Enhanched binding and mass renormalization of nonrelativistic QED

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Abstract

The Pauli-Fierz Hamiltonian of the nonrelativistic QED is defined as a self-adjoint operator H_Λ with ultraviolet cutoff $\Lambda > 0$, which describes an interaction between an electron and photons with momentum $< \Lambda$. Spectral properties of H_Λ are investigated for a sufficiently large Λ. In particular enhanced binding, stability of matter and asymptotic behavior of effective mass for $\Lambda \to \infty$ are studied.

1 The Pauli-Fierz Hamiltonian

This is a joint work with Herbert Spohn [20, 21].[1] We consider spectral properties of a system of one spinless electron minimally coupled to a quantized radiation field quantized in the Coulomb gauge. The system is called the Pauli-Fierz model [26]. The Pauli-Fierz Hamiltonian with ultraviolet cutoff Λ is defined as a self-adjoint operator on a Hilbert space. In this paper we analyze the Hamiltonian for a sufficiently large Λ.

Since a photon is a transversely polarized wave, one particle state space of a photon is defined by $L^2(\mathbb{R}^3 \times \{1,2\})$. Here $\mathbb{R}^3 \times \{1,2\} \ni (k,j)$ expresses momentum and transversal component of one photon, respectively. The Boson Fock space $\mathcal{F}$ describing a state space of photons is defined by

$$\begin{aligned}\mathcal{F} &= \bigoplus_{n=0}^{\infty} \left[\otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\})\right] \\ &= \{\Psi = \{\Psi^{(n)}\}_{n=0}^\infty | \Psi^{(n)} \in \otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\}), \|\Psi\|^2 = \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty\},\end{aligned}$$

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where $\otimes_s^n L^2(\mathbb{R}^3\times\{1,2\})$, $n \geq 1$, denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}^3\times\{1,2\})$ and we set

$$\otimes_s^0 L^2(\mathbb{R}^3\times\{1,2\}) = \mathbb{C}.$$

The creation operator $a^*(f)$ smeared by $f \in L^2(\mathbb{R}^3\times\{1,2\})$ is defined by

$$(a^*(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}),$$

where S_n denotes the symmetrization operator, i.e.,

$$S_n[\otimes^n L^2(\mathbb{R}^3 \times \{1,2\})] = \otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\}).$$

The annihilation operator is given by

$$a(f) = (a^*(\bar{f}))^* \lceil_{\mathcal{F}_0},$$

where $\mathcal{F}_0$ denotes the finite particle subspace of $\mathcal{F}$. Formally we often write $a^\sharp(f)$ as

$$a^\sharp(f) = \sum_{j=1,2} \int f(k,j) a^\sharp(k,j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1,2\}).$$

Note that we do not give any rigorous mathematical meaning to formal kernel $a^\sharp(k,j)$ in this paper. $a^\sharp(k,j)$ is just a symbol. $a^\sharp(f)$ satisfy CCR,

$$\begin{aligned} &[a(f), a^*(g)] = (f,g), \\ &[a(f), a(g)] = 0, \\ &[a^*(f), a^*(g)] = 0. \end{aligned}$$

We see that

the liner hull of $\{a^*(f_1)\cdots a^*(f_n)\Omega, \Omega | f_j \in L^2(\mathbb{R}^3\times\{1,2\}), 1 \leq j \leq n, n \geq 1\}$

is dense in $\mathcal{F}$. The free Hamiltonian H_f of $\mathcal{F}$ is defined by

$$\begin{aligned} &H_\mathrm{f}\Omega = 0, \\ &H_\mathrm{f} a^*(f_1)\cdots a^*(f_n)\Omega = \sum_{j=1}^n a^*(f_1)\cdots a^*(\omega f_j)\cdots a^*(f_n)\Omega, \\ &\qquad\qquad f_j \in D(\omega), \quad j = 1, ..., n, \end{aligned}$$

and which is formally written as

$$H_\mathrm{f} = \sum_{j=1,2} \int \omega(k) a^*(k,j) a(k,j) dk,$$

where the dispersion relation is given by

$$\omega(k) = |k|.$$

Let us denote the spectrum (resp. discrete spectrum, point spectrum, essential spectrum) of self-adjoint operator T by $\sigma(T)$ (resp. $\sigma_{\rm disc}(T)$, $\sigma_{\rm p}(T)$, $\sigma_{\rm ess}(T)$). It is well known that

$$\sigma(H_{\rm f}) = [0, \infty), \quad \sigma_{\rm p}(H_{\rm f}) = \{0\}.$$

Inequalities

$$\|a(f)\Psi\| \le \|f/\sqrt{\omega}\| \|H_{\rm f}^{1/2}\Psi\|,$$
$$\|a^*(f)\Psi\| \le \|f/\sqrt{\omega}\| \|H_{\rm f}^{1/2}\Psi\| + \|f\|\Psi\|$$

are well known. The Pauli-Fierz Hamiltonian H is defined as a self-adjoint operator acting on

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} dx \tag{1.1}$$

by

$$H = \frac{1}{2m}(p_x \otimes 1 - eA_{\hat{\varphi}})^2 + V \otimes 1 + 1 \otimes H_{\rm f},$$

where $\int_{\mathbb{R}^3}^{\oplus} \cdots dx$ denotes a constant fiber direct integral, m and e the mass and the charge of electron, respectively,

$$p_x = \left(-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, -i\frac{\partial}{\partial x_3}\right)$$

and V an external potential. We regard e as a coupling constant. Under identification (1.1), quantized radiation field $A_{\hat{\varphi}}$ is defined by

$$A_{\hat{\varphi}} = \int_{\mathbb{R}^3}^{\oplus} A_{\hat{\varphi}}(x) dx,$$

where

$$A_{\hat{\varphi}}(x) = \sum_{j=1,2} \int \frac{\hat{\varphi}(k)}{\sqrt{2\omega(k)}} e(k,j) \left\{ e^{-ikx} a^*(k,j) + e^{ikx} a(k,j) \right\} dk,$$

and, $e(k,1)$, $e(k,2)$ and $k/|k|$ form a three dimensional right-handed orthonormal system, i.e.,

$$e(k,j) \cdot k = 0, \quad e(k,i) \cdot e(k,j) = \delta_{ij}, \quad e(k,1) \times e(k,2) = k/|k|. \tag{1.2}$$

Note that

$$e(-k,1) = -e(k,1), \quad e(-k,2) = e(k,2).$$

Finally $\hat{\varphi}$ denotes a form factor. $A_{\hat{\varphi}}$ acts for $\Psi \in \mathcal{H}$ as

$$(A_{\hat{\varphi}}\Psi)(x) = A_{\hat{\varphi}}(x)\Psi(x), \quad x \in \mathbb{R}^3.$$

By (1.2), we have

$$p_x \cdot A_{\hat{\varphi}}(x) = 0.$$

The decoupled Hamiltonian is given by H with e replaced by 0, i.e.,

$$H_0 = \left(\frac{1}{2m} p_x^2 + V\right) \otimes 1 + 1 \otimes H_{\rm f}.$$

Theorem 1.1 *Assume that $\hat{\varphi}/\omega, \sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3)$ and V is relatively bounded with respect to p_x^2 with a relative bound < 1. Then, for arbitrary values of e, H is self-adjoint on $D(p_x^2 \otimes 1) \cap D(1 \otimes H_{\rm f})$ and bounded from below. Moreover it is essentially self-adjoint on any core of $D(H_0)$.*

Proof: See [15, 16]. □

Note that

$$D(H_0) = D(p_x^2 \otimes 1) \cap D(1 \otimes H_{\rm f}).$$

Quantized radiation field A_Λ with a sharp ultraviolet cutoff is defined by $A_{\hat{\varphi}}$ with $\hat{\varphi}$ replaced by

$$\chi_\Lambda(k) = \begin{cases} 0, & |k| < \kappa, \\ 1/\sqrt{(2\pi)^3}, & \kappa \le |k| \le \Lambda, \\ 0, & |k| > \Lambda. \end{cases}$$

Here $\kappa > 0$ is called infrared cutoff, and which is fixed throughout this paper. Hence the Hamiltonian under consideration is

$$H_\Lambda = \frac{1}{2m}(p_x \otimes 1 - eA_\Lambda)^2 + V \otimes 1 + 1 \otimes H_{\rm f}.$$

In this paper we will review recent advances in analysis of the spectral properties of H_Λ for sufficiently large Λ. In particular we will discuss 1.-3.

1. Enhanced binding for a sufficiently large Λ.

2. Stability of matter as $\Lambda \to \infty$.

3. The asymptotic behavior of an effective mass as $\Lambda \to \infty$.

2 Enhanced binding

It is proven that, *if* $\frac{1}{2m}p_x^2 + V$ *has a ground state*, then H_Λ has a ground state and it is unique, under suitable conditions on V and e. See e.g., [1, 3, 8, 12, 13, 14]. We want to show, however, the existence of a ground state *without* assumption "if $\frac{1}{2m}p_x^2 + V$ has a ground state". On a formal level we expect that bare mass m of an electron amounts to effective mass $m_{\rm eff}$ by a coupling with a quantized radiation field, i.e.,

$$m \to m_{\rm eff} = m_{\rm eff}(\Lambda) = m + \delta m(\Lambda)$$

Roughly speaking, H_Λ may be replaced by

$$H_\Lambda \sim H_{\rm eff} = \left(\frac{1}{2m_{\rm eff}(\Lambda)}p_x^2 + V\right) \otimes 1 + 1 \otimes H_{\rm f} + \text{ remainders }. \tag{2.1}$$

Since it is expected that effective mass $m_{\rm eff}(\Lambda)$ increases as Λ does, a ground state of H_Λ could be appear for a sufficiently large Λ even when H_0 has no

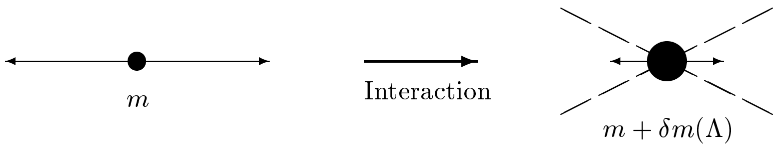

Figure 1: Effective mass

ground states. This kind of phenomena is called *enhanced binding*.
Enhanced binding for coupling constant e has been done in Hiroshima and Spohn [20], and developed by e.g., [2, 4, 5, 10]. Catto and Hainzl [4], Chen, Vougalter and Vugalter [5], Hainzl, Vougalter and Vugalter [10] study a more physically reasonable case. Arai and Kawano [2] proved the similar result as ours, i.e., enhanced binding for Λ, in a general framework.

In this section, we take the dipole approximation, i.e., $A_\Lambda(x)$ in the definition of H_Λ is replaced as

$$A_\Lambda(x) \longrightarrow 1 \otimes A_\Lambda(0).$$

Then the Hamiltonian under consideration is

$$H_{\rm dip} = \frac{1}{2m}(p_x \otimes 1 + 1 \otimes A_\Lambda(0))^2 + V \otimes 1 + 1 \otimes H_{\rm f}.$$

For notational convenience we omit the tensor notation $\otimes$ unless confusions may arise, i.e., $H_{\rm dip}$ is simply written as

$$H_{\rm dip} = \frac{1}{2m}(p_x - eA_\Lambda(0))^2 + V + H_{\rm f}.$$

Assumption (V) is as follow.

Assumption V

(a) $V \in C_0^\infty(\mathbb{R}^3)$.

(b) $V \le 0$.

(c) There exists $\mu_0 > 1$ and $r > 0$ such that for $\mu > \mu_0$,

$$\inf \sigma(\frac{1}{2m}p_x^2 + \mu V) < -r.$$

Since V is relatively compact with respect to $\frac{1}{2m}p_x^2$, it holds that

$$\sigma_{\rm ess}(\frac{1}{2m}p_x^2 + \mu V) = [0, \infty).$$

Hence $\frac{1}{2m}p_x^2 + \mu V$, $\mu > \mu_0$, has a ground state.

Remark 2.1 *We do not assume the existence of ground states of* $\frac{1}{2m}p_x^2 + V$.

A typical example of V is sufficiently shallow nonpositive potentials. By the Lieb-Thirring inequality [24],

$$
\begin{aligned}
&\#\{\text{bound states of } \frac{1}{2m}p_x^2 + V\} \leq L_3 \int |mV_-(x)|^{3/2} d^3x, \\
&(V_- : \text{ the negative part of } V)
\end{aligned}
$$

with some constant L_3 independent of V, we see that for a sufficiently shallow nonpositive potential V, $\frac{1}{2m}p_x^2 + V$ has no bound states. In particular it has no ground states. Thus H_0 has also no ground state.

Proposition 2.2 *There exists a unitary operator U such that*

$$
U : D(p_x^2) \cap D(H_{\mathrm{f}}) \to D(p_x^2) \cap D(H_{\mathrm{f}})
$$

and

$$
U^{-1} H_{\mathrm{dip}} U = \frac{1}{2m_{\mathrm{eff}}} p_x^2 + V(\cdot - K/m_{\mathrm{eff}}) + H_{\mathrm{f}} + g(\Lambda),
$$

where

$$
\begin{aligned}
m_{\mathrm{eff}} &= m + \frac{8\pi}{3} \frac{1}{(2\pi)^3} (\Lambda - \kappa), \\
g(\Lambda) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\chi_\Lambda/(t^2 + \omega^2)\|^2}{m + \frac{2}{3}\|\chi_\Lambda/\sqrt{t^2+\omega^2}\|^2} dt,
\end{aligned}
$$

and $K = (K_1, K_2, K_3)$ with

$$
K_\mu = \sum_{j=1,2} \frac{1}{\sqrt{2}} \int \left\{ \varrho_\mu(k,j) a^*(k,j) + \overline{\varrho_\mu(k,j)} a(k,j) \right\} d^3k,
$$

and $\varrho_\mu(\cdot, j)$ satisfies that

$$
\|\omega^{n/2} \varrho_\mu(\cdot, j)\| \leq C \|\omega^{(n-3)/2} \chi_\Lambda\| \tag{2.2}
$$

with some constant C.

Proof: See [20, 18]. □

We set

$$
\begin{aligned}
\delta V &= V(\cdot - K/m_{\mathrm{eff}}) - V, \\
H_{\mathrm{eff}} &= \frac{1}{2m_{\mathrm{eff}}} p_x^2 + V,
\end{aligned}
$$

and

$$
\widehat{H}_\Lambda = U^{-1} H_{\mathrm{dip}} U = H_{\mathrm{eff}} + \delta V + H_{\mathrm{f}} + g(\Lambda).
$$

Lemma 2.3 *Let Λ be such that*

$$\Lambda > (2\pi)^3 \frac{3}{8\pi}(\mu_0 - 1)m. \tag{2.3}$$

Then $H_{\rm eff}$ has a ground state, and

$$\#\{\text{bound states of } H_{\rm eff}\} \le L_3 \left(m + \frac{8\pi}{3}\frac{1}{(2\pi)^3}(\Lambda - \kappa)\right)^{3/2} \int |V(x)|^{3/2} d^3x. \tag{2.4}$$

In particular $H_{\rm eff}$ has a finite number of bound states.

Proof: By Hypothesis (V),

$$H_{\rm eff} = \frac{1}{2m_{\rm eff}} p_x^2 + V = \frac{m}{m_{\rm eff}} \left(\frac{1}{2m} p_x^2 + \frac{m_{\rm eff}}{m} V \right)$$

implies that if

$$\frac{m_{\rm eff}}{m} > \mu_0, \tag{2.5}$$

then $H_{\rm eff}$ has a ground state. (2.5) is identical with (2.3). (2.4) follows from the Lieb-Thirring inequality. Then the lemma follows. □

We introduce an artificial parameter $\nu > 0$, and define

$$\widehat{H}_\Lambda^\nu = H_{\rm eff} + \delta V^\nu + H_{\rm f}^\nu + g(\Lambda),$$

where δV^ν and $H_{\rm f}^\nu$ are defined by δV and $H_{\rm f}$ with ω replaced by $\omega + \nu$, respectively. It is easily seen that

$$\|\delta V^\nu \Psi\| \le \theta(\Lambda)(\|H_{\rm f}^{\nu 1/2}\Psi\| + \|\Psi\|)$$

with some constant $\theta(\Lambda)$ independent of ν. Actually it is presented as

$$\theta(\Lambda) = \frac{\|\nabla V\|_\infty}{m_{\rm eff}} (\|\chi_\Lambda/\omega^2\| + \|\chi_\Lambda/\omega^{3/2}\|) \times \text{const}.$$

Note that

$$m_{\rm eff} \sim \Lambda, \quad \|\chi_\Lambda/\omega^2\| \sim \Lambda^{1/2}, \quad \|\chi_\Lambda/\omega^{3/2}\| \sim \log \Lambda,$$

as $\Lambda \to \infty$, we have

$$\lim_{\Lambda \to \infty} \theta(\Lambda) = 0. \tag{2.6}$$

Lemma 2.4 *Suppose that* $\min\{|\inf \sigma(H_{\rm eff})|/3, 2\} > \theta(\Lambda)$. *Then*

$$\sigma(\widehat{H}_\Lambda^\nu) \cap [\inf \sigma(\widehat{H}_\Lambda^\nu), \inf \sigma(\widehat{H}_\Lambda^\nu) + \nu) \subset \sigma_{\rm disc}(\widehat{H}_\Lambda^\nu).$$

In particular $\widehat{H}_\Lambda^\nu$ has a ground state.

Proof: See [18, Lemma 10]. □

The number operator N of $\mathcal{F}$ is defined by

$$N = \sum_{j=1,2} \int a^*(k,j) a(k,j) d^3k.$$

I.e.,

$$(N\Psi)^{(n)} = n\Psi^{(n)},$$
$$D(N) = \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \sum_{n=0}^{\infty} n^2 \|\Psi^{(n)}\|^2 < \infty\}.$$

A ground state of $\widehat{H}_\Lambda^\nu$ is denoted by $\varphi_{\rm g}(\nu)$.

Lemma 2.5 *Suppose that* $\min\{|\inf \sigma(H_{\rm eff})|/3, 2\} > \theta(\Lambda)$. *Then, for* ν *such that* $|\inf \sigma(H_{\rm eff})| > 3\theta(\Lambda) + \nu$,

$$\frac{\|N^{1/2}\varphi_{\rm g}(\nu)\|}{\|\varphi_{\rm g}(\nu)\|} \le C(\max_\mu \|\nabla_\mu V\|_\infty) \frac{\|\chi_\Lambda/\omega^{5/2}\|}{m_{\rm eff}} \tag{2.7}$$

with some constant C.

Proof: We set $E = \inf \sigma(\widehat{H}_\Lambda^\nu)$. Since

$$[\widehat{H}_\Lambda^\nu, a(k,j)] = -(\omega(k)+\nu) a(k,j) + [\delta V^\nu, a(k,j)],$$

we have

$$(\widehat{H}_\Lambda^\nu - E + \omega(k) + \nu) a(k,j) \varphi_{\rm g}(\nu) = [\delta V^\nu, a(k,j)] \varphi_{\rm g}(\nu).$$

Note that

$$V(\cdot - K^\nu/m_{\rm eff}) = e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}} V e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}},$$

where K^ν is defined by K with ω replaced by $\omega + \nu$. Then we see that

$$[\delta V^\nu, a(k,j)] = e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}} [V, e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}} a(k,j) e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}}] e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}}.$$

Since

$$e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}} a(k,j) e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}} = a(k,j) - \frac{i}{\sqrt{2} m_{\rm eff}} p \cdot \varrho^\nu(k,j),$$

it follows that

$$[\delta V^\nu, a(k,j)] = e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}} \left(\frac{1}{\sqrt{2} m_{\rm eff}} (\nabla V) \cdot \varrho^\nu(k,j) \right) e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}}.$$

Thus we obtain that

$$\begin{aligned} a(k,j)\varphi_{\rm g}(\nu) &= (\widehat{H}_\Lambda^\nu - E + \omega(k) + \nu)^{-1} \times \\ &\times e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}} \left(\frac{1}{\sqrt{2} m_{\rm eff}} (\nabla V) \cdot \varrho^\nu(k,j) \right) e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}} \varphi_{\rm g}(\nu). \end{aligned}$$

Using this identity we see that

$$\begin{aligned}
&(N^{1/2}\varphi_{\rm g}(\nu), N^{1/2}\varphi_{\rm g}(\nu)) \\
&= \sum_{j=1,2}\int \|a(k,j)\varphi_{\rm g}(\nu)\|^2 d^3k \\
&= \sum_{j=1,2}\int \Bigg\| (\widehat{H}^{\nu}_{\Lambda} - E + \omega(k) + \nu)^{-1} \times \\
&\qquad\qquad \times e^{-i\frac{p\cdot K^{\nu}}{m_{\rm eff}}}\left(\frac{1}{\sqrt{2}m_{\rm eff}}(\nabla V)\varrho^{\nu}(k,j)\right) e^{i\frac{p\cdot K^{\nu}}{m_{\rm eff}}}\varphi_{\rm g}(\nu)\Bigg\|^2 d^3k \\
&\le 3\sum_{\mu=1}^{3}\sum_{j=1,2}\int \left(\frac{1}{\omega(k)}\|\nabla_\mu V\|_\infty\right)^2 \left|\frac{1}{\sqrt{2}m_{\rm eff}}\varrho^{\nu}_{\mu}(k,j)\right|^2 d^3k \|\varphi_{\rm g}(\nu)\|^2 \\
&\le C\left((\max_\mu \|\nabla_\mu V\|_\infty)\right)^2 \frac{\|\chi_\Lambda/\omega^{5/2}\|^2}{{m_{\rm eff}}^2}\|\varphi_{\rm g}(\nu)\|^2.
\end{aligned}$$

Hence the lemma follows. □

Remark 2.6 *Although we used a formal calculation of $a(k,j)$ in the proof of Lemma 2.5, (2.7) can be justified in [19] rigorously.*

We normalize $\varphi_{\rm g}(\nu)$, i.e.,

$$\|\varphi_{\rm g}(\nu)\| = 1.$$

Take a subsequence ν' such that $\varphi_{\rm g}(\nu')$ weakly converges to a vector $\varphi_{\rm g}$ as $\nu' \to \infty$.

Proposition 2.7 *Assume that $\varphi_{\rm g} \neq 0$. Then $\varphi_{\rm g}$ is a ground state of $H_{\rm dip}$.*

Proof: See [1, Lemma 4.9]. □

Theorem 2.8 *There exists Λ_* such that for $\Lambda > \Lambda_*$, $H_{\rm dip}$ has a ground state.*

Proof: It is enough to prove $\varphi_{\rm g} \neq 0$ by Proposition 2.7. Let E_B denote the spectral projection of $H_{\rm eff}$ to a Borel set $B \subset \mathbb{R}$. Let P_Ω be the projection onto the one-dimensional subspace $\{\alpha\Omega \mid \alpha \in \mathbb{C}\}$, and we set

$$Q = E_{[\Sigma+\delta,\infty)} \otimes P_\Omega$$

with some $\delta > 0$ such that

$$\delta > \frac{3}{2}\theta(\Lambda).$$

Note that $1 \otimes N + 1 \otimes P_\Omega \ge 1$. Hence

$$E_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega \ge 1 - 1 \otimes N - Q. \tag{2.8}$$

Suppose that $\min\{|\inf\sigma(H_{\rm eff})|/3, 2\} > \theta(\Lambda)$. Then it is established in [18, Lemma 12] that

$$\frac{\|Q\varphi_{\rm g}(\nu')\|}{\|\varphi_{\rm g}(\nu')\|} \le \sqrt{\frac{\theta(\Lambda)}{\delta - \frac{3}{2}\theta(\Lambda)}} \tag{2.9}$$

for ν' such that

$$|\inf\sigma(H_{\rm eff})| > 3\theta(\Lambda) + \nu'. \tag{2.10}$$

Then for ν' such as (2.10), we have by (2.8),

$$\begin{aligned}
&(\varphi_{\rm g}(\nu'), E_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega \varphi_{\rm g}(\nu')) \\
&\ge \|\varphi_{\rm g}(\nu')\|^2 - (\varphi_{\rm g}(\nu'), N\varphi_{\rm g}(\nu')) - (\varphi_{\rm g}(\nu'), Q\varphi_{\rm g}(\nu')) \\
&= 1 - \left\{ \frac{C\|\chi_\Lambda/\omega^{5/2}\|}{m_{\rm eff}} (\max_\mu \|\nabla_\mu V\|_\infty) \right\}^2 - \frac{\theta(\Lambda)}{\delta - \frac{3}{2}\theta(\Lambda)}.
\end{aligned}$$

Note that by (2.6),

$$\lim_{\Lambda\to\infty} \frac{\|\chi_\Lambda/\omega^{5/2}\|}{m_{\rm eff}} = 0, \quad \lim_{\Lambda\to\infty} \frac{\theta(\Lambda)}{\delta - \frac{3}{2}\theta(\Lambda)} = 0.$$

Hence for sufficiently large Λ,

$$(\varphi_{\rm g}(\nu'), (E_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega)\varphi_{\rm g}(\nu')) > \epsilon$$

uniformly in ν' with some $\epsilon > 0$. Take $\nu' \to \infty$ on the both sides above. Since $E_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega$ is a finite rank operator, we have

$$(\varphi_{\rm g}, (E_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega)\varphi_{\rm g}) > \epsilon,$$

which implies $\varphi_{\rm g} \neq 0$. Then $\varphi_{\rm g}$ is a ground state of $\widehat{H}_\Lambda$. Hence $H_{\rm dip}$ has a ground state. □

Remark 2.9 *The uniqueness of of the ground state of $H_{\rm dip}$ can be also established. See [14].*

3 Stability of matter

As a corollary of Proposition 2.2 we can see a stability of matter with respect to Λ. Stability of matter investigated in this section is pointed out in e.g., Lieb and Loss [22, 23] and Fefferman, Fröhlich and Graf [7].

3.1 $g(\Lambda)/\Lambda^z$

In the case of $V = 0$, from Proposition 3.3 it follows that

$$g(\Lambda) = \inf\sigma(H_{\rm dip}).$$

We want to see the asymptotic behavior of $g(\Lambda)$ as $\Lambda \to \infty$.

Remark 3.1 *From a formal perturbation theory it follows that*

$$g(\Lambda) \sim (f\otimes\Omega, H_{\rm dip} f\otimes\Omega) = (f\otimes\Omega, \frac{1}{2m}(p_x^2 + e^2 A_\Lambda(0)^2) f\otimes\Omega) \sim \Lambda^2$$

as $\Lambda\to\infty$*. As will be seen later, this is, however, incorrect.*

Since

$$\|\chi_\Lambda/\sqrt{t^2+\omega^2}\|^2 = \frac{4\pi}{(2\pi)^3}\left\{(\Lambda-\kappa) + t\left(\tan^{-1}\frac{\kappa}{t} - \tan^{-1}\frac{\Lambda}{t}\right)\right\},$$

and

$$\begin{aligned}&\|\chi_\Lambda/(t^2+\omega^2)\|^2\\ &= \frac{4\pi}{(2\pi)^3}\left\{\frac{1}{2t}\left(\tan^{-1}\frac{\Lambda}{t} - \tan^{-1}\frac{\kappa}{t}\right) + \frac{1}{2}\left(\frac{\kappa}{t^2+\kappa^2} - \frac{\Lambda}{t^2+\Lambda^2}\right)\right\},\end{aligned}$$

we have

$$g(\Lambda) = 4\Lambda^2\int_0^\infty \frac{(\tan^{-1} r - \frac{r}{1+r^2}) - \left(\tan^{-1} r\left(\frac{\kappa}{\Lambda}\right) - \frac{r\left(\frac{\kappa}{\Lambda}\right)}{1+r^2\left(\frac{\kappa}{\Lambda}\right)^2}\right)}{(2\pi)^3 m r + \frac{8\pi}{3}\Lambda\left\{(r-\tan^{-1} r) - (r\left(\frac{\kappa}{\Lambda}\right) - \tan^{-1} r\left(\frac{\kappa}{\Lambda}\right))\right\}}\frac{dr}{r^2}.$$

In [18] the following proposition is established.

Proposition 3.2 *Assume that* $(2\pi)^3 m > 8\pi\kappa/3$*. Then*

$$\frac{8}{3}\left(\frac{3}{8\pi}\frac{1}{(2\pi)^3}\frac{1}{m}\right)^{1/2}\frac{\pi}{2} \le \lim_{\Lambda\to\infty}\frac{g(\Lambda)}{\Lambda^{3/2}} \le \frac{8}{3}\left(\frac{9}{8\pi}\frac{1}{(2\pi)^3}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}.$$

3.2 $g(\Lambda, N)/N^z$

We consider an N particle system. We assume simply that each particle has mass m and there is no external potential. The Hamiltonian, $H_{\rm dip}^N$, is defined as a self-adjoint operator acting on $L^2(\mathbb{R}^{3N})\otimes\mathcal{F}$, and is given by

$$H_{\rm dip}^N = \sum_{j=1}^N \frac{1}{2m}(p_j + A_{j\Lambda}(0))^2 + H_{\rm f},$$

where

$$A_{j\Lambda}(0) = \sum_{j'=1,2}\int \frac{\chi_{j\Lambda}(k)}{\sqrt{2\omega(k)}} e(k,j')\left\{a^*(k,j') + a(k,j')\right\} d^3k.$$

Let

$$\inf\sigma(H_{\rm dip}^N) = g(\Lambda, N).$$

We consider the two cases such as

(1) $\chi_{j\Lambda}(k) = \chi_\Lambda(k), \quad j = 1, ..., N,$

(2) $\chi_{j\Lambda}$, $j=1,...,N$, are characteristic functions on closed sets in $\mathbb{R}^3$ such as

$$\mathrm{supp}\chi_{j\Lambda}\cap\mathrm{supp}\chi_{i\Lambda}\cap\{0\}=\emptyset,\quad (i\neq j).$$

Intuitively (1) describes that N electrons interact each others by exchanging photons, but in (2), they do not. We expect that $g(\Lambda,N)\sim N$ for a sufficiently large N in case (2). We have a proposition.

Proposition 3.3 *In the case of (1),*

$$g(\Lambda,N)=\frac{N}{\pi}\int_{-\infty}^{\infty}\frac{t^2\|\chi_\Lambda/(t^2+\omega^2)\|^2}{m+\frac{2}{3}N\|\chi_\Lambda/\sqrt{t^2+\omega^2}\|^2}dt,$$

in the case of (2),

$$g(\Lambda,N)=\sum_{j=1}^{N}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t^2\|\chi_{j\Lambda}/(t^2+\omega^2)\|^2}{m+\frac{2}{3}\|\chi_{j\Lambda}/\sqrt{t^2+\omega^2}\|^2}dt.$$

Proof: See [17]. □

In the case of (1), in a similar manner as in Proposition 3.2 we can prove the following proposition.

Proposition 3.4 *We assume case (1) and* $(2\pi)^3m>8\pi\kappa/3$. *Then*

$$\frac{8}{3}\left(\frac{3}{8\pi}\frac{1}{(2\pi)^3}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}\leq\lim_{\Lambda,N\to\infty}\frac{g(\Lambda,N)}{\sqrt{N}\Lambda^{3/2}}\leq\frac{8}{3}\left(\frac{9}{8\pi}\frac{1}{(2\pi)^3}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}.$$

Proof: see [18]. □

In the case of (2), if we adjust $\chi_{j\Lambda}$ such as

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t^2\|\chi_{j\Lambda}/(t^2+\omega^2)\|^2}{m+\frac{2}{3}\|\chi_{j\Lambda}/\sqrt{t^2+\omega^2}\|^2}dt=g$$

with some constant g independent of j. Then

$$g(\Lambda,N)=Ng.$$

4 Effective mass

In this section, instead of H_{dip}, we revive H_Λ.

4.1 Translation invariance

The momentum of the quantized radiation field is given by

$$P_{\mathrm{f}} = \sum_{j=1,2} \int k a^*(k,j) a(k,j) dk$$

and the total moment by

$$P_{\mathrm{total}} = p_x \otimes 1 + 1 \otimes P_{\mathrm{f}}.$$

Let us assume that $V \equiv 0$. Then we see that

$$[H_\Lambda, P_{\mathrm{total}\mu}] = 0, \quad \mu = 1,2,3.$$

Hence H_Λ and $\mathcal{H}$ can be decomposable with respect to $\sigma(P_{\mathrm{total}}) = \mathbb{R}^3$, i.e.,

$$\mathcal{H} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}(p) dp, \quad H_\Lambda = \int_{\mathbb{R}^3}^{\oplus} H_\Lambda(p) dp.$$

Note that

$$\begin{aligned} e^{-ix\otimes P_{\mathrm{f}}} P_{\mathrm{total}} e^{ix\otimes P_{\mathrm{f}}} &= p_x, \\ e^{-ix\otimes P_{\mathrm{f}}} H_\Lambda e^{ix\otimes P_{\mathrm{f}}} &= \frac{1}{2m}(p_x \otimes 1 - 1 \otimes P_{\mathrm{f}} - e1 \otimes A_\Lambda(0)) + 1 \otimes H_{\mathrm{f}}. \end{aligned}$$

From this we obtain that for each $p \in \mathbb{R}^3$,

$$\begin{aligned} \mathcal{H}(p) &\cong \mathcal{F}, \\ H_\Lambda(p) &\cong \frac{1}{2m}(p - P_{\mathrm{f}} - eA_\Lambda(0)) + H_{\mathrm{f}}. \end{aligned}$$

Let

$$E_{m,\Lambda}(p) = \inf \sigma(H_\Lambda(p)), \quad p \in \mathbb{R}^3.$$

Lemma 4.1 *There exist constants p_* and e_* such that for*

$$(p,e) \in \mathcal{O} = \{(p,e) \in \mathbb{R}^3 \times \mathbb{R} | |p| < p_*, |e| < e^*\},$$

a ground state $\varphi_{\mathrm{g}}(p)$ of $H_\Lambda(p)$ exists and it is unique. Moreover $\varphi_{\mathrm{g}}(p) = \varphi_{\mathrm{g}}(p,e)$ is strongly analytic and $E_{m,\Lambda}(p) = E_{m,\Lambda}(p,e)$ analytic with respect to $(p,e) \in \mathcal{O}$.

Proof: See [21]. □

Remark 4.2 *Note that $E_{m,\Lambda}(p) \in \sigma_{\mathrm{disc}}(H_\Lambda(p))$ for $(p,e) \in \mathcal{O}$ and*

$$E_{m,\Lambda}(p) = E_{m,\Lambda}(-p).$$

In what follows we assume that $(p, e) \in \mathcal{O}$. The effective mass $m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)$ is the inverse of the curvature of energy-momentum graph $(p, E(p))$ in $\mathbb{R}^3 \times \mathbb{R}$ at $p = 0$. Precisely m_{eff} is given by

$$E_{m,\Lambda}(p) - E_{m,\Lambda}(0) = \frac{1}{2m_{\text{eff}}}|p|^2 + O(|p|^3),$$

or

$$\frac{1}{m_{\text{eff}}} = \frac{1}{3}\Delta_p E_{m,\Lambda}(p, e)\lceil_{p=0}.$$

Removal of the ultraviolet cutoff Λ through mass renormalization means to find sequences

$$\Lambda \to \infty, \quad m \to 0$$

such that $E_{m,\Lambda}(p) - E_{m,\Lambda}(0)$ has a nondegenerate limit. In order to find such sequences, we want to find constants

$$\beta < 0, \quad 0 < b \tag{4.1}$$

such that

$$\lim_{\Lambda\to\infty} m_{\text{eff}}(e^2, \Lambda, \kappa\Lambda^\beta, (b\Lambda)^\beta) = m_{\text{ph}}, \tag{4.2}$$

where m_{ph} is a given constant. It is well known that

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3}\sum_{\mu=1,2,3} \times$$
$$\times \frac{(\varphi_{\text{g}}(0), (P_{\text{f}} + eA_\Lambda(0))_\mu (H_\Lambda(0) - E_{m,\Lambda}(0))^{-1}(P_{\text{f}} + eA_\Lambda(0))_\mu \varphi_{\text{g}}(0))}{(\varphi_{\text{g}}(0), \varphi_{\text{g}}(0))}. \tag{4.3}$$

From this we see that m_{eff}/m is a function of e^2, Λ/m and κ/m. Let

$$\frac{m_{\text{eff}}}{m} = f(e^2, \Lambda/m, \kappa/m).$$

To find constants (4.1), it is enough to find constants

$$0 \le \gamma < 1, \quad 0 < b_0$$

such that

$$\lim_{\Lambda\to\infty} \frac{f(e^2, \Lambda/m, \kappa/m)}{(\Lambda/m)^\gamma} = b_0.$$

Actually, taking

$$\beta = \frac{-\gamma}{1-\gamma} < 0, \quad b = 1/b_1^{1/\gamma},$$

we see that

$$\lim_{\Lambda\to\infty} m_{\text{eff}}(e^2, \Lambda, \kappa\Lambda^\beta, (b\Lambda)^\beta) = b_0 b_1,$$

where b_1 is a parameter, which is adjusted such as

$$b_0 b_1 = m_{\text{ph}}.$$

Hence (4.2) has been established. It is seen by (4.3) that

$$f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha\frac{8}{3\pi}\log(\frac{\Lambda/m+2}{\kappa/m+2}) + O(\alpha^2), \tag{4.4}$$

where

$$\alpha = \frac{e^2}{4\pi}.$$

By (4.4) one may assume that

$$f(e^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{\alpha(8/3\pi)+\alpha^2 c}$$

for sufficiently small α and large Λ with some constant c. Then by expanding m_{eff}/m to order α^2 one may expect that

$$f(e^2, \Lambda/m, \kappa/m) \approx 1+\alpha\frac{8}{3\pi}\log(\frac{\Lambda}{m})+\frac{1}{2}\alpha^2\left(\frac{8}{3\pi}\log(\frac{\Lambda}{m})\right)^2+c\alpha^2\log(\frac{\Lambda}{m})+O(\alpha^3). \tag{4.5}$$

Hence the coefficient of α^2 may diverge as $[\log(\Lambda/m)]^2$ as $\Lambda \to \infty$. It is, however, that (4.5) is not confirmed. Instead of (4.5) we prove in this section that the coefficient of α^2 diverge as $\sqrt{\Lambda/m}$ as $\Lambda \to \infty$, i.e., there exists a constant $C > 0$ such that

$$f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha\frac{8}{3\pi}\log(\frac{\Lambda/m+2}{\kappa/m+2}) + \alpha^2 C\sqrt{\Lambda/m} + O(\alpha^3).$$

The effective mass and its renormalization have been studied from a mathematical point of view by many authors. Spohn [27] investigates the effective mass of the Nelson model [25] from a functional integral point of view. Lieb and Loss [23] studied mass renormalization and binding energies of models of matter coupled to radiation fields including the Pauli-Fierz model. Hainzl and Seiringer [9] computed exactly the leading order in α of the effective mass of the Pauli-Fierz Hamiltonian with spin 1/2.

4.2 Asymptotics

We split $H_\Lambda(0)$ as

$$H(0) = H_0 + eH_1 + \frac{e^2}{2}H_2,$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}{P_{\rm f}}^2 + H_{\rm f}, \\ H_1 &= \frac{1}{2}(P_{\rm f}\cdot A_\Lambda(0) + A_\Lambda(0)\cdot P_{\rm f}), \\ H_2 &= A_\Lambda(0)\cdot A_\Lambda(0). \end{aligned}$$

Let

$$\varphi_{\rm g}(0)=\sum_{n=0}^\infty \frac{e^n}{n!}\varphi_n,\quad E(0)=\sum_{n=0}^\infty \frac{e^{2n}}{(2n)!}E_{2n}.$$

Directly we see that

$$E_0=E_1=E_2=E_3=0, \tag{4.6}$$

$$\varphi_0=\Omega,\quad \varphi_1=0,\quad \varphi_2=-H_0^{-1}H_2\Omega,\quad \varphi_3=3H_0^{-1}H_1H_0^{-1}H_2\Omega. \tag{4.7}$$

Substitute (4.6) and (4.7) into formula (4.3). Then we obtain that

$$\frac{m}{m_{\rm eff}}=1-e^2\frac{2}{3}\sum_{\mu=1}^3\left(\Omega,A_\mu H_0^{-1}A_\mu\Omega\right)$$

$$-e^4\frac{2}{3}\sum_{\mu=1}^3\left\{2\left(\Psi_3^\mu,H_0^{-1}\Psi_1^\mu\right)+\left(\Psi_2^\mu,H_0^{-1}\Psi_2^\mu\right)-2\left(\Psi_2^\mu,H_0^{-1}H_1H_0^{-1}\Psi_1^\mu\right)\right.$$

$$\left.-\frac{1}{2}\left(\Psi_1^\mu,H_0^{-1}H_2H_0^{-1}\Psi_1^\mu\right)+\left(\Psi_1^\mu,H_0^{-1}H_1H_0^{-1}H_1H_0^{-1}\Psi_1^\mu\right)\right\}+O(e^6), \tag{4.8}$$

where

$$\Psi_1^\mu=A_\mu\Omega,$$

$$\Psi_2^\mu=-\frac{1}{2}P_{{\rm f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega,$$

$$\Psi_3^\mu=\frac{1}{2}\left\{-A_\mu H_0^{-1}(A^+\cdot A^+)\Omega+\frac{1}{2}P_{{\rm f}\mu}H_0^{-1}(P_{\rm f}\cdot A+A\cdot P_{\rm f})H_0^{-1}(A^+\cdot A^+)\Omega\right\},$$

and

$$A^-=\sum_{j=1,2}\int\frac{\chi_\Lambda(k)}{\sqrt{2\omega(k)}}e(k,j)a(k,j)dk,$$

$$A^+=\sum_{j=1,2}\int\frac{\chi_\Lambda(k)}{\sqrt{2\omega(k)}}e(k,j)a^*(k,j)dk.$$

We compute the coefficients of e^2 and e^4 in (4.8). Let

$$\frac{1}{F_j}=\frac{1}{r_j^2/2+r_j},\quad j=1,2,$$

$$\frac{1}{F_{12}}=\frac{1}{(r_1^2+2r_1r_2X+r_2^2)/2+r_1+r_2},\quad r_1,r_2\ge 0,\quad -1\le X\le 1.$$

A direct calculation shows that

$$\frac{m}{m_{\rm eff}}=1-\alpha a_1(\Lambda/m,\kappa/m)-\alpha^2a_2(\Lambda/m,\kappa/m)+O(\alpha^3),$$

where

$$a_1(\Lambda/m,\kappa/m)=\frac{8}{3\pi}\log\left(\frac{\Lambda/m+2}{\kappa/m+2}\right)$$

and

$$a_2(\Lambda/m, \kappa/m) = \frac{(4\pi)^2}{(2\pi)^6} \frac{2}{3} \sum_{j=1}^{6} b_j(\Lambda/m, \kappa/m), \tag{4.9}$$

$$\begin{aligned}
b_1(\Lambda/m, \kappa/m) &= -\int (1+X^2) \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \frac{1}{F_{12}}, \\
b_2(\Lambda/m, \kappa/m) &= \int (1+X^2) \left(\frac{1}{F_{12}} \right)^3 \frac{r_1^2 + 2r_1 r_2 X + r_2^2}{2}, \\
b_3(\Lambda/m, \kappa/m) &= \int X(-1+X^2) r_1 r_2 \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \left(\frac{1}{F_{12}} \right)^2, \\
b_4(\kappa/m \Lambda/m) &= -\int (1+X^2) \frac{1}{F_1} \frac{1}{F_2}, \\
b_5(\Lambda/m, \kappa/m) &= \int (1-X^2) \left(\frac{r_1^2}{F_1^2} + \frac{r_2^2}{F_2^2} \right) \frac{1}{F_{12}}, \\
b_6(\Lambda/m, \kappa/m) &= \int X(-1+X^2) r_1 r_2 \frac{1}{F_1} \frac{1}{F_2} \frac{1}{F_{12}},
\end{aligned}$$

and

$$\int = \int_{-1}^{1} \mathrm{d}X \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_1 \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_2 \pi r_1 r_2.$$

The main theorem in this section is as follows.

Theorem 4.3 *There exist strictly positive constants $C_{\min}$ and $C_{\max}$ such that*

$$C_{\min} \le \lim_{\Lambda \to \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \le C_{\max}.$$

Proof: We show an outline of a proof. See [21] for details. We can prove that there exists a constant $C > 0$ such that

$$\begin{aligned}
&|b_j(\Lambda/m)| \le C[\log(\chi_\Lambda/m)]^2, \quad j = 1, 4, \\
&|b_2(\Lambda/m)| \le C(\Lambda/m)^{1/2}, \\
&|b_j(\Lambda/m)| \le C \log(\Lambda/m), \quad j = 3, 5, 6.
\end{aligned}$$

Hence there exists a constant $C_{\max}$ such that

$$\lim_{\Lambda \to \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \le C_{\max}.$$

Next we can show that there exists a positive constant $\xi > 0$ such that

$$\lim_{\Lambda \to \infty} \sqrt{\Lambda/m} \frac{d}{d(\Lambda/m)} b_2(\Lambda/m) > \xi,$$

which implies that there exists a constant ξ' such that

$$\xi' \leq \lim_{\Lambda\to\infty} \frac{b_2(\chi_\Lambda/m)}{\sqrt{\chi_\Lambda/m}}.$$

Thus we have

$$C_{\min} \leq \lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m,\kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\max}.$$

□

Remark 4.4 *(1)* $a_2(\Lambda/m,\kappa/m)/\sqrt{\Lambda/m}$ *converges to a nonnegative constant as* $\Lambda\to\infty$. *(2) By (4.9), we can define* $a_2(\Lambda/m,0)$ *since* $b_j(\Lambda/m)$ *with* $\kappa=0$ *are finite. Moreover* $a_2(\Lambda/m,0)$ *also satisfies Theorem 4.3. (3) In the case of* $\kappa=0$, *Chen [6] established that* $H(0)$ *has a ground state* $\varphi_{\rm g}(0)$ *but does not for* $H_\Lambda(p)$ *with* $p\neq 0$.

4.3 Concluding remarks

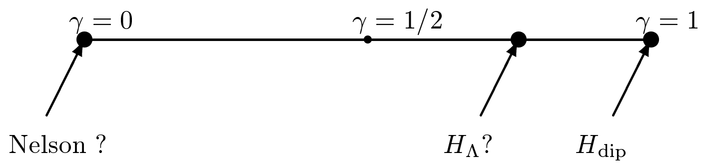

Figure 2: Mass renormalization

(H_Λ) Theorem 4.3 may suggests $\gamma\geq 1/2$ uniformly in e but $e\neq 0$.

(Nelson models) It is expected that the effective mass of the Nelson model can be trivially renormalized, i.e., $\gamma=0$. See [11].

($H_{\rm dip}$) Let $V\equiv 0$. Note that

$$[H_{\rm dip}, P_{\rm total}]\neq 0.$$

It has been seen, however, that

$$[UH_{\rm dip}U^{-1}, P_{\rm total}]=0.$$

Then we can define the effective mass $m_{\rm eff}$ for $UH_{\rm dip}U^{-1}$, and which is

$$m_{\rm eff}/m = 1+\alpha\frac{4}{3\pi}(\Lambda/m-\kappa/m).$$

Hence $\gamma=1$, then the mass renormalization for $H_{\rm dip}$ is not available.

See Fig. 2.

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