# Effective mass and mass renormalization of nonrelativistic QED

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November 29, 2003

**Abstract**

The effective mass $m_{\text{eff}}$ of the nonrelativistic QED is considered. $m_{\text{eff}}$ is defined as the inverse of curvature of the ground state energy with total momentum zero. The effective mass $m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)$ is a function of bear mass $m > 0$, ultraviolet cutoff $\Lambda > 0$, infrared cutoff $\kappa > 0$, and the square of charge $e$ of an electron. Introduce a scaling $m \to m(\Lambda) = (b\Lambda)^\beta$, $\beta < 0$. Then asymptotics behavior of $m_{\text{eff}}$ as $\Lambda \to \infty$ is studied.

# 1 Introduction

## 1.1 The Pauli-Fierz Hamiltonian

This is a joint work with Herbert Spohn.[1] We consider a single, spinless free electron coupled to a quantized radiation field (photons). The Hilbert space of states of photons is the symmetric Fock space:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \left[ \otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\}) \right],$$

where $\otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\})$ denotes the $n$-fold symmetric tensor product of $L^2(\mathbb{R}^3 \times \{1,2\})$ with $\otimes_s^0 L^2(\mathbb{R}^3 \times \{1,2\}) = \mathbb{C}$. The inner product in $\mathcal{F}$ is denoted by $(\cdot,\cdot)$ and the Fock vacuum by $\Omega$. On $\mathcal{F}$ we introduce the Bose field

$$a(f) = \sum_{j=1,2} \int f(k,j)^* a(k,j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1,2\}), \tag{1.1}$$

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†This work is partially supported by Grant-in-Aid for Science Reserch C 1554019 from MEXT.

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where $a(f)$ and $a^*(f) = a(\bar{f})^*$ are densely defined and satisfy the CCR

$$[a(f), a^*(g)] = (f, g)_{L^2(\mathbb{R}^3 \times \{1,2\})},$$
$$[a(f), a(g)] = 0,$$
$$[a^*(f), a^*(g)] = 0.$$

The free Hamiltonian of $\mathcal{F}$ is read as

$$H_{\rm f} = \sum_{j=1,2} \int \omega(k) a^*(k, j) a(k, j) dk, \tag{1.2}$$

where the dispersion relation is given by

$$\omega(k) = |k|.$$

The free Hamiltonian $H_{\rm f}$ acts as

$$H_{\rm f}\Omega = 0,$$
$$H_{\rm f} a^*(f_1) \cdots a^*(f_n)\Omega = \sum_{j=1}^{n} a^*(f_1) \cdots a^*(\omega f_j) \cdots a^*(f_n)\Omega.$$

The Pauli-Fierz Hamiltonian $H$ is defined as a self-adjoint operator acting on

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} dx$$

by

$$H = \frac{1}{2m}(p_x \otimes 1 - eA_{\hat{\varphi}})^2 + V \otimes 1 + 1 \otimes H_{\rm f},$$

where $m$ and $e$ denote the mass and charge of electron, respectively,

$$p_x = \left( -i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, -i\frac{\partial}{\partial x_3} \right)$$

and $V$ an external potential. The quantized radiation field $A_{\hat{\varphi}}$ is defined by

$$A_{\hat{\varphi}} = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3}^{\oplus} (a(f_x) + a^*(\bar{f}_x)) dx, \tag{1.3}$$

where

$$f_x(k, j) = \frac{1}{\sqrt{\omega}} \hat{\varphi}(k) e(k, j) e^{ikx}, \tag{1.4}$$

$e(k, 1), e(k, 2), k/|k|$ form a right-handed dreibain, and $\hat{\varphi}$ is a form factor. $A_{\hat{\varphi}}$ acts for $\Psi \in \mathcal{H}$ as

$$(A_{\hat{\varphi}}\Psi)(x) = (a(f_x) + a^*(\bar{f}_x))\Psi(x), \quad x \in \mathbb{R}^3.$$

**Theorem 1.1** *Assume that $\hat{\varphi}/\omega, \hat{\varphi}/\sqrt{\omega}, \sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3)$ and $V$ is relatively bounded with respect to $-\Delta$ with a relative bound $< 1$. Then, for arbitrary values of $e$, $H$ is self-adjoint on $D(\Delta \otimes 1) \cap D(1 \otimes H_{\rm f})$ and bounded from below.*

*Proof:* See Hiroshima [3, 4]. □

## 1.2 Effective mass

The momentum of the photon field is given by

$$P_{\rm f} = \sum_{j=1,2} \int k a^*(k,j) a(k,j) dk \tag{1.5}$$

and the total moment by

$$P_{\rm total} = p_x \otimes 1 + 1 \otimes P_{\rm f}.$$

Let as assume that

$$V \equiv 0.$$

Then we see that

$$[H, P_{\rm total \mu}] = 0, \quad \mu = 1,2,3.$$

Hence $H$ and $\mathcal{H}$ can be decomposable with respect to $\mathrm{Spec}(P_{\rm total}) = \mathbb{R}^3$, i.e.,

$$\begin{aligned} \mathcal{H} &= \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}(p) dp, \\ H &= \int_{\mathbb{R}^3}^{\oplus} H(p) dp. \end{aligned}$$

Note that

$$\begin{aligned} e^{-ix\otimes P_{\rm f}} P_{\rm total} e^{ix\otimes P_{\rm f}} &= p_x, \\ e^{-ix\otimes P_{\rm f}} H e^{ix\otimes P_{\rm f}} &= \frac{1}{2m}(p_x \otimes 1 - 1 \otimes P_{\rm f} - e1 \otimes A_{\hat{\varphi}}(0)) + 1 \otimes H_{\rm f}, \end{aligned}$$

where

$$A_{\hat{\varphi}}(0) = \frac{1}{\sqrt{2}}(a(f_0) + a(\bar{f}_0)).$$

From this we obtain that for each $p \in \mathbb{R}^3$,

$$\begin{aligned} \mathcal{H}(p) &\cong \mathcal{F}, \\ H(p) &\cong \frac{1}{2m}(p - P_{\rm f} - eA_{\hat{\varphi}}(0)) + H_{\rm f}, \end{aligned}$$

Let

$$E_{m,\Lambda}(p) = \inf \mathrm{Spec}(H(p)). \tag{1.6}$$

Let us assume sharp ultraviolet cutoff $\Lambda$ and infrared cutoff $\kappa$, which means

$$\hat{\varphi}(k) = \begin{cases} 0 & \text{for } |k| < \kappa, \\ (2\pi)^{-3/2} & \text{for } \kappa \le |k| \le \Lambda, \\ 0 & \text{for } |k| > \Lambda. \end{cases} \tag{1.7}$$

**Lemma 1.2** *There exists constants $p_*$ and $e_*$ such that for*

$$(p,e) \in \mathcal{O} = \{(p,e) \in \mathbb{R}^3 \times \mathbb{R} | |p| < p_*, |e| < e^*\},$$

*$H(p)$ has a ground state $\psi_{\rm g}(p)$ and it is unique. Moreover $\psi_{\rm g}(p) = \psi_{\rm g}(p,e)$ is strongly analytic and $E_{m,\Lambda}(p) = E_{m,\Lambda}(p,e)$ analytic with respect to $(p,e) \in \mathcal{O}$.*

*Proof:* See Hiroshima and Spohn [6, 7]. □

In what follows we assume that $(p, e) \in \mathcal{O}$.

**Definition 1.3** *The effective mass* $m_{\rm eff} = m_{\rm eff}(e^2, \Lambda, \kappa, m)$ *is defined by*

$$\frac{1}{m_{\rm eff}} = \frac{1}{3}\Delta_p E(p, e)\lceil_{p=0}. \tag{1.8}$$

## 1.3 Mass renormalization

Removal of the ultraviolet cutoff $\Lambda$ through mass renormalization means to find sequences

$$\Lambda \to \infty, \quad m \to 0 \tag{1.9}$$

such that $E_{m,\Lambda}(p) - E_{m,\Lambda}(0)$ has a nondegenerate limit. To achive this, as a first step we want to find constants

$$\beta < 0, \quad 0 < b$$

such that

$$\lim_{\Lambda\to\infty} m_{\rm eff}(e^2, \Lambda, \kappa\Lambda^\beta, (b\Lambda)^\beta) = m_{\rm ph}, \tag{1.10}$$

where $m_{\rm ph}$ is a given constant. Actually $m_{\rm ph}$ is a physical mass. Namely in the mass renormalization the scaled bare mass goes to zero and the effective mass goes to a physical mass as the ultraviolet cutoff $\Lambda$ goes to infinity.

We will see later that $m_{\rm eff}/m$ is a function of $e^2$, $\Lambda/m$ and $\kappa/m$. Let

$$\frac{m_{\rm eff}}{m} = f(e^2, \Lambda/m, \kappa/m), \tag{1.11}$$

where $f(0, \Lambda/m, \kappa/m) = 1$ holds. An analysis of (1.10) can be reduce to investigate the asymptotic behavior of $f$ as $\Lambda \to \infty$. Namely we want to find constants

$$0 \le \gamma < 1, \quad 0 < b_0$$

such that

$$\lim_{\Lambda\to\infty} \frac{f(e^2, \Lambda/m, \kappa/m)}{(\Lambda/m)^\gamma} = b_0. \tag{1.12}$$

If we succeed to find constants $\gamma$ and $b_0$ such as in (1.12) then by

$$m_{\rm eff}(e^2, \Lambda, \kappa, m) = m f(e^2, \Lambda/m, \kappa/m),$$

we have

$$m_{\rm eff}(e^2, \Lambda, \kappa\Lambda^\beta, (b\Lambda)^\beta) = (b\Lambda)^\beta f(e^2, \Lambda/(b\Lambda)^\beta, \kappa/b^\beta) \approx b_0(b\Lambda)^\beta(\Lambda/(b\Lambda)^\beta)^\gamma. \tag{1.13}$$

Taking

$$\beta = \frac{-\gamma}{1-\gamma} < 0, \quad b = 1/b_1^{1/\gamma},$$

we see that by (1.13)

$$\lim_{\Lambda\to\infty} m_{\rm eff}(e^2,\Lambda,\kappa\Lambda^\beta,(b\Lambda)^\beta)=\lim_{\Lambda\to\infty} b_0\left(\frac{\Lambda}{b_1^{1/\gamma}}\right)^\beta\left(\frac{\Lambda}{(\Lambda/(b_1)^{1/\gamma})^\beta}\right)^\gamma=b_0b_1,$$

where $b_1$ is a parameter, which is adjusted such as

$$b_0b_1=m_{\rm ph}.$$

Hence we will be able to establish (1.10). It is easily seen that

$$f(e^2,\Lambda/m,\kappa/m)=1+\alpha\frac{8}{3\pi}\log(\frac{\Lambda/m+2}{\kappa/m+2})+O(\alpha^2),$$

where $\alpha=e^2/4\pi$, which suggests

$$f(e^2,\Lambda/m,\kappa/m)\approx(\Lambda/m)^{8\alpha/3\pi},$$

for sufficiently small $\alpha$ and large $\Lambda$, and therefore

$$\gamma=8\alpha/3\pi.$$

One may assume that

$$f(e^2,\Lambda/m,\kappa/m)\approx(\Lambda/m)^{\alpha(8/3\pi)+\alpha^2 b}$$

for sufficiently small $\alpha$ with some constant $b$. Then by expading $m_{\rm eff}/m$ to order $\alpha^2$ one may expect that

$$f(e^2,\Lambda/m,\kappa/m)\approx 1+\alpha\frac{8}{3\pi}\log(\frac{\Lambda}{m})+\frac{1}{2}\alpha^2\left(\frac{8}{3\pi}\log(\frac{\Lambda}{m})\right)^2+b\alpha^2\log(\frac{\Lambda}{m})+O(\alpha^3) \quad (1.14)$$

for sufficiently small $\alpha$ and large $\Lambda$. It is, however, that (1.14) is not confirmed. Instead of (1.14) we prove that there exists a constant $C>0$ such that

$$f(e^2,\Lambda/m,\kappa/m)=1+\alpha\frac{8}{3\pi}\log(\frac{\Lambda/m+2}{\kappa/m+2})+\alpha^2C\sqrt{\Lambda/m}+O(\alpha^3).$$

The effective mass and its renormalization have been studied from a mathematical point of viwe by many authors. Spohn [10] investigates the effective mass of the Nelson model [9] from a functional integral point of view. Lieb and Loss [8] studied mass renormalization and binding energies of models of matter coupled to radiation fields including the Pauli-Fierz model. Hainzl and Seiringer [2] computed exactly the leading order in $\alpha$ of the effective mass of the Pauli-Fierz Hamiltonian with spin.

# 2 Perturbative expansions

The effective masses for $H(p)$ and

$$\frac{1}{2m} :(p - P_\mathrm{f} - eA_{\hat\varphi}(0))^2: + H_\mathrm{f}$$

are identical. Then in what follows we redefine $H(p)$ as

$$H(p) = \frac{1}{2m} :(p - P_\mathrm{f} - eA_{\hat\varphi}(0))^2: + H_\mathrm{f}.$$

Furthermore for notational convenience we write $A$ and $E(p)$ for $A_{\hat\varphi}(0)$ and $E_{m,\Lambda}(p)$, respectively.

## 2.1 Formulae

**Lemma 2.1** *We have*

$$\frac{m}{m_\mathrm{eff}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{(\psi_\mathrm{g}(0), (P_\mathrm{f} + eA)_\mu (H(0) - E(0))^{-1} (P_\mathrm{f} + eA)_\mu \psi_\mathrm{g}(0))}{(\psi_\mathrm{g}(0), \psi_\mathrm{g}(0))}.$$

*Proof:* It is seen that $E(p,e) = E(p,-e) = E(-p,e)$. Then

$$\frac{\partial}{\partial p_\mu} E(p,e) \bigg\lceil_{p_\mu = 0} = 0, \quad \mu = 1,2,3, \tag{2.1}$$

follows. Moreover it is seen that $E(p,e)$ is a function of $e^2$ and

$$\frac{d^{2m-1}}{de^{2m-1}} E(p,e) \bigg\lceil_{e=0} = 0. \tag{2.2}$$

In this proof, $f'(p)_\mu$ means the strong derivative of $f(p)$ with respect to $p_\mu$. Since

$$H(p)\psi_\mathrm{g}(p) = E(p)\psi_\mathrm{g}(p),$$

we have

$$H'(p)_\mu \psi_\mathrm{g}(p) + H(p)\psi'_\mathrm{g}(p)_\mu = E'(p)_\mu \psi_\mathrm{g}(p) + E(p)\psi'_\mathrm{g}(p)_\mu \tag{2.3}$$

and

$$\begin{aligned} &H''(p)_\mu \psi_\mathrm{g}(p) + 2H'(p)_\mu \psi'_\mathrm{g}(p)_\mu + H(p)\psi''_\mathrm{g}(p)_\mu \\ &= E''(p)_\mu \psi_\mathrm{g}(p) + 2E'(p)_\mu \psi'_\mathrm{g}(p)_\mu + E(p)\psi''_\mathrm{g}(p)_\mu. \end{aligned} \tag{2.4}$$

By (2.1) it follows that $E'(0)_\mu = 0$, and by (2.3) with $p = 0$,

$$\begin{aligned} &(P_\mathrm{f} + eA)_\mu \psi_\mathrm{g}(0) \in D((H(0) - E(0))^{-1}), \\ &\psi'_\mathrm{g}(0)_\mu = (H(0) - E(0))^{-1} (P_\mathrm{f} + eA)_\mu \psi_\mathrm{g}(0). \end{aligned}$$

Then we have by (2.3) and (2.4),

$$\begin{aligned}\frac{m}{m_{\rm eff}} &= \frac{1}{3}\sum_{\mu=1,2,3}\frac{(\psi_{\rm g}(0), E''(0)_\mu\psi_{\rm g}(0))}{(\psi_{\rm g}(0),\psi_{\rm g}(0))}\\ &= 1-\frac{2}{3}\sum_{\mu=1,2,3}\frac{((P_{\rm f}+eA)_\mu\psi_{\rm g}(0),(H(0)-E(0))^{-1}(P_{\rm f}+eA)_\mu\psi_{\rm g}(0))}{(\psi_{\rm g}(0),\psi_{\rm g}(0))}\end{aligned}$$

Thus the lemma follows. □

Let

$$\psi_{\rm g}(0)=\sum_{n=0}^\infty \frac{e^n}{n!}\varphi_n,\quad E(0)=\sum_{n=0}^\infty\frac{e^{2n}}{(2n)!}E_{2n}.$$

Note that

$$\varphi_{2m}\in\bigoplus_{m=0}^\infty \mathcal{F}^{(2m)},\quad \varphi_{2m+1}\in\bigoplus_{m=0}^\infty\mathcal{F}^{(2m+1)}.$$

We want to get the explicit form of $\varphi_n$. Let

$$\begin{aligned}&\mathcal{F}_{\rm fin}=\{\{\Psi^{(n)}\}_{n=0}^\infty\in\mathcal{F}|\Psi^{(m)}=0 \text{ for } m\geq\ell \text{ with some } \ell\},\\ &\mathcal{F}_0=\Big\{\{\Psi^{(n)}\}_{n=0}^\infty\in\mathcal{F}_{\rm fin}\Big| \text{(i) } \Psi^{(0)}=0,\\ &\qquad \text{(ii) } {\rm supp}_{(k_1,\dots,k_n)\in\mathbb{R}^{3n}}\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n)\not\ni\{(0,...,0)\}\Big\}.\end{aligned}$$

**Lemma 2.2** *We see that* $\mathcal{F}_0\subset D(H_0^{-1})$.

*Proof:* Let $\Psi=\{\Psi^{(n)}\}_{n=0}^\infty\in\mathcal{F}_0$. Since

$$\begin{aligned}&(H_0\Psi)^{(n)}(k_1,...,k_n,j_1,...,j_n)\\ &=\left[\frac{1}{2}(k_1+\cdots+k_n)^2+\sum_{j=1}^n\omega(k_j)\right]\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n),\end{aligned}$$

we see that

$$\begin{aligned}&(H_0{}^{-1}\Psi)^{(n)}(k_1,...,k_n,j_1,...,j_n)\\ &=\left[\frac{1}{2}(k_1+\cdots+k_n)^2+\sum_{j=1}^n\omega(k_j)\right]^{-1}\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n).\end{aligned}$$

Since ${\rm supp}_{(k_1,\dots,k_n)\in\mathbb{R}^{3n}}\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n)\not\ni\{(0,...,0)\}$, we obtain that

$$\|H_0{}^{-1}\Psi\|_{\mathcal{F}}^2=\sum_{n=1}^{\rm finite}\|(H_0{}^{-1}\Psi)^{(n)}\|^2_{\mathcal{F}^{(n)}}<\infty.$$

Then the lemma follows. □

We split $H(0)$ as

$$H(0)=H_0+eH_1+\frac{e^2}{2}H_2,$$

where

$$H_0 = \frac{1}{2}{P_{\rm f}}^2 + H_{\rm f},$$
$$H_1 = \frac{1}{2}(P_{\rm f}\cdot A + A\cdot P_{\rm f}) = P_{\rm f}\cdot A = A\cdot P_{\rm f},$$
$$H_2 = :A^2: .$$

**Lemma 2.3** *We have $E_0 = E_1 = E_2 = E_3 = 0$ and*

$$\varphi_0 = \Omega, \quad \varphi_1 = 0, \quad \varphi_2 = -{H_0}^{-1}H_2\Omega, \quad \varphi_3 = 3{H_0}^{-1}H_1{H_0}^{-1}H_2\Omega.$$

*In particular $\varphi_2 \in \mathcal{F}^{(2)}$ and $\varphi_3 \in \mathcal{F}^{(1)} \cap \mathcal{F}^{(3)}$.*

*Proof:* Let us set $H(0)$, $E(0)$ and $\psi_{\rm g}(0)$ as $H$, $E$ and $\psi_{\rm g}$, respectively. It is obvious that $E_0 = 0$ and $\varphi_0 = a\Omega$ with arbitrary $a \in \mathbb{C}$, and by (2.2), $E_1 = E_3 = 0$. Set $a = 1$. We denote the strong derivative of $f = f(e)$ with respect to $e$ by $f'$. We have

$$H'\psi_{\rm g} + H\psi_{\rm g}' = E'\psi_{\rm g} + E\psi_{\rm g}' \tag{2.5}$$

and

$$H''\psi_{\rm g} + 2H'\psi_{\rm g}' + H\psi_{\rm g}'' = E''\psi_{\rm g} + 2E'\psi_{\rm g}' + E\psi_{\rm g}''. \tag{2.6}$$

From (2.6) it follows that

$$(\psi_{\rm g}, H''\psi_{\rm g}) + (\psi_{\rm g}, 2H'\psi_{\rm g}') + (\psi_{\rm g}, H\psi_{\rm g}'') = E''(\psi_{\rm g}, \psi_{\rm g}) + (\psi_{\rm g}, 2E'\psi_{\rm g}') + (\psi_{\rm g}, E\psi_{\rm g}''). \tag{2.7}$$

Put $e = 0$ in (2.7). Then

$$(\Omega, H_2\Omega) + (\Omega, 2H_1\Omega) + (\Omega, H_0\varphi_2) = E_2(\Omega, \Omega). \tag{2.8}$$

Since the left-hand side of (2.8) vanishes, we have $E_2 = 0$. From (2.5) with $e = 0$ and the fact $E_0 = E_1 = 0$, it follows that

$$H_1\Omega + H_0\varphi_1 = 0,$$

from which it holds that $H_0\varphi_1 = 0$. Since $H_0$ has the unique eigenvector $\Omega$ (the ground state) with eigenvalue zero, it follows that $\varphi_1 = b\Omega$ with some constant $b$. $\varphi_1 \in \bigoplus_{m=0}^{\infty}\mathcal{F}^{(2m+1)}$ which implies $b = 0$. Hence $\varphi_1 = 0$ follows. By (2.6) with $e = 0$, we have

$$H_2\Omega + 2H_1\varphi_1 + H_0\varphi_2 = 0.$$

Since $H_2\Omega \in \mathcal{F}_0$, we see that by Lemma 2.2, $H_2\Omega \in D({H_0}^{-1})$. Thus we have $\varphi_2 = -{H_0}^{-1}H_2\Omega$. From the identity

$$H'''\psi_{\rm g} + 3H''\psi_{\rm g}' + 3H'\psi_{\rm g}'' + H\psi_{\rm g}''' = E'''\psi_{\rm g} + 3E''\psi_{\rm g}' + 3E'\psi_{\rm g}'' + E\psi_{\rm g}''' \tag{2.9}$$

it follows that at $e = 0$,

$$3H_1\varphi_2 + H_0\varphi_3 = 0.$$

Since $H_1\varphi_2 = -H_1{H_0}^{-1}H_2\Omega \in \mathcal{F}_0$, Lemma 2.2 ensures that $H_1\varphi_2 \in D({H_0}^{-1})$. Hence $\varphi_3 = -3{H_0}^{-1}H_1\varphi_2 = 3{H_0}^{-1}H_1{H_0}^{-1}H_2\Omega$. Then the lemma is proven. □

## 2.2 Order $e^4$

In this subsection we expand $m/m_{\rm eff}$ up to order $e^4$. We define $A^-$ and $A^+$ by

$$A^- = \frac{1}{\sqrt{2}} a(f), \quad A^+ = \frac{1}{\sqrt{2}} a^*(f).$$

Then $A = A^+ + A^-$.

**Lemma 2.4** *We have*

$$\begin{aligned}\frac{m}{m_{\rm eff}} &= 1 - e^2 \frac{2}{3} \sum_{\mu=1}^{3} \left(\Omega, A_\mu {H_0}^{-1} A_\mu \Omega\right) \\ &-e^4 \frac{2}{3} \sum_{\mu=1}^{3} \Big\{ 2\left(\Psi_3^\mu, {H_0}^{-1} \Psi_1^\mu\right) + \left(\Psi_2^\mu, {H_0}^{-1}\Psi_2^\mu\right) - 2\left(\Psi_2^\mu, {H_0}^{-1} H_1 {H_0}^{-1} \Psi_1^\mu\right) \\ &-\frac{1}{2}\left(\Psi_1^\mu, {H_0}^{-1} H_2 {H_0}^{-1} \Psi_1^\mu\right) + \left(\Psi_1^\mu, {H_0}^{-1} H_1 {H_0}^{-1} H_1 {H_0}^{-1} \Psi_1^\mu\right)\Big\} + O(e^6),\end{aligned} \tag{2.10}$$

*where*

$$\begin{aligned}\Psi_1^\mu &= A_\mu \Omega, \\ \Psi_2^\mu &= -\frac{1}{2} P_{{\rm f}\mu} {H_0}^{-1} (A^+ \cdot A^+)\Omega, \\ \Psi_3^\mu &= \frac{1}{2}\left\{ -A_\mu {H_0}^{-1}(A^+ \cdot A^+)\Omega + \frac{1}{2} P_{{\rm f}\mu} {H_0}^{-1} (P_{\rm f} \cdot A + A \cdot P_{\rm f}) {H_0}^{-1} (A^+ \cdot A^+)\Omega \right\}.\end{aligned}$$

*Proof:* In Lemma 2.1 we have seen that

$$\frac{m}{m_{\rm eff}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_{\rm f} + eA)_\mu \psi_{\rm g}(0), (H(0) - E(0))^{-1} (P_{\rm f} + eA)_\mu \psi_{\rm g}(0))}{(\psi_{\rm g}(0), \psi_{\rm g}(0))}. \tag{2.11}$$

We can strongly expand $(H(0) - E(0))^{-1}$ as

$$\begin{aligned}(H(0) - E(0))^{-1} &= {H_0}^{-1} - e{H_0}^{-1} H_1 {H_0}^{-1} \\ &+ e^2 \left( -\frac{1}{2} {H_0}^{-1} H_2 {H_0}^{-1} + {H_0}^{-1} H_1 {H_0}^{-1} H_1 {H_0}^{-1} \right) + O(e^3).\end{aligned} \tag{2.12}$$

Here we set

$$H_j = \begin{cases} H_j, & j = 1, 2, \\ -E_j, & j \geq 3. \end{cases}$$

Note that

$$\varphi_0 \in \mathcal{F}^{(0)}, \varphi_2 \in \mathcal{F}^{(2)}, \varphi_3 \in \mathcal{F}^{(3)} \cap \mathcal{F}^{(1)}, \varphi_4 \in \mathcal{F}^{(4)} \cap \mathcal{F}^{(2)}.$$

In particular

$$\frac{1}{(\psi_{\rm g},\psi_{\rm g})}=1-e^4(\frac{1}{2}\varphi_2,\frac{1}{2}\varphi_2)-e^4(\Omega,\frac{1}{24}\varphi_4)+O(e^6)=1-e^4\frac{1}{4}(\varphi_2,\varphi_2)+O(e^6). \tag{2.13}$$

Moreover we have

$$\begin{aligned}(P_{\rm f}+eA)_\mu\psi_{\rm g}(0)&=eA_\mu\Omega+e^2(\frac{1}{2}P_{{\rm f}\mu}\varphi_2)+e^3(\frac{1}{2}A_\mu\varphi_2+\frac{1}{6}P_{{\rm f}\mu}\varphi_3)+O(e^4)\\ &=e\Psi_1^\mu+e^2\Psi_2^\mu+e^3\Psi_3^\mu+O(e^4). \end{aligned}\tag{2.14}$$

Substitute (2.12), (2.13) and (2.14) into (2.11). Then the lemma follows. □

For each $k\in\mathbb{R}^3$ let us define the projection $Q(k)$ on $\mathbb{R}^3$ by

$$Q(k)=\sum_{j=1,2}|e_j(k)\rangle\langle e_j(k)|.$$

We set

$$\hat\varphi_j=\hat\varphi(k_j),\quad \omega_j=\omega(k_j),\quad Q(k_j)=Q_j,\quad j=1,2.$$

Let

$$\begin{aligned}\frac{1}{F_j}&=\frac{1}{r_j^2/2+r_j},\quad j=1,2,\\ \frac{1}{F_{12}}&=\frac{1}{(r_1^2+2r_1r_2X+r_2^2)/2+r_1+r_2},\quad r_1,r_2\geq 0,\quad -1\leq X\leq 1.\end{aligned}$$

**Lemma 2.5** *We have*

$$\frac{m}{m_{\rm eff}}=1-\alpha a_1(\Lambda/m,\kappa/m)-\alpha^2a_2(\Lambda/m,\kappa/m)+O(\alpha^3),$$

*where*

$$a_1(\Lambda/m,\kappa/m)=\frac{8}{3\pi}\log\left(\frac{\Lambda/m+2}{\kappa/m+2}\right) \tag{2.15}$$

*and*

$$\begin{aligned}&a_2(\Lambda/m,\kappa/m)\\ &=\frac{(4\pi)^2}{(2\pi)^6}\frac{2}{3}\int_{-1}^1{\rm d}X\int_{\kappa/m}^{\Lambda/m}{\rm d}r_1\int_{\kappa/m}^{\Lambda/m}{\rm d}r_2\pi r_1r_2\times\\ &\times\left\{-\left(\frac{1}{F_1}+\frac{1}{F_2}\right)\frac{1}{F_{12}}(1+X^2)+\left(\frac{1}{F_{12}}\right)^3\frac{r_1^2+2r_1r_2X+r_2^2}{2}(1+X^2)\right.\\ &+\left(\frac{1}{F_1}+\frac{1}{F_2}\right)\left(\frac{1}{F_{12}}\right)^2r_1r_2X(-1+X^2)-\frac{1}{F_1}\frac{1}{F_2}(1+X^2)\\ &\left.+\left(\frac{r_1^2}{F_1^2}+\frac{r_2^2}{F_2^2}\right)\frac{1}{F_{12}}(1-X^2)+\frac{1}{F_1}\frac{1}{F_2}\frac{1}{F_{12}}r_1r_2X(-1+X^2)\right\}.\end{aligned}\tag{2.16}$$

*Proof:* Note that

$$
\begin{aligned}
a_1(\Lambda,\kappa) &= \frac{2}{3}(\sqrt{4\pi})^2(A_\mu^+\Omega, H_0^{-1}A_\mu^+\Omega) \\
&= \frac{8}{3\pi}\log\left(\frac{\Lambda/m+2}{\kappa/m+2}\right).
\end{aligned}
$$

Thus (2.15) follows. To see $a_2(\Lambda,\kappa)$ we exactly compute the five terms on the right-hand side of (2.10) separately. Let

$$
\begin{aligned}
\frac{1}{E_j} &= \frac{1}{|k_j|^2/2+\omega_j}, \quad j=1,2, \\
\frac{1}{E_{12}} &= \frac{1}{|k_1+k_2|^2/2+\omega_1+\omega_2}.
\end{aligned}
$$

(1) We have

$$
\begin{aligned}
&2\left(\Psi_3^\mu, H_0^{-1}\Psi_1^\mu\right) = \left(\Omega, -(A^-\cdot A^-)H_0^{-1}A_\mu H_0^{-1}A_\mu^+\Omega\right) \\
&+\frac{1}{2}\left(\Omega, (A^-\cdot A^-)H_0^{-1}(P_{\mathrm{f}}\cdot A + A\cdot P_{\mathrm{f}})H_0^{-1}P_{\mathrm{f}\mu}H_0^{-1}A_\mu^+\Omega\right). \\
&= -\iint \mathrm{d}k_1^3 \mathrm{d}k_2^3 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_{12}}(\frac{1}{E_1}+\frac{1}{E_2})\mathrm{tr}(Q_1Q_2).
\end{aligned}
\tag{2.17}
$$

(2) We have

$$
\begin{aligned}
&\left(\Psi_2^\mu, H_0^{-1}\Psi_2^\mu\right) \\
&= \left(\frac{1}{2}\right)^2\left(P_{\mathrm{f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega, H_0^{-1}P_{\mathrm{f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega\right) \\
&= \left(\frac{1}{2}\right)^2\iint \mathrm{d}k_1^3 \mathrm{d}k_2^3 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\left(\frac{1}{E_{12}}\right)^3|k_1+k_2|^2 2\mathrm{tr}(Q_1Q_2).
\end{aligned}
\tag{2.18}
$$

(3) We have

$$
\begin{aligned}
&-2\left(\Psi_2^\mu, H_0^{-1}H_1H_0^{-1}\Psi_1^\mu\right) \\
&= \frac{1}{2}\left(P_{\mathrm{f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega, H_0^{-1}(P_{\mathrm{f}}\cdot A + A\cdot P_{\mathrm{f}})H_0^{-1}A_\mu^+\Omega\right) \\
&= \iint \mathrm{d}k_1^3 \mathrm{d}k_2^3 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\left(\frac{1}{E_{12}}\right)^2(\frac{1}{E_1}+\frac{1}{E_2})(k_2, Q_1Q_2k_1).
\end{aligned}
\tag{2.19}
$$

(4) We have

$$
\begin{aligned}
&-\frac{1}{2}\left(\Psi_1^\mu, H_0^{-1}H_2H_0^{-1}\Psi_1^\mu\right) \\
&= -\frac{1}{2}\left(A_\mu^+\Omega, H_0^{-1}((A^+\cdot A^+)+2(A^+\cdot A^-)+(A^-\cdot A^-))H_0^{-1}A_\mu^+\Omega\right) \\
&= -\iint \mathrm{d}k_1^3 \mathrm{d}k_2^3 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_1}\frac{1}{E_2}\mathrm{tr}(Q_1Q_2).
\end{aligned}
\tag{2.20}
$$

(5) We have

$$
\begin{aligned}
&\left(\Psi_1^\mu, H_0{}^{-1}H_1H_0{}^{-1}H_1H_0{}^{-1}\Psi_1^\mu\right)\\
&= \left(\frac{1}{2}\right)^2 \left(A_\mu^+\Omega, H_0{}^{-1}(P_\mathrm{f}\cdot A + A\cdot P_\mathrm{f})H_0{}^{-1}(P_\mathrm{f}\cdot A + A\cdot P_\mathrm{f})H_0{}^{-1}A_\mu^+\Omega\right)\\
&= \iint \mathrm{d}k_1^3 \mathrm{d}k_2^3 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_{12}}\left\{\left(\frac{1}{E_1}\right)^2 (k_1, Q_2k_1) + \left(\frac{1}{E_2}\right)^2 (k_2, Q_1k_2)\right\}\\
&+ \iint \mathrm{d}k_1^3 \mathrm{d}k_2^3 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_{12}}\frac{1}{E_1}\frac{1}{E_2}(k_2, Q_1Q_2k_1). \qquad (2.21)
\end{aligned}
$$

Changing variables to the polar coordinate, we obtain (2.16) from Lemma 2.4, (2.17), (2.18), (2.19), (2.20), (2.21) and the facts

$$
\begin{aligned}
&\mathrm{tr}[Q_1Q_2] = 1 + (\hat k_1, \hat k_2)^2,\\
&(k_1, Q_2Q_1k_2) = (k_1, k_2)((\hat k_1, \hat k_2)^2 - 1),\\
&(k_1, Q_2k_1) = |k_1|^2(1 - (\hat k_1, \hat k_2)^2).
\end{aligned}
$$

Thus the proof is complete. □

# 3 Main theorem

The main theorem is as follows.

**Theorem 3.1** *There exist strictly positive constants $C_{\min}$ and $C_{\max}$ such that*

$$C_{\min} \le \lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \le C_{\max}.$$

*Proof:* We show an outline of a proof. See Hiroshima and Spohn [7] for details. By (2.16) we can see that

$$a_2(\Lambda, \kappa) = \frac{(4\pi)^2}{(2\pi)^6}\frac{2}{3}\sum_{j=1}^6 b_j(\Lambda/m), \qquad (3.1)$$

where

$$
\begin{aligned}
b_1(\Lambda/m) &= -\int (1+X^2)\left(\frac{1}{F_1} + \frac{1}{F_2}\right)\frac{1}{F_{12}},\\
b_2(\Lambda/m) &= \int (1+X^2)\left(\frac{1}{F_{12}}\right)^3 \frac{r_1^2 + 2r_1r_2X + r_2^2}{2},\\
b_3(\Lambda/m) &= \int X(-1+X^2)r_1r_2\left(\frac{1}{F_1} + \frac{1}{F_2}\right)\left(\frac{1}{F_{12}}\right)^2,\\
b_4(\Lambda/m) &= -\int (1+X^2)\frac{1}{F_1}\frac{1}{F_2},\\
b_5(\Lambda/m) &= \int (1-X^2)\left(\frac{r_1^2}{F_1^2} + \frac{r_2^2}{F_2^2}\right)\frac{1}{F_{12}},\\
b_6(\Lambda/m) &= \int X(-1+X^2)r_1r_2\frac{1}{F_1}\frac{1}{F_2}\frac{1}{F_{12}},
\end{aligned}
$$

where

$$\int = \int_{-1}^{1} \mathrm{d}X \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_1 \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_2 \pi r_1 r_2.$$

Let $\rho_\Lambda(\cdot,\cdot) : [0,\infty) \times [-1,1] \to \mathbb{R}$ be defined by

$$\rho_\Lambda = \rho_\Lambda(r,X) = r^2 + 2\Lambda r X + \Lambda^2 + 2r + 2\Lambda = (r + \Lambda X + 1)^2 + \Delta,$$

where

$$\Delta = \Lambda^2(1 - X^2) + 2\Lambda(1 - X) - 1. \tag{3.2}$$

Then we can show that there exist constants $C_1, C_2, C_3$ and $C_4$ such that for sufficiently large $\Lambda > 0$,

$$\begin{aligned}
&(1)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \frac{1}{\rho_\Lambda(r,X)} \le C_1 \frac{1}{\Lambda},\\
&(2)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \left(\frac{1}{\rho_\Lambda(r,X)}\right)^2 \le C_2 \frac{1}{\Lambda^{5/2}},\\
&(3)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \frac{1}{\rho_\Lambda(r,X)} \frac{1}{r+2} \le C_3 \frac{\log\Lambda}{\Lambda^2},\\
&(4)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \left(\frac{1}{\rho_\Lambda(r,X)}\right)^2 (1 - X^2) \le C_4 \frac{1}{\Lambda^3}.
\end{aligned}$$

Using (1)–(4) we can prove that there exists a constant $C > 0$ such that

$$\begin{aligned}
&|b_j(\Lambda/m)| \le C[\log(\Lambda/m)]^2, \quad j = 1, 4,\\
&|b_2(\Lambda/m)| \le C(\Lambda/m)^{1/2},\\
&|b_j(\Lambda/m)| \le C\log(\Lambda/m), \quad j = 3, 5, 6.
\end{aligned}$$

Hence there exists a constant $C_{\max}$ such that

$$\lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \le C_{\max}.$$

Next we can show that there exists a positive constant $\xi > 0$ such that

$$\lim_{\Lambda\to\infty} \sqrt{\Lambda/m} \frac{d}{d(\Lambda/m)} b_2(\Lambda/m) > \xi,$$

which implies that there exists a constan $\xi'$ such that

$$\xi' \le \lim_{\Lambda\to\infty} \frac{b_2(\Lambda/m)}{\sqrt{\Lambda/m}}.$$

Thus we have

$$C_{\min} \le \lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \le C_{\max}.$$

□

**Remark 3.2** *Theorem 3.1 may suggests* $\gamma \geq 1/2$ *uniformly in* $e$ *but* $e \neq 0$.

**Remark 3.3** *(1)* $a_2(\Lambda/m, \kappa/m)/\sqrt{\Lambda/m}$ *converges to a nonnegative constant as* $\Lambda \to \infty$. *(2) By (3.1), we can define* $a_2(\Lambda/m, 0)$ *since* $b_j(\Lambda/m)$ *with* $\kappa = 0$ *are finite. Moreover* $a_2(\Lambda/m, 0)$ *also satisfies Theorem 3.1. (3) In the case of* $\kappa = 0$, *Chen [1] established that* $H(0)$ *has a ground state* $\psi_{\mathrm{g}}(0)$ *but does not for* $H(p)$ *with* $p \neq 0$.

# 4 Concluding remarks

The Pauli-Fierz Hamitonian with the dipole approximation, $H_{\mathrm{dip}}$, is defined by $H$ with $A_{\hat{\varphi}}$ replaced by $1 \otimes A_{\hat{\varphi}}(0)$, i.e.,

$$H_{\mathrm{dip}} = \frac{1}{2m}(p \otimes 1 - e1 \otimes A_{\hat{\varphi}}(0))^2 + V \otimes 1 + 1 \otimes H_{\mathrm{f}}.$$

Set $V \equiv 0$. Note that

$$[H_{\mathrm{dip}}, P_{\mathrm{total}}] \neq 0.$$

It is established in [5] that there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that

$$UH_{\mathrm{dip}}U^{-1} = -\frac{1}{2(m + \delta m)}\Delta \otimes 1 + 1 \otimes H_{\mathrm{f}} + e^2 G,$$

where

$$\delta m = m + e^2 \frac{2}{3}\|\hat{\varphi}/\omega\|^2,$$

$$G = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t^2\|\hat{\varphi}/(t^2+\omega^2)\|^2}{m + (2e^2/3)\|\hat{\varphi}/\sqrt{t^2+\omega^2}\|^2} dt.$$

Hence

$$[UH_{\mathrm{dip}}U^{-1}, P_{\mathrm{total}}] = 0.$$

Then we can define the effective mass $m_{\mathrm{eff}}$ for $UH_{\mathrm{dip}}U^{-1}$, and which is

$$m_{\mathrm{eff}}/m = 1 + \alpha \frac{4}{3\pi}(\Lambda/m - \kappa/m).$$

Hence $\gamma = 1$, then the mass renormalization for $H_{\mathrm{dip}}$ is not available.

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