Ground state measure and its applications

Fumio Hiroshima*

1 Introduction

In this paper we shall consider structures of ground states of a model describing an interaction between a particle and a quantized scalar boson field, which is called the “Nelson model”[15],[18]. Basic ideas in this paper is due to a fairly nice work of H.Spohn [22], in which he studies the spin-boson model. The Hamiltonian, $H$, of the Nelson model is defined as a self-adjoint operator acting on Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}$, where $\mathcal{F}$ denotes a Boson Fock space. The existence of the ground states, $\Psi_g$, of $H$ is established in e.g., [2],[4],[12],[23]. The main results presented here is to give the expectation-value of the number of bosons of $\Psi_g$ and its boson distribution by means of a ground state measure constructed in this paper. Especially the localization of bosons of $\Psi_g$ is proved. The ground state measure, $\mu$, on the set of paths, $\Omega$, gives an integral representation of the expectation-value of certain operator $A$ in $\mathcal{H}$, i.e.,

$$(\Psi_g, A\Psi_g) = \int_{\Omega} f_A(q) \mu(dq),$$

where $f_A$ is a density function corresponding to $A$. This integral representation leads us to the goal of this paper. Detailed arguments shall be published elsewhere [2], and refer to see [17],[21],[22]. This paper is organized as follows: section 2 gives a definition of models considered in this paper. In section 3 we review the second quantizations. Section 4 is devoted to investigating the ground states. In section 5 we give further problems on the Pauli-Fierz model in nonrelativistic quantum electrodynamics.

*I thank Japan Society for the Promotion of Science for the financial support.
2 Scalar quantum field models

Let $\mathcal{F} := \bigoplus_{n=0}^{\infty} \otimes^n L^2(\mathbb{R}^d) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n$, where $\otimes^n$ denotes the $n$-fold symmetric tensor product with $\otimes_0^0 L^2(\mathbb{R}^d) := \mathbb{C}$. The bare vacuum, $\Omega \in \mathcal{F}$, is defined by $\Omega := \{1, 0, 0, \ldots\}$. Let $a^\dagger(f)$ and $a(g)$ be the creation operator and the annihilation operator smeared by $f, g \in L^2(\mathbb{R}^d)$, respectively, which are linear in $f$ and $g$. Let $\mathcal{F}_\text{fin}$ be the finite particle subspace of $\mathcal{F}$:

$$\mathcal{F}_\text{fin} := \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F} \mid \text{there exists } n_0 \text{ such that } \Psi^{(m)} = 0, m \geq n_0\}.$$  

They satisfy canonical commutation relations (CCR), i.e.,

$$[a(f), a^\dagger(g)] = (\bar{f}, g)_{L^2(\mathbb{R}^d)}, \quad [a^z(f), a^z(g)] = 0,$$

on $\mathcal{F}_\text{fin}$, where $a^z$ denotes $a$ or $a^\dagger$, and $(\cdot, \cdot)_\mathcal{K}$ the scalar product on Hilbert space $\mathcal{K}$. We denote by $|| \cdot ||_\mathcal{K}$ its associated norm. Unless confusion arises we omit $\mathcal{K}$ in $(\cdot, \cdot)_\mathcal{K}$ and $|| \cdot ||_\mathcal{K}$, respectively. $a^z$ also satisfies that $(\Psi, a(f)\Phi) = (a^\dagger(\bar{f})\Psi, \Phi)$ for $\Psi, \Phi \in \mathcal{F}_\text{fin}$. For dense subset $\mathcal{K} \subset L^2(\mathbb{R}^d)$,

$$\mathcal{F}(\mathcal{K}) := \text{l.h.}\{a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega, \Omega f_j \in \mathcal{K}, j = 1, \ldots, n, n \in \mathbb{N}\}$$

is dense in $\mathcal{F}$. We define the free Hamiltonian, $H_f$, in $\mathcal{F}$ by

$$H_f \Omega := 0,$$

$$H_f a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega := \sum_{j=1}^{n} a^\dagger(f_1) \cdots a^\dagger(\omega f_j) \cdots a^\dagger(f_n)\Omega, \quad f_j \in D(\omega), \quad j = 1, \ldots, n, \quad n \in \mathbb{N},$$

where $D(T)$ denotes the domain of $T$, $\omega := \omega(k) := \sqrt{|k|^2 + m^2}$, $m \geq 0$. Here $m$ denotes the mass of the quantized scalar boson field. Field operators $\phi(f)$ are defined by

$$\phi(f) := \frac{1}{\sqrt{2}}(a^\dagger(\bar{f}) + a(f)), \quad f \in L^2(\mathbb{R}^d).$$

Note that $H_f|_{\mathcal{F}(D(\omega))}$ and $\phi(f)|_{\mathcal{F}_\text{fin}}$ are essentially self-adjoint, respectively. It is known that $\sigma(H_f) = [0, \infty)$ and $\sigma_p(H_f) = \{0\}$. The Hamiltonian, $H$, considered in this paper is defined by

$$H := H_p \otimes 1 + 1 \otimes H_f + aH_f$$
on \( \mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F} \cong L^2(\mathbb{R}^d; \mathcal{F}) \), where \( \alpha \in \mathbb{R} \) is a coupling constant, and
\[
H_1 := \phi(e^{ikx} \hat{\lambda}),
H_p := -\Delta/2 + V,
\]
where \( \hat{\lambda} \) is the Fourier transform of \( \lambda \). A reasonable physical choice of \( \hat{\lambda} \) is of the form
\[
\hat{\lambda} = \hat{\rho}/\sqrt{(2\pi)^d \omega},
\]
where \( \rho \) describes a charge distribution, i.e.,
\[
\sqrt{(2\pi)^d} \hat{\rho}(0) = \int_{\mathbb{R}^d} \rho(x) dx = \alpha.
\]
For simplicity we assume that external potential \( V = V_+ - V_- \) satisfies that \( V_+ \in L^1_{\text{loc}}(\mathbb{R}^d) \) and that \( V_- \) is infinitesimally small with respect to \( \Delta \) in the sense of form. Throughout this paper we assume that
\[
\hat{\lambda}(k) = \hat{\lambda}(-k).
\]
Let \( \hat{\lambda}, \hat{\lambda}/\sqrt{\omega} \in L^2(\mathbb{R}^d) \). Then it is known that, for arbitrary \( \alpha \), \( H \) is self-adjoint on \( D(H_p \otimes 1) \cap D(1 \otimes H_f) \) and bounded from below. Moreover it is essentially self-adjoint on any core of \( H_p \otimes 1 + 1 \otimes H_f \).

**Proposition 2.1 ([2],[12])** Let \( \hat{\lambda}/\sqrt{\omega}, \hat{\lambda}/\sqrt{\omega}, \hat{\lambda} \in L^2(\mathbb{R}^d) \). Then there exists \( \alpha_\ast \) such that for \( |\alpha| \leq \alpha_\ast \) the ground states, \( \Psi_g \), of \( H \) exist. Moreover \( (f \otimes \Omega, \Psi_g) > 0 \) for arbitrary nonnegative \( f \in L^2(\mathbb{R}^d) \) with \( f \neq 0 \).

See Figure 2 for more explicit results on the existence of the ground states of \( H \).

## 3 Second quantizations

For later use we review the second quantization of operator \( T \) on \( L^2(\mathbb{R}^d) \). Let \( T \) be a contraction operator on \( L^2(\mathbb{R}^d) \), i.e., \( ||T|| \leq 1 \). Then we define \( \Gamma(T) : \mathcal{F}_{\text{fin}} \rightarrow \mathcal{F}_{\text{fin}} \) by
\[
\Gamma(T) \Omega := \Omega,
\]
\[ \Gamma(T) a^\dagger(f_1) \cdots a^\dagger(f_n) \Omega := a^\dagger(T f_1) \cdots a^\dagger(T f_n) \Omega, \]
\[ f_j \in L^2(\mathbb{R}^d), \quad j = 1, \ldots, n, \quad n \in \mathbb{N}. \]

For \( \Phi \in \mathcal{F}_{\text{fin}} \) we have \( \| \Gamma(T) \Phi \| \leq \| \Phi \| \). Thus \( \Gamma(T) \) extends to a contraction operator on \( \mathcal{F} \). We denote its extension by the same symbol. It is seen that \( \Gamma(\cdot) \) is linear in \( \cdot \) and that \( \Gamma(T)^* = \Gamma(T^*) \). Let \( h \) be a nonnegative self-adjoint operator in \( L^2(\mathbb{R}^d) \). Then we see that \( \Gamma(e^{-th}) \) is a strongly continuous symmetric contraction one-parameter semigroup in \( t \geq 0 \). The second quantization of \( h \), \( d\Gamma(h) \), is defined by the generator of \( \Gamma(e^{-th}) \), i.e.,

\[ \Gamma(e^{-th}) = e^{-td\Gamma(h)}, \quad t \geq 0. \]

Actually \( H_f \) is the second quantization of multiplication operator \( \omega \). For nonnegative multiplication operator \( h \) in \( L^2(\mathbb{R}^d) \), formally, it is written as

\[ d\Gamma(h) = \int h(k) a^\dagger(k) a(k) dk. \quad (3.1) \]

The number operator, \( N \), in \( \mathcal{F} \) is defined by the second quantization of the identity operator in \( L^2(\mathbb{R}^d) \), i.e.,

\[ D(N) := \left\{ \Psi = \{ \Psi^{(n)} \}_{n=0}^\infty \in \mathcal{F} \left| \sum_{n=0}^\infty n^2 \| \Psi^{(n)} \|_{\mathcal{F}_n}^2 < \infty \right. \right\}, \]

\[ (N\Psi)^{(n)} := n\Psi^{(n)}. \]

Let \( h \) be a multiplication operator in \( L^2(\mathbb{R}^d) \) such that \( s = s_{R+} - s_{R-} + i(s_{I+} - s_{I-}) \), where \( s_{R+} \) (resp. \( s_{R-}, s_{I+}, s_{I-} \)) denotes the real positive (resp. real nonpositive, imaginary positive, imaginary nonpositive) part of \( s \). Then we define

\[ d\Gamma(h) := d\Gamma(s_{R+}) - d\Gamma(s_{R-}) + i(d\Gamma(h_{I+}) - d\Gamma(h_{I-})), \]

\[ D(d\Gamma(h)) := D(d\Gamma(s_{R+})) \cap D(d\Gamma(s_{R-})) \cap D(d\Gamma(h_{I+})) \cap D(d\Gamma(h_{I-})). \]

### 4 Ground state measures

Let \( \Omega := (\mathbb{R}^d)^{-\infty,\infty} \) be the set of \( \mathbb{R}^d \)-valued paths and \( \mathcal{B}(\Omega) \) the \( \sigma \)-field constructed by cylinder sets. For \( T : \mathcal{H} \to \mathcal{H} \), we define

\[ \langle T \rangle := (\Psi_g, T\Psi_g)_{\mathcal{H}}. \]
For a convenience we denote by $\langle S \rangle$ for $\langle 1 \otimes S \rangle$, for $S : \mathcal{F} \to \mathcal{F}$. Our fundamental theorem is as follows:

**Theorem 4.1 ([2])** Let $s$ be such that $\sup_{k \in \mathbb{R}} |s(k)| < \infty$. Let $\hat{\lambda}/\omega$, $\hat{\lambda}/\sqrt{\omega}$, $\hat{\lambda} \in L^2(\mathbb{R}^d)$, and $|\alpha| \leq \alpha_s$. We assume that $A_1, \ldots, A_m$ are measurable sets in $\mathbb{R}^d$ and let $1_A$ denote the characteristic function of $A$. Then there exists a probability measure $\mu$ on $(\Omega, \mathcal{B}(\Omega))$ such that, for $t_1 \leq \cdots \leq t_m$,

$$
\langle 1_A e^{-(t_2-t_1)H_1 A_t \cdots e^{-(t_m-t_{m-1})H_1 A_m} = \int_\Omega 1_A(q(t_1)) \cdots 1_A(q(t_m))\mu(dq),
$$

$$
\langle e^{-\beta d \Gamma(s)} \rangle = \int_\Omega e^{(\alpha^2/2)Z(\beta)}\mu(dq), \quad \beta > 0,
$$

where

$$
Z(\beta) := \int_{-\infty}^0 dt \int_0^\infty ds \int_{\mathbb{R}^d} |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} (e^{-\beta s(k)} - 1) e^{i\kappa(q(t)-q(s))}dk.
$$

We give a remark on $Z(\beta)$. Since $\|\hat{\lambda}/\omega\| < \infty$, we see that

$$
|Z(\beta)| \leq 2 \|\hat{\lambda}/\omega\|^2 < \infty
$$

uniformly in paths $q \in \Omega$. Thus $Z(\beta)$ is well defined. It is proved in [2] that $\mu$ is a Gibbs measure. We call $\mu$ the “ground state measure for $H$”. It is easily seen that the right-hand side of (4.1) is analytically continued to $\beta \in \mathbb{C}$. Although it does not imply that $\langle e^{-\beta d \Gamma(s)} \rangle$ is well defined for all $\beta \in \mathbb{C}$, we have the following theorem:

**Theorem 4.2 ([2])** Let $s, \hat{\lambda}$ and $\alpha$ be in Theorem 4.1. Then we have $\Psi_\beta \in D(1 \otimes e^{-\beta d \Gamma(s)})$ for all $\beta \in \mathbb{C}$, and (4.1) holds true for all $\beta \in \mathbb{C}$.

We immediately have the following corollary.

**Corollary 4.3** Let $\hat{\lambda}$ and $\alpha$ be in Theorem 4.1. Then, for arbitrary $\epsilon \in \mathbb{R}$, we have $\Psi_\epsilon \in D(1 \otimes e^{\epsilon N})$. Moreover

$$
\langle N \rangle = \frac{\alpha^2}{2} \int_{-\infty}^0 dt \int_0^\infty ds \int_{\mathbb{R}^d} dk |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} \int_\Omega e^{i\kappa(q(t)-q(s))}\mu(dq). \quad (4.2)
$$
Proof: Putting $s = 1$ in Theorem 4.2, we get $\Psi_g \in D(1 \otimes e^{\epsilon N})$ for all $\epsilon \in \mathbb{R}$. (4.2) follows from (4.1) and

$$\langle N \rangle = -\frac{d(e^{-\beta N})}{d\beta} \bigg|_{\beta=0}.$$

The proof is complete. Q.E.D.

Corollary 4.3 implies that

$$\sum_{n=0}^{\infty} e^{2\epsilon n} \|\Psi^{(n)}_g\|_{L^2(\mathbb{R}^d) \otimes F_n} < \infty, \quad \text{for all } \epsilon > 0.$$

Hence we conclude that $\|\Psi^{(n)}_g\|$ decays super-exponentially as $n \to \infty$; it decays faster than $e^{-\epsilon n}$ for arbitrary $\epsilon > 0$. Let $s \in C_0^{\infty}(\mathbb{R}^d)$. Then, by Theorem 4.2, we see that $\Psi_g \in D(d\Gamma(s))$ and

$$|\langle d\Gamma(s) \rangle| \leq (\alpha^2/2)\|s\|_{\infty}||\hat{\lambda}/\omega||^2.$$

Thus map

$$\mathcal{D} : C_0^{\infty}(\mathbb{R}^d) \ni s \mapsto \langle d\Gamma(s) \rangle \in \mathbb{C}$$

defines a distribution on $C_0^{\infty}(\mathbb{R}^d)$. Taking into account of the formal expression of $d\Gamma(s)$ (3.1), we denote by $\langle a^\dagger(k)a(k) \rangle$ the kernel of $\mathcal{D}$. From Corollary 4.3 it immediately follows:

**Corollary 4.4** Let $\hat{\lambda}$ and $\alpha$ be in Theorem 4.1. Then for a.e. $k \in \mathbb{R}^d$,

$$\langle a^\dagger(k)a(k) \rangle = \frac{\alpha^2}{2} |\hat{\lambda}(k)|^2 \int_{-\infty}^{0} dt \int_{0}^{\infty} ds \ e^{-t-s} |\omega(k)| \int_{\Omega} e^{ik(q(t)-q(s))} \mu(dq).$$

Note that

$$\int_{\mathbb{R}^d} \langle a^\dagger(k)a(k) \rangle dk = \langle N \rangle.$$

Moreover we see that

$$|\langle a^\dagger(k)a(k) \rangle| \leq \frac{\alpha^2}{2} \frac{|\hat{\lambda}(k)|^2}{\omega(k)^2}, \quad \text{a.e. } k \in \mathbb{R}^d.$$

See Figure 1.
5 Nonrelativistic QED

5.1 The Pauli-Fierz model

The Pauli-Fierz model \([1],[3],[5]-[11],[19],[20]\) in nonrelativistic QED describes an interaction of particles (electrons) and a quantized radiation field (photons). The quantized radiation field is quantized in a Coulomb gage. We assume that the number of the electrons is one and that the electron has spineless.

Let

\[
\mathcal{F}_{PF} := \bigoplus_{n=0}^{\infty} \bigotimes_{s} L^2(\mathbb{R}^d) \oplus \cdots \oplus L^2(\mathbb{R}^d) \cong \mathcal{F} \otimes \cdots \otimes \mathcal{F}.
\]

Let \(\{b^r(f), b^r(g)\}_{r=1}^{d-1}\) be the annihilation operators and the creation operators, respectively, which satisfy CCR:

\[
[b^r(f), b^{s*}(g)] = \delta_{rs} (\bar{f}, g)_{L^2(\mathbb{R}^d)}, \quad [b^{sr}(f), b^{st}(g)] = 0.
\]

Let \(H_{\mathrm{PF}}^{\mathrm{f}}\) be the free Hamiltonian in \(\mathcal{F}_{PF}\), i.e.,

\[
H_{\mathrm{PF}}^{\mathrm{f}} := \sum_{r=1}^{d-1} \int \omega(k) b^{hr}(k) b^r(k) dk.
\]
The Hamiltonian of the Pauli-Fierz model is defined as an operator in $\mathcal{H}_{PF} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_{PF} \cong L^2(\mathbb{R}; \mathcal{F}_{PF})$ and reads

$$H_{PF} := \frac{1}{2} (-i \nabla \otimes 1 - e \mathbf{A}(x))^2 + 1 \otimes H_i^{PF} + V \otimes 1,$$

where $e$ is a coupling constant, $\mathbf{A}(x) := (A_1(x), \cdots, A_d(x))$,

$$A_{\mu}(x) := \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \left( b^r_{\mu} (e_{\mu}^r \lambda e^{-ikr}) + b^r_{\mu} (e_{\mu}^r \lambda e^{ikr}) \right),$$

and $e^r := (e_1^r, \cdots, e_d^r)$, polarization vectors; $e^r(k) \cdot e^s(k) = \delta_{rs}$ and $e^r(k) \cdot k = 0$. Note that

$$\text{div}\mathbf{A} = 0.$$

For the Nelson model, the self-adjointness of $H$ for arbitrary $\alpha$ is trivial, since $H_1$ is infinitesimally small with respect to $H_p \otimes 1 + 1 \otimes H_i$. It is not so easy to show self-adjointness of $H_{PF}$ for arbitrary $e \in \mathbb{R}$. Let $N_{PF}$ be the number operator in $\mathcal{F}_{PF}$. We have the following proposition:

**Proposition 5.1 ([9])** ¹ Let $\hat{\lambda}, \omega^2 \lambda \in L^2(\mathbb{R}^d)$. We assume that $V$ is relatively bounded with respect to $\Delta$. Then, for arbitrary $\epsilon \in \mathbb{R}$, $H_{PF}$ is essentially self-adjoint on

$$D(\Delta \otimes 1) \cap D(1 \otimes (H_i^{PF})^2) \cap_{k=1}^\infty D(1 \otimes N_k^{PF}).$$

The existence of the ground states of $H_{PF}$ are studied in [1],[6], and their multiplicities in [7],[11]. Moreover $\inf \sigma(H_{PF})$ is investigated in [3],[16].

### 5.2 Ground states of $H$ and $H_{PF}$

Let

$$\text{gap}(T) := \inf \sigma_{\text{ess}}(T) - \inf \sigma(T).$$

The existence of the ground states of $H$ and $H_{PF}$ are deeply related to conditions on $m$, gap, $\hat{\lambda}$ and coupling constants. Let $\hat{\lambda}/\omega \in L^2(\mathbb{R}^d).$² Then sufficient conditions for the existence of the ground states of $H$ and $H_{PF}$, as far as we know, are in Figures 2 and 3, respectively.
<table>
<thead>
<tr>
<th></th>
<th>$m &gt; 0$</th>
<th>$m = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{gap}(H) = \infty$</td>
<td>$\alpha \in \mathbb{R}$</td>
<td>$\alpha \in \mathbb{R}$</td>
</tr>
<tr>
<td>$0 &lt; \text{gap}(H) &lt; \infty$</td>
<td>$</td>
<td>\alpha</td>
</tr>
</tbody>
</table>

Figure 2: $\alpha$ for the existence of the ground states of $H$.

<table>
<thead>
<tr>
<th></th>
<th>$m &gt; 0$</th>
<th>$m = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{gap}(H_{PF}) = \infty$</td>
<td>$e \in \mathbb{R}$</td>
<td>$</td>
</tr>
<tr>
<td>$0 &lt; \text{gap}(H_{PF}) &lt; \infty$</td>
<td>$</td>
<td>e</td>
</tr>
</tbody>
</table>

Figure 3: $e$ for the existence of the ground states of $H_{PF}$.

Note that see [4],[23] for a proof of the existence of ground states for case $\text{gap}(H) = \infty$ and $m \geq 0$ in Figure 2, and [8],[9] for case $\text{gap}(H_{PF}) = \infty$ and $m > 0$ in Figure 3. In [13],[14] the authors give examples such that the ground states of $H$ and $H_{PF}$ exist for the case where $\text{gap}(H) = 0$ and $\text{gap}(H_{PF}) = 0$, respectively. In [17] no existence of the ground states of $H$ for arbitrary $\alpha \neq 0$ is proved if $||\hat{\lambda}/\omega|| = \infty$.

5.3 Distribution of bosons for $\Psi_{PF}$

Let $\Psi_{PF}$ be the ground state of $H_{PF}$ and

$$\langle T \rangle_{PF} := (\Psi_{PF}, T \Psi_{PF})_{H_{PF}}.$$  

1 In [9] essential self-adjointness of $H_{PF}$ is proved only for the case where the number of the electrons is one. As far as we know it is not clear whether the statement in Proposition 5.1 with $N$-electrons holds true or not. In [19] self-adjointness of $H_{PF}$ on $D(\Delta \otimes 1) \cap D(1 \otimes H_{PF}^{d})$ is proved for sufficiently small $|e|$.

2 It is not necessarily to assume $\hat{\lambda}/\omega \in L^2(\mathbb{R})$ for $H_{PF}$. See [1].

9
Our next problem is to study the distribution of bosons of \( \Psi_{PF} \), e.g., \( \langle N_{PF} \rangle_{PF} \), \( \langle e^{-\beta N_{PF}} \rangle_{PF} \), etc. In [10] a ground state measure, \( \mu_{PF} \), on \((\Omega, B(\Omega))\) for \( H_{PF} \) is constructed, which satisfies

\[
\langle 1_{A_1} e^{-(t_2-t_1)H_{PF}} 1_{A_2} \cdots 1_{A_m} e^{-(t_m-t_{m-1})H_{PF}} 1_{A_m} \rangle_{PF}
= \int_{\Omega} 1_{A_1}(q(t_1)) \cdots 1_{A_m}(q(t_m)) \mu_{PF}(dq).
\]

Moreover a “formal” calculation gives a “formal” expression [5],[21]:

\[
\langle e^{-\beta N_{PF}} \rangle_{PF} = \int_{\Omega} e^{(-e^2/2)Z_{PF}(\beta)} \mu_{PF}(dq),
\]

where

\[
Z_{PF}(\beta) := (e^{-\beta} - 1) \sum_{\mu,\nu=1}^{d} \int_{-\infty}^{0} dq_{\mu}(t) \int_{0}^{\infty} dq_{\nu}(s) \times \int_{\mathbb{R}^d} d_{\mu\nu}(k) |\hat{\lambda}(k)|^2 e^{-|t-s| |\omega(k)|} e^{ik(q(t)-q(s))} dk.
\]

Here \( d_{\mu\nu}(k) := \sum_{r=1}^{d-1} e_{\mu}^r(k) e_{\nu}^r(k) \) and \( f \cdots dq_{\mu}(t) \) denotes a stochastic integral. For the Nelson model \( |Z(\beta)| \leq 2 \| \hat{\lambda} / \omega \|^2 < \infty \) guarantees that \( \int_{\Omega} e^{(\alpha^2/2)Z(\beta)} \mu(dq) \) is well defined. We do not have such an estimate for \( Z_{PF}(\beta) \), which is a crucial points to study \( \langle N_{PF} \rangle_{PF} \) in terms of the ground state measure. Actually the definition of \( Z_{PF}(\beta) \) is not clear, e.g., it is needed to give a rigorous definition of \( \int_{-\infty}^{0} dq_{\mu}(t) \int_{0}^{\infty} dq_{\nu}(s) \).

### 5.4 Conjectures and problems

In view of subsections 5.1-5.3, we give the following conjectures. We assume some conditions on \( \hat{\lambda} \) and \( V \).

**Conjecture 5.2** For arbitrary \( e \in \mathbb{R} \), \( H_{PF} \) is self-adjoint and bounded from below on \( D(\Delta \otimes 1) \cap D(1 \otimes H_{\rm{PF}}^{1}) \).

**Conjecture 5.3** Let \( \text{gap}(H_{PF}) = \infty \) and \( m \geq 0 \). Then the ground states of \( H_{PF} \) exist for arbitrary \( e \in \mathbb{R} \).

**Conjecture 5.4** \( \Psi_{PF} \in D(1 \otimes e^{eN_{PF}}) \) for all \( e \in \mathbb{R} \).
References


