Ground state measure and its applications

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1 Introduction

In this paper we shall consider structures of ground states of a model describing an interaction between a particle and a quantized scalar bose field, which is called the "Nelson model"[15],[18]. Basic ideas in this paper is due to a fairly nice work of H.Spohn [22], in which he studies the spin-boson model. The Hamiltonian, H, of the Nelson model is defined as a self-adjoint operator acting on Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}$, where $\mathcal{F}$ denotes a Boson Fock space. The existence of the ground states, Ψ_{g}, of H is established in e.g., [2],[4],[12],[23]. The main results presented here is to give the expectation-value of the number of bosons of Ψ_{g} and its boson distribution by means of a ground state measure constructed in this paper. Especially the localization of bosons of Ψ_{g} is proved. The ground state measure, μ, on the set of paths, $\mathbf{\Omega}$, gives an integral representation of the expectation-value of certain operator A in $\mathcal{H}$, i.e.,

$$(\Psi_{\mathrm{g}}, A\Psi_{\mathrm{g}}) = \int_{\mathbf{\Omega}} f_A(q)\mu(dq),$$

where f_A is a density function corresponding to A. This integral representation leads us to the goal of this paper. Detailed arguments shall be published elsewhere [2], and refer to see [17],[21],[22]. This paper is organized as follows: section 2 gives a definition of models considered in this paper. In section 3 we review the second quantizations. Section 4 is devoted to investigating the ground states. In section 5 we give further problems on the Pauli-Fierz model in nonrelativistic quantum electrodynamics.

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2 Scalar quantum field models

Let $\mathcal{F} := \oplus_{n=0}^{\infty} \otimes_s^n L^2(\mathbb{R}^d) := \oplus_{n=0}^{\infty}\mathcal{F}_n$, where $\otimes_s^n$ denotes the n-fold symmetric tensor product with $\otimes_s^0 L^2(\mathbb{R}^d) := \mathbb{C}$. The bare vacuum, $\Omega \in \mathcal{F}$, is defined by $\Omega := \{1, 0, 0, ...\}$. Let $a^\dagger(f)$ and $a(g)$ be the creation operator and the annihilation operator smeared by $f, g \in L^2(\mathbb{R}^d)$, respectively, which are linear in f and g. Let $\mathcal{F}_{\text{fin}}$ be the finite particle subspace of $\mathcal{F}$:

$$\mathcal{F}_{\text{fin}} := \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F} \,\middle|\, \text{there exists } n_0 \text{ such that } \Psi^{(m)} = 0, m \geq n_0 \right\}.$$

They satisfy canonical commutation relations (CCR), i.e.,

$$[a(f), a^\dagger(g)] = (\bar{f}, g)_{L^2(\mathbb{R}^d)}, \quad [a^\sharp(f), a^\sharp(g)] = 0,$$

on $\mathcal{F}_{\text{fin}}$, where $a^\sharp$ denotes a or $a^\dagger$, and $(\cdot, \cdot)_{\mathcal{K}}$ the scalar product on Hilbert space $\mathcal{K}$. We denote by $\|\cdot\|_{\mathcal{K}}$ its associated norm. Unless confusion arises we omit $\mathcal{K}$ in $(\cdot, \cdot)_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}}$, respectively. $a^\sharp$ also satisfies that $(\Psi, a(f)\Phi) = (a^\dagger(\bar{f})\Psi, \Phi)$ for $\Psi, \Phi \in \mathcal{F}_{\text{fin}}$. For dense subset $\mathcal{K} \subset L^2(\mathbb{R}^d)$,

$$\mathcal{F}(\mathcal{K}) := l.h.\{a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega, \Omega | f_j \in \mathcal{K}, j = 1, ..., n, n \in \mathbb{N}\}$$

is dense in $\mathcal{F}$. We define the free Hamiltonian, H_{f}, in $\mathcal{F}$ by

$$H_{\text{f}}\Omega := 0,$$
$$H_{\text{f}} a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega := \sum_{j=1}^{n} a^\dagger(f_1) \cdots a^\dagger(\omega f_j) \cdots a^\dagger(f_n)\Omega,$$
$$f_j \in D(\omega), \quad j = 1, ..., n, \quad n \in \mathbb{N},$$

where $D(T)$ denotes the domain of T, $\omega := \omega(k) := \sqrt{|k|^2 + m^2}$, $m \geq 0$. Here m denotes the mass of the quantized scalar bose field. Field operators $\phi(f)$ are defined by

$$\phi(f) := \frac{1}{\sqrt{2}}(a^\dagger(\bar{f}) + a(f)), \quad f \in L^2(\mathbb{R}^d).$$

Note that $H_{\text{f}}\lceil_{\mathcal{F}(D(\omega))}$ and $\phi(f)\lceil_{\mathcal{F}_{\text{fin}}}$ are essentially self-adjoint, respectively. It is known that $\sigma(H_{\text{f}}) = [0, \infty)$ and $\sigma_{\text{p}}(H_{\text{f}}) = \{0\}$. The Hamiltonian, H, considered in this paper is defined by

$$H := H_{\text{p}} \otimes 1 + 1 \otimes H_{\text{f}} + \alpha H_{\text{I}}$$

on $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F} \cong L^2(\mathbb{R}^d; \mathcal{F})$, where $\alpha \in \mathbb{R}$ is a coupling constant, and

$$H_\mathrm{I} := \phi(e^{ikx}\hat{\lambda}),$$

$$H_\mathrm{p} := -\Delta/2 + V,$$

where $\hat{\lambda}$ is the Fourier transform of λ. A reasonable physical choice of $\hat{\lambda}$ is of the form

$$\hat{\lambda} = \hat{\rho}/\sqrt{(2\pi)^d \omega},$$

where ρ describes a charge distribution, i.e.,

$$\sqrt{(2\pi)^d}\hat{\rho}(0) = \int_{\mathbb{R}^d} \rho(x)dx = \alpha.$$

For simplicity we assume that external potential $V = V_+ - V_-$ satisfies that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and that V_- is infinitesimally small with respect to Δ in the sense of form. Throughout this paper we assume that

$$\overline{\hat{\lambda}(k)} = \hat{\lambda}(-k).$$

Let $\hat{\lambda}, \hat{\lambda}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. Then it is known that, for arbitrary α, H is self-adjoint on $D(H_\mathrm{p} \otimes 1) \cap D(1 \otimes H_\mathrm{f})$ and bounded from below. Moreover it is essentially self-adjoint on any core of $H_\mathrm{p} \otimes 1 + 1 \otimes H_\mathrm{f}$.

Proposition 2.1 ([2],[12]) *Let $\hat{\lambda}/\omega, \hat{\lambda}/\sqrt{\omega}, \hat{\lambda} \in L^2(\mathbb{R}^d)$. Then there exists α_* such that for $|\alpha| \le \alpha_*$ the ground states, Ψ_g, of H exist. Moreover $(f \otimes \Omega, \Psi_\mathrm{g}) > 0$ for arbitrary nonnegative $f \in L^2(\mathbb{R}^d)$ with $f \not\equiv 0$.*

See Figure 2 for more explicit results on the existence of the ground states of H.

3 Second quantizations

For later use we review the second quantization of operator T on $L^2(\mathbb{R}^d)$. Let T be a contraction operator on $L^2(\mathbb{R}^d)$, i.e., $\|T\| \le 1$. Then we define $\Gamma(T) : \mathcal{F}_{\mathrm{fin}} \to \mathcal{F}_{\mathrm{fin}}$ by

$$\Gamma(T)\Omega := \Omega,$$

$$\Gamma(T)a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega := a^\dagger(Tf_1)\cdots a^\dagger(Tf_n)\Omega,$$
$$f_j \in L^2(\mathbb{R}^d), \quad j=1,...,n, \quad n \in \mathbb{N}.$$

For $\Phi \in \mathcal{F}_{\mathrm{fin}}$ we have $\|\Gamma(T)\Phi\| \le \|\Phi\|$. Thus $\Gamma(T)$ extends to a contraction operator on $\mathcal{F}$. We denote its extension by the same symbol. It is seen that $\Gamma(\cdot)$ is linear in $\cdot$ and that $\Gamma(T)^* = \Gamma(T^*)$. Let h be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^d)$. Then we see that $\Gamma(e^{-th})$ is a strongly continuous symmetric contraction one-parameter semigroup in $t \ge 0$. The second quantization of h, $d\Gamma(h)$, is defined by the generator of $\Gamma(e^{-th})$, i.e.,

$$\Gamma(e^{-th}) = e^{-td\Gamma(h)}, \quad t \ge 0.$$

Actually H_{f} is the second quantization of multiplication operator ω. For nonnegative multiplication operator h in $L^2(\mathbb{R}^d)$, formally, it is written as

$$d\Gamma(h) = \int h(k)a^\dagger(k)a(k)dk. \tag{3.1}$$

The number operator, N, in $\mathcal{F}$ is defined by the second quantization of the identity operator in $L^2(\mathbb{R}^d)$, i.e.,

$$D(N) := \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F} \,\middle|\, \sum_{n=0}^\infty n^2 \left\|\Psi^{(n)}\right\|_{\mathcal{F}_n}^2 < \infty \right\},$$
$$(N\Psi)^{(n)} := n\Psi^{(n)}.$$

Let h be a multiplication operator in $L^2(\mathbb{R}^d)$ such that $s = s_{R+} - s_{R-} + i(s_{I+} - s_{I-})$, where s_{R+} (resp. s_{R-}, s_{I+}, s_{I-}) denotes the real positive (resp. real nonpositive, imaginary positive, imaginary nonpositive) part of s. Then we define

$$d\Gamma(h) := d\Gamma(s_{R+}) - d\Gamma(s_{R-}) + i(d\Gamma(h_{I+}) - d\Gamma(h_{I-})),$$
$$D(d\Gamma(h)) := D(d\Gamma(s_{R+})) \cap D(d\Gamma(s_{R-})) \cap D(d\Gamma(h_{I+})) \cap D(d\Gamma(h_{I-})).$$

4 Ground state measures

Let $\mathbf{\Omega} := (\mathbb{R}^d)^{(-\infty,\infty)}$ be the set of $\mathbb{R}^d$-valued paths and $\mathcal{B}(\mathbf{\Omega})$ the σ-field constructed by cylinder sets. For $T : \mathcal{H} \to \mathcal{H}$, we define

$$\langle T \rangle := (\Psi_{\mathrm{g}}, T\Psi_{\mathrm{g}})_{\mathcal{H}}.$$

For a convenience we denote by $\langle S\rangle$ for $\langle 1\otimes S\rangle$, for $S:\mathcal{F}\to\mathcal{F}$. Our fundamental theorem is as follows:

Theorem 4.1 ([2]) *Let s be such that $\sup_{k\in\mathbb{R}^d}|s(k)|<\infty$. Let $\hat\lambda/\omega$, $\hat\lambda/\sqrt{\omega}$, $\hat\lambda\in L^2(\mathbb{R}^d)$, and $|\alpha|\le\alpha_*$. We assume that $A_1,...,A_m$ are measurable sets in $\mathbb{R}^d$ and let 1_A denote the characteristic function of A. Then there exists a probability measure μ on $(\mathbf{\Omega},\mathcal{B}(\mathbf{\Omega}))$ such that, for $t_1\le\cdots\le t_m$,*

$$\langle 1_{A_1}e^{-(t_2-t_1)H}1_{A_2}\cdots e^{-(t_m-t_{m-1})H}1_{A_m}\rangle=\int_{\mathbf{\Omega}}1_{A_1}(q(t_1))\cdots 1_{A_m}(q(t_m))\mu(dq),$$

$$\langle e^{-\beta d\Gamma(s)}\rangle=\int_{\mathbf{\Omega}}e^{(\alpha^2/2)Z(\beta)}\mu(dq),\quad \beta>0, \tag{4.1}$$

where

$$Z(\beta):=\int_{-\infty}^0 dt\int_0^\infty ds\int_{\mathbb{R}^d}|\hat\lambda(k)|^2e^{-|t-s|\omega(k)}\left(e^{-\beta s(k)}-1\right)e^{ik(q(t)-q(s))}dk.$$

We give a remark on $Z(\beta)$. Since $\left\|\hat\lambda/\omega\right\|<\infty$, we see that

$$|Z(\beta)|\le 2\left\|\hat\lambda/\omega\right\|^2<\infty$$

uniformly in paths $q\in\mathbf{\Omega}$.Thus $Z(\beta)$ is well defined. It is proved in [2] that μ is a Gibbs measure. We call μ the "ground state measure for H". It is easily seen that the right-hand side of (4.1) is analytically continued to $\beta\in\mathbb{C}$. Although it does *not* imply that $\langle e^{-\beta d\Gamma(s)}\rangle$ is well defined for all $\beta\in\mathbb{C}$, we have the following theorem:

Theorem 4.2 ([2]) *Let s, $\hat\lambda$ and α be in Theorem 4.1. Then we have $\Psi_{\rm g}\in D(1\otimes e^{-\beta d\Gamma(s)})$ for all $\beta\in\mathbb{C}$, and (4.1) holds true for all $\beta\in\mathbb{C}$.*

We immediately have the following corollary.

Corollary 4.3 *Let $\hat\lambda$ and α be in Theorem 4.1.Then, for arbitrary $\epsilon\in\mathbb{R}$, we have $\Psi_{\rm g}\in D(1\otimes e^{\epsilon N})$. Moreover*

$$\langle N\rangle=\frac{\alpha^2}{2}\int_{-\infty}^0 dt\int_0^\infty ds\int_{\mathbb{R}^d}dk|\hat\lambda(k)|^2e^{-|t-s|\omega(k)}\int_{\mathbf{\Omega}}e^{ik(q(t)-q(s))}\mu(dq). \tag{4.2}$$

Proof: Putting $s=1$ in Theorem 4.2, we get $\Psi_{\rm g} \in D(1\otimes e^{\epsilon N})$for all $\epsilon \in \mathbb{R}$. (4.2) follows from (4.1) and

$$\langle N\rangle = -\frac{d\langle e^{-\beta N}\rangle}{d\beta}\Big\lceil_{\beta=0}.$$

The proof is complete. Q.E.D.

Corollary 4.3 implies that

$$\sum_{n=0}^{\infty} e^{2\epsilon n}\|\Psi_{\rm g}^{(n)}\|^2_{L^2(\mathbb{R}^d)\otimes\mathcal{F}_n} < \infty, \quad \text{for all } \epsilon > 0.$$

Hence we conclude that $\|\Psi_{\rm g}^{(n)}\|$ decays super-exponentially as $n\to\infty$; it decays faster than $e^{-\epsilon n}$ for arbitrary $\epsilon > 0$. Let $s \in C_0^\infty(\mathbb{R}^d)$. Then, by Theorem 4.2, we see that $\Psi_{\rm g} \in D(d\Gamma(s))$ and

$$|\langle d\Gamma(s)\rangle| \leq (\alpha^2/2)\|s\|_\infty \|\hat{\lambda}/\omega\|^2.$$

Thus map

$$\mathcal{D} : C_0^\infty(\mathbb{R}^d) \ni s \to \langle d\Gamma(s)\rangle \in \mathbb{C}$$

defines a distribution on $C_0^\infty(\mathbb{R}^d)$. Taking into account of the formal expression of $d\Gamma(s)$ (3.1), we denote by $\langle a^\dagger(k)a(k)\rangle$ the kernel of $\mathcal{D}$. From Corollary 4.3 it immediately follows:

Corollary 4.4 *Let $\hat{\lambda}$ and α be in Theorem 4.1. Then for* a.e.$k \in \mathbb{R}^d$,

$$\langle a^\dagger(k)a(k)\rangle = \frac{\alpha^2}{2}|\hat{\lambda}(k)|^2 \int_{-\infty}^0 dt \int_0^\infty ds\ e^{-|t-s|\omega(k)} \int_{\mathbf{\Omega}} e^{ik(q(t)-q(s))}\mu(dq).$$

Note that

$$\int_{\mathbb{R}^d} \langle a^\dagger(k)a(k)\rangle dk = \langle N\rangle.$$

Moreover we see that

$$|\langle a^\dagger(k)a(k)\rangle| \leq \frac{\alpha^2}{2}\frac{|\hat{\lambda}(k)|^2}{\omega(k)^2}, \quad \text{a.e. } k \in \mathbb{R}^d.$$

See Figure 1.

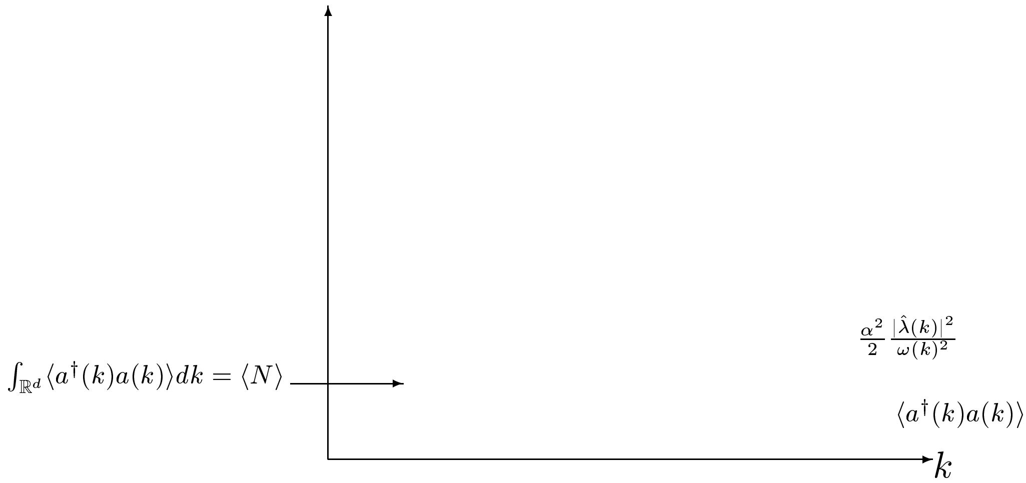

Figure 1: Infrared cutoff $\|\hat{\lambda}/\omega\| < \infty$ and $\langle N\rangle < \infty$

5 Nonrelativistic QED

5.1 The Pauli-Fierz model

The Pauli-Fierz model [1],[3],[5]-[11],[19],[20] in nonrelativistic QED describes an interaction of particles (electrons) and a quantized radiation field (photons). The quantized radiation field is quantized in a Coulomb gage. We assume that the number of the electrons is one and that the electron has spineless. Let

$$\mathcal{F}_{\rm PF} := \oplus_{n=0}^\infty \otimes_s^n \underbrace{L^2(\mathbb{R}^d)\oplus\cdots\oplus L^2(\mathbb{R}^d)}_{d-1} \cong \underbrace{\mathcal{F}\otimes\cdots\otimes\mathcal{F}}_{d-1}.$$

Let $\{b^r(f), b^{\dagger r}(g)\}_{r=1}^{d-1}$ be the annihilation operators and the creation operators, respectively, which satisfy CCR:

$$[b^r(f), b^{\dagger s}(g)] = \delta_{rs}(\bar{f}, g)_{L^2(\mathbb{R}^d)}, \quad [b^{\sharp r}(f), b^{\sharp s}(g)] = 0.$$

Let $H_{\rm f}^{\rm PF}$ be the free Hamiltonian in $\mathcal{F}_{\rm PF}$, i.e.,

$$H_{\rm f}^{\rm PF} := \sum_{r=1}^{d-1}\int \omega(k) b^{\dagger r}(k) b^r(k) dk.$$

The Hamiltonian of the Pauli-Fierz model is defined as an operator in $\mathcal{H}_{\rm PF} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\rm PF} \cong L^2(\mathbb{R}^d; \mathcal{F}_{\rm PF})$ and reads

$$H_{\rm PF} := \frac{1}{2}\left(-i\nabla \otimes 1 - e\mathbf{A}(x)\right)^2 + 1 \otimes H_{\rm f}^{\rm PF} + V \otimes 1,$$

where e is a coupling constant, $\mathbf{A}(x) := (\mathbf{A}_1(x), \cdots, \mathbf{A}_d(x))$,

$$\mathbf{A}_\mu(x) := \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \left(b^{\dagger r}(e_\mu^r \bar{\hat{\lambda}} e^{-ikx}) + b^r(e_\mu^r \hat{\lambda} e^{ikx}) \right),$$

and $e^r := (e_1^r, \cdots, e_d^r)$, polarization vectors; $e^r(k)\cdot e^s(k) = \delta_{rs}$ and $e^r(k)\cdot k = 0$. Note that

$$\mathrm{div}\mathbf{A} = 0.$$

For the Nelson model, the self-adjointness of H for arbitrary α is trivial, since $H_{\rm I}$ is infinitesimally small with respect to $H_{\rm p} \otimes 1 + 1 \otimes H_{\rm f}$. It is not so easy to show self-adjointness of $H_{\rm PF}$ for *arbitrary* $e \in \mathbb{R}$. Let $N_{\rm PF}$ be the number operator in $\mathcal{F}_{\rm PF}$. We have the following proposition:

Proposition 5.1 ([9]) [1] *Let $\hat{\lambda}, \omega^2\hat{\lambda} \in L^2(\mathbb{R}^d)$. We assume that V is relatively bounded with respect to Δ. Then, for arbitrary $\epsilon \in \mathbb{R}$, $H_{\rm PF}$ is essentially self-adjoint on*

$$D(\Delta \otimes 1) \cap D(1 \otimes (H_{\rm f}^{\rm PF})^2) \cap_{k=1}^\infty D(1 \otimes N_{\rm PF}^k).$$

The existence of the ground states of $H_{\rm PF}$ are studied in [1],[6], and their multiplicities in [7],[11]. Moreover $\inf \sigma(H_{\rm PF})$ is investigated in [3],[16].

5.2 Ground states of H and $H_{\rm PF}$

Let

$$\mathrm{gap}(T) := \inf \sigma_{\rm ess}(T) - \inf \sigma(T).$$

The existence of the ground states of H and $H_{\rm PF}$ are deeply related to conditions on m, gap, $\hat{\lambda}$ and coupling constants. Let $\hat{\lambda}/\omega \in L^2(\mathbb{R}^d)$.[2] Then sufficient conditions for the existence of the ground states of H and $H_{\rm PF}$, as far as we know, are in Figures 2 and 3, respectively.

	$m > 0$	$m = 0$
$\mathrm{gap}(H) = \infty$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{R}$
$0 < \mathrm{gap}(H) < \infty$	$\lvert\alpha\rvert \ll 1$	$\lvert\alpha\rvert \ll 1$

Figure 2: α for the existence of the ground states of H.

	$m > 0$	$m = 0$
$\mathrm{gap}(H_{\mathrm{PF}}) = \infty$	$e \in \mathbb{R}$	$\lvert e\rvert \ll 1$
$0 < \mathrm{gap}(H_{\mathrm{PF}}) < \infty$	$\lvert e\rvert \ll 1$	$\lvert e\rvert \ll 1$

Figure 3: e for the existence of the ground states of H_{PF}.

Note that see [4],[23] for a proof of the existence of ground states for case $\mathrm{gap}(H) = \infty$ and $m \geq 0$ in Figure 2, and [8],[9] for case $\mathrm{gap}(H_{\mathrm{PF}}) = \infty$ and $m > 0$ in Figure 3. In [13],[14] the authors give examples such that the ground states of H and H_{PF} exist for the case where $\mathrm{gap}(H) = 0$ and $\mathrm{gap}(H_{\mathrm{PF}}) = 0$, respectively. In [17] no existence of the ground states of H for arbitrary $\alpha \neq 0$ is proved if $\|\hat{\lambda}/\omega\| = \infty$.

5.3 Distribution of bosons for Ψ_{PF}

Let Ψ_{PF} be the ground state of H_{PF} and

$$\langle T \rangle_{\mathrm{PF}} := (\Psi_{\mathrm{PF}}, T\Psi_{\mathrm{PF}})_{\mathcal{H}_{\mathrm{PF}}}.$$

[1] In [9] essential self-adjointness of H_{PF} is proved only for the case where the number of the electrons is *one*. As far as we know it is not clear whether the statement in Proposition 5.1 with N-electrons holds true or not. In [19] self-adjointness of H_{PF} on $D(\Delta \otimes 1) \cap D(1 \otimes H_{\mathrm{f}}^{\mathrm{PF}})$ is proved for sufficiently small $|e|$.

[2] It is not necessarily to assume $\hat{\lambda}/\omega \in L^2(\mathbb{R}^d)$ for H_{PF}. See [1].

Our next problem is to study the distribution of bosons of $\Psi_{\rm PF}$, e.g., $\langle N_{\rm PF}\rangle_{\rm PF}$, $\langle e^{-\beta N_{\rm PF}}\rangle_{\rm PF}$, etc. In [10] a ground state measure, $\mu_{\rm PF}$, on $(\boldsymbol{\Omega}, \mathcal{B}(\boldsymbol{\Omega}))$ for $H_{\rm PF}$ is constructed, which satisfies

$$\langle 1_{A_1} e^{-(t_2-t_1)H_{\rm PF}} 1_{A_2} \cdots e^{-(t_m-t_{m-1})H_{\rm PF}} 1_{A_m}\rangle_{\rm PF}$$
$$= \int_{\boldsymbol{\Omega}} 1_{A_1}(q(t_1)) \cdots 1_{A_m}(q(t_m)) \mu_{\rm PF}(dq).$$

Moreover a "formal" calculation gives a "formal" expression [5],[21]:

$$\langle e^{-\beta N_{\rm PF}}\rangle_{\rm PF} = \int_{\boldsymbol{\Omega}} e^{(-e^2/2) Z_{\rm PF}(\beta)} \mu_{\rm PF}(dq),$$

where

$$Z_{\rm PF}(\beta) := \left(e^{-\beta}-1\right) \sum_{\mu,\nu=1}^{d} \int_{-\infty}^{0} dq_\mu(t) \int_0^\infty dq_\nu(s) \times$$
$$\times \int_{\mathbb{R}^d} d_{\mu\nu}(k) |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} e^{ik(q(t)-q(s))} dk.$$

Here $d_{\mu\nu}(k) := \sum_{r=1}^{d-1} e_\mu^r(k) e_\nu^r(k)$ and $\int \cdots dq_\mu(t)$ denotes a stochastic integral. For the Nelson model $|Z(\beta)| \leq 2\left\|\hat{\lambda}/\omega\right\|^2 < \infty$ guarantees that $\int_{\boldsymbol{\Omega}} e^{(\alpha^2/2)Z(\beta)}\mu(dq)$ is well defined. We do not have such an estimate for $Z_{\rm PF}(\beta)$, which is a crucial points to study $\langle N_{\rm PF}\rangle_{\rm PF}$ in terms of the ground state measure. Actually the definition of $Z_{\rm PF}(\beta)$ is not clear, e.g., it is needed to give a rigorous definition of $\int_{-\infty}^0 dq_\mu(t) \int_0^\infty dq_\nu(s)$.

5.4 Conjectures and problems

In view of subsections 5.1-5.3, we give the following conjectures. We assume some conditions on $\hat{\lambda}$ and V.

Conjecture 5.2 *For* **arbitrary** $e \in \mathbb{R}$, $H_{\rm PF}$ *is self-adjoint and bounded from below on* $D(\Delta \otimes 1) \cap D(1 \otimes H_{\rm f}^{\rm PF})$.

Conjecture 5.3 *Let* $\text{gap}(H_{\rm PF}) = \infty$ *and* $m \geq 0$. *Then the ground states of* $H_{\rm PF}$ *exist for* **arbitrary** $e \in \mathbb{R}$.

Conjecture 5.4 $\Psi_{\rm PF} \in D(1 \otimes e^{\epsilon N_{\rm PF}})$ *for all* $\epsilon \in \mathbb{R}$.

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