q_0)} \neq 0$. Hence

$$E(0,e^2) = \lim_{t\to\infty} -\frac{1}{t}\log(\Omega, e^{-tH(0)}\Omega)_{\mathcal{F}_{\rm b}} = \lim_{t\to\infty} -\frac{1}{t}\log\int_W e^{-\frac{e^2}{2}q_0(\mathcal{K}_1^{[0,t]}(0),\mathcal{K}_1^{[0,t]}(0))}db.$$

Since $e^{-\frac{e^2}{2}q_0(\mathcal{K}_1^{[0,t]}(0),\mathcal{K}_1^{[0,t]}(0))}$ is log convex on e^2, $E(0,e^2)$ is concave. Then $E(0,e^2)$ is continuous on $(0,e_0)$. Since $E(0,e^2)$ is also continuous at $e^2=0$ by the fact that $H(0)$ converges as $e^2 \to 0$ in the uniform resolvent sense, $E(0,e^2)$ is continuous on $[0,e_0)$. Then $E(0,e^2)$ can be expressed as $E(0,e^2) = \int_0^{e^2}\phi(t)dt$ with some positive function ϕ. Thus $E(0,e^2)$ is monotonously increasing on e^2. Then (2) is obtained. We have

$$(F,(1\otimes\vartheta)e^{-tH}(1\otimes\vartheta^{-1})G)_{\mathcal{H}} = \int_{\mathbb{R}^d} dP(F(P),\vartheta e^{-tH(P)}\vartheta^{-1}G(P)))_{\mathcal{F}_{\rm b}}.$$

Then by (2.17) it is seen that

$$|(F,(1\otimes\vartheta)e^{-tH}(1\otimes\vartheta^{-1})F)_{\mathcal{H}}| \le e^{-tE(0,e^2)}\int_{\mathbb{R}^d} dP\|F(P)\|^2_{\mathcal{F}_{\rm b}} = e^{-tE(0,e^2)}\|F\|^2_{\mathcal{H}}.$$

Thus (3) follows. **qed**

3 The n point Euclidean Green functions

The functional integral representations derived in the previous section can be extended to the n point Euclidean Green functions.

Theorem 3.1 *Let $K = d\Gamma(h)$ with a multiplication operator h in $L^2(\mathbb{R}^d)$. We assume that $\Phi_0, \Phi_m \in \mathcal{F}_{\rm b}$ and $\Phi_j \in \mathcal{F}_{\rm b}^\infty$ for $j=1,...,m-1$ with $\Phi_j = \Phi_j(A(f_1^j),\cdots,A(f_{n_j}^j))$. Then for $P_0,\cdots,P_{m-1}\in\mathbb{R}^d$,*

$$\begin{aligned}&(\Phi_0, \prod_{j=1}^m e^{-(s_j-s_{j-1})K}e^{-(t_j-t_{j-1})H(P_{j-1})}\Phi_j)_{\mathcal{F}_{\rm b}}\\ &= \int_W (\hat\Phi_0, e^{-ie\mathcal{A}_2(\mathcal{K}_2(0))}\prod_{j=1}^m\hat\Phi_j)_{L^2(Q_2)} e^{+i\sum_{j=1}^m(b(t_j)-b(t_{j-1}))P_{j-1}}db,\end{aligned} \tag{3.1}$$

where $\mathcal{K}_2(0) := \oplus_{\mu=1}^d \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \xi_{s_j} j_s \tilde{\hat\varphi}(\cdot - b(s))db_\mu(s)$ and

$$\hat\Phi_j := \Xi_{s_j}J_{t_j}e^{-iP_{\rm f}\cdot b(t_j)}\Phi_j = \Phi_j\left(\mathcal{A}_2\left(\xi_{s_j}j_{t_j}f_1^j(\cdot-b(t_j))\right),\cdots,\mathcal{A}_2\left(\xi_{s_j}j_{t_j}f_{n_j}^j(\cdot-b(t_j))\right)\right).$$

Proof: See [H06] for detail.

We shall show some applications of Theorem 3.1, by which we can construct a sequence of measures on W converging to $(\varphi_{\rm g}(P), T\varphi_{\rm g}(P))_{\mathcal{F}_{\rm b}}$ for some bounded operator T. In particular $T = e^{-\beta N}$ and $T = e^{-iA(f)}$ are taken as examples. It is known that $H(P)$ has a unique ground state $\varphi_{\rm g}(P)$ and $(\varphi_{\rm g}(P), \Omega)_{\mathcal{F}_{\rm b}} \neq 0$ for sufficiently small e.

Corollary 3.2 *We suppose that $H(P)$ has the unique ground state $\varphi_{\rm g}(P)$ and it satisfies $(\varphi_{\rm g}(P), \Omega)_{\mathcal{F}_{\rm b}} \neq 0$. Then for $\beta > 0$,*

$$(\varphi_{\rm g}(P), e^{-\beta N}\varphi_{\rm g}(P)) = \lim_{t\to\infty} \int_W e^{(e^2/2)(1-e^{-\beta})D(t)} e^{iP\cdot b(2t)} d\mu_{2t},$$

where $D(t) := q_1(\mathcal{K}_1^{[0,t]}(0), \mathcal{K}_1^{[t,2t]}(0))$ and μ_{2t} is a measure on W given by

$$d\mu_{2t} := \frac{1}{Z} e^{-(e^2/2)q_1(\mathcal{K}_1^{[0,2t]}(0), \mathcal{K}_1^{[0,2t]}(0))} db$$

with normalizing constant Z such that $\int_W e^{iP\cdot b(2t)} d\mu_{2t} = 1$.

Proof: We define the family of isometries $\xi_s = \xi_s(1)$, $s \in \mathbb{R}$, by (2.3). By Theorem 3.1 we have

$$\begin{aligned}(e^{-tH(P)}\Omega, e^{-\beta N} e^{-tH(P)}\Omega)_{\mathcal{F}_{\rm b}} &= \int_W db e^{iP\cdot b(2t)} (1, e^{-ie\mathcal{A}_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0))} 1)_{L^2(Q_2)} \\ &= \int_W db e^{iP\cdot b(2t)} e^{-(e^2/2) q_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0))}.\end{aligned}$$

Noticing that $q_2(\xi_s f, \xi_t g) = e^{-|s-t|} q_1(f, g)$, we have

$$q_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0)) = q_1(\mathcal{K}_1^{[0,2t]}(0), \mathcal{K}_1^{[0,2t]}(0)) - (1 - e^{-\beta}) q_1(\mathcal{K}_1^{[0,t]}(0), \mathcal{K}_1^{[t,2t]}(0)).$$

Then

$$\frac{(e^{-tH(P)}\Omega, e^{-\beta N} e^{-tH(P)}\Omega)}{(e^{-tH(P)}\Omega, e^{-tH(P)}\Omega)} = \int_W e^{(e^2/2)(1-e^{-\beta})D(t)} e^{iP\cdot b(2t)} d\mu_{2t}. \tag{3.2}$$

The corollary follows from (3.2) and

$$s - \lim_{t\to\infty} \frac{e^{-tH(P)}\Omega}{\|e^{-tH(P)}\Omega\|_{\mathcal{F}_{\rm b}}} = \frac{(\varphi_{\rm g}(P), \Omega)_{\mathcal{F}_{\rm b}}}{|(\varphi_{\rm g}(P), \Omega)_{\mathcal{F}_{\rm b}}|} \cdot \varphi_{\rm g}(P)$$

qed

Corollary 3.3 *Assume the same assumptions as in Corollary 3.2. Then*

$$(\varphi_{\rm g}(P), e^{-iA(f)}\varphi_{\rm g}(P))_{\mathcal{F}_{\rm b}} = \lim_{t\to\infty} \int_W e^{-eq_1(\mathcal{K}_1^{[0,2t]}(0), f^t) - \frac{1}{2}q_0(f,f)} e^{iP\cdot b(2t)} d\mu_{2t}, \tag{3.3}$$

where $f^t := \oplus_{\mu=1}^d j_t f_\alpha(\cdot - b(t))$.

Proof: We have by Theorem 3.1

$$
\begin{aligned}
(\varphi_{\rm g}(P), e^{-iA(f)}\varphi_{\rm g}(P))_{\mathcal{F}_{\rm b}} &= \lim_{t\to\infty} \frac{(e^{-tH(P)}\Omega, e^{-iA(f)}e^{-tH(P)}\Omega)_{\mathcal{F}_{\rm b}}}{(e^{-tH(P)}\Omega, e^{-tH(P)}\Omega)_{\mathcal{F}_{\rm b}}} \\
&= \lim_{t\to\infty} \frac{1}{Z}\int_W db e^{iP\cdot b(2t)} (1, e^{-i(e\mathcal{A}_1(\mathcal{K}_1^{[0,2t]}(0))+\mathcal{A}_1(j_t f))}1)_{L^2(Q_1)} \\
&= \lim_{t\to\infty} \frac{1}{Z}\int_W db e^{iP\cdot b(2t)} e^{-\frac{1}{2}q_1(e\mathcal{K}_1^{[0,2t]}(0)+f^t)}.
\end{aligned}
$$

Note that $q_1(f^t, f^t) = q_0(f, f)$. Then the corollary follows. **qed**

Remark 3.4 *It is informally written as*

$$
\begin{aligned}
& q_1(\mathcal{K}_1^{[S,T]}(0), \mathcal{K}_1^{[S',T']}(0)) \\
&= \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \int_S^T db_\alpha(s) \int_{S'}^{T'} db_\beta(r) \int_{\mathbb{R}^d} \delta^\perp_{\alpha\beta}(k) \frac{|\hat{\varphi}(k)|^2}{\omega(k)} e^{-|s-r|\omega(k)} e^{-ik(b(s)-b(r))} dk.
\end{aligned}
$$

and

$$
q_1(\mathcal{K}_1^{[0,2t]}(0), f^t) = \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \int_0^{2t} db_\alpha(s) \int_{\mathbb{R}^d} \delta^\perp_{\alpha\beta}(k) \frac{\overline{\hat{\varphi}(k)}}{\sqrt{\omega(k)}} \hat{f}_\beta(k) e^{ik\cdot(b(s)-b(t))} e^{-|s-t|\omega(k)} dk.
$$

4 The Pauli-Fierz Hamiltonian with spin $1/2$

Let us include the spin of the electron. Let $d = 3$ and $\sigma_1, \sigma_2, \sigma_3$ be the 2×2 Pauli matrices given by

$$
\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

The Pauli-Fierz Hamiltonian with spin 1/2 is defined by

$$
H_\sigma(P) = \frac{1}{2}(P - P_{\rm f} - eA(0))^2 + H_{\rm rad} - \frac{e}{2}\sum_{\mu=1}^{3} \sigma_\mu B_\mu(0),
$$

where $B(0) = {\rm rot}A(x)$. Although $H_\sigma(P)$ acts on $\mathbb{C}^2 \otimes \mathcal{F}_{\rm b}$, it can be reduced to the self-adjoint operator on $L^2(\mathbb{Z}/2\mathbb{Z}; Q_0)$. The functional integral representation of $e^{-tH_\sigma(P)}$ can be also constructed by making use of $3+1$ dimensional Lévy process $(b(t), N_t)$ with values in $\mathbb{R}^3 \times (\mathbb{N}\cup\{0\})$, where N_t denotes the Poisson process on a measure space $(S, \Sigma, P_{\rm P})$ with $\mathbb{E}_{P_{\rm P}}[N_t = N] = e^{-t}t^N/N!$. For $\sigma \in \mathbb{Z}/2\mathbb{Z}$ we define $\sigma_t = \sigma(-1)^{N_t}$. Let $\mathcal{B}_q(x) = {\rm rot}\mathcal{A}_q(x)$. The net result is

Theorem 4.1 *Let* $\Phi, \Psi \in L^2(\mathbb{Z}/2\mathbb{Z}; Q_0)$. *Then*

$$(\Phi, e^{-tH_\sigma(P)}\Psi) = \lim_{\epsilon\to 0} e^t \sum_{\sigma\in\mathbb{Z}/2\mathbb{Z}} \int_{W\times S} db \otimes dP_{\mathrm{P}} \left[e^{iP\cdot b(t)} \int_{Q_1} d\mu_1 \overline{J_0\Phi(\sigma)} e^{X_t^\epsilon} J_t e^{-iP_{\mathrm{f}}\cdot b(t)} \Psi(\sigma_t) \right], \tag{4.1}$$

where

$$X_t = -ie\sum_{\mu=1}^3 \int_0^t \mathcal{A}_{1,\mu}(\lambda(\cdot - b(s)))db_s^\mu - \int_0^t (-\frac{e}{2})\sigma_s \mathcal{B}_{1,3}(j_s\lambda(\cdot - b(s)))ds$$
$$+ \int_0^{t+} \log\left(-H_{\mathrm{od}}(b(s), -\sigma_{s-}, s) - \epsilon\psi_\epsilon(H_{\mathrm{od}}(b(s), -\sigma_{s-}, s))\right) dN_s$$

and

$$H_{\mathrm{od}}(x, -\sigma, s) = \frac{e}{2}(\mathcal{B}_{1,1}(j_s\lambda(\cdot - b(s)) - i\sigma\mathcal{B}_{1,2}(j_s\lambda(\cdot - b(s)))$$

with the indicator function $\psi_\epsilon(x) = \begin{cases} 1, & |x| < \epsilon/2, \\ 0, & |x| \ge \epsilon/2. \end{cases}$

Proof: See [HL07] for detail. **qed**

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