Strong and Weak Coupling Limits of Interaction Models of Quantum Fields and Particles

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1 INTRODUCTION

Asymptotic behaviors of scaling Hamiltonians which describe interactions of particles and quantized fields are considered. In a mathematical formulation, interaction Hamiltonians of the particles and the quantized fields are described by the theory of self-adjoint operators acting in the tensor product of two Hilbert spaces over the complex field $\mathbb{C}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. We define a self-adjoint operator $\mathbf{H}$ acting in the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, by

$$\mathbf{H} = H_1 \otimes I + \alpha H_{int} + I \otimes H_2.$$

Here H_1 and H_2 are self-adjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, H_{int} is a symmetric operator in $\mathcal{H}$ and $\alpha \in \mathbb{R}$ is a coupling constant. Then, for the given self-adjoint operator $\mathbf{H}$, we define "β-coupling Hamiltonian, $\mathbf{H}_\beta(\Lambda)$", by

$$\mathbf{H}_\beta(\Lambda) = H_1 \otimes I + \Lambda\alpha H_{int} + \Lambda^\beta I \otimes H_2, \quad 1 \leq \beta. \tag{1. 1}$$

Introducing a renormalization $E_\beta(\Lambda)$ which goes to infinity or minus infinity as $\Lambda \to \infty$ in some sense, we want to investigate the following asymptotic behaviors

$$s-\lim_{\Lambda\to\infty} e^{-it\left(\mathbf{H}_\beta(\Lambda)-E_\beta(\Lambda)\right)} = \mathbf{U}\left(e^{-itH_{eff}} \otimes \mathbf{P}\right)\mathbf{U}^{-1}, \quad t \in \mathbb{R}. \tag{1. 2}$$

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Here H_{eff} is a self-adjoint operator in $\mathcal{H}_1$, which is called "effective Hamiltonian", $\mathbf{U}$ is a unitary operator in $\mathcal{H}$ and $\mathbf{P}$ a projection operator onto a one-dimensional subspace in $\mathcal{H}_2$. It seems to be useful to readers to collect some background ingredient. Motivation of this paper is [1] and [3]. In [1], in order to give an interpretation of a physical phenomenon "Lamb shift" without formal perturbation theory, A.Arai elaborates a scaling limit of the Pauli-Fierz model. The scaling limit corresponds to the case $\beta = 1$ in (1.1). In [3], E.B.Davies studies a scaling limit of the Nelson model to derive a Schrödinger Hamiltonian (effective Hamiltonian) with a scalar potential. The scaling limit corresponds to the case $\beta = 2$ in (1.1). In this paper, we deal with the Nelson model [2,3,4,5,8,10], the Pauli-Fierz model [1,6,7,8,9,10] and the spin-boson model [1]. Thus considering scaling limits as in (1.2) for these models is an extension of those considered in [1,2,3,6,7,10]. We organize this paper as follows. In section 2, we overview an abstract theory of a scaling limit of self-adjoint operators. In section 3,4 and 5, we study the Nelson model, the Pauli-Fierz model and the spin-boson model,respectively. In section 6, we give some remarks.

2 FUNDAMENTAL FACTS

2.1 An abstract Boson Fock space

In this subsection we define an abstract Boson Fock space and basic notations. For a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$, we denote the scalar product by $< f, g >_{\mathcal{H}}$ and the associated norm by $||f||_{\mathcal{H}}$, where the scalar product is linear in g and antilinear in f. For the tempered distributions f and g, the notation $\bar{f}$ denotes the complex conjugate of f, and $\hat{f}$ (resp.$\check{g}$) the Fourier transform of f (resp.the inverse Fourier transform of g). We denote the domain of an operator A by $D(A)$. The Boson Fock space over the Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} [\otimes_s^n \mathcal{H}] \, ,$$

where $\otimes_s^n \mathcal{H}, n \geq 1$, denotes the n-fold symmetric tensor product of $\mathcal{H}$, and $\otimes_s^0 \mathcal{H} = \mathbb{C}$. Define $\Omega_{\mathcal{H}} = \{1, 0, 0, ..\}$. Let the annihilation and creation operators in the Boson Fock space denoted by $a_{\mathcal{H}}(f), f \in \mathcal{H}$ and $a_{\mathcal{H}}^{\dagger}(g), g \in \mathcal{H}$, respectively. It is well known that

$$\mathcal{F}_{\mathcal{H}}^{\infty} \equiv \mathbf{L} \left\{ a_{\mathcal{H}}^{\dagger}(f_1) ... a_{\mathcal{H}}^{\dagger}(f_n) \Omega, \Omega | f_j \in \mathcal{H}, j = 1, ..., n, n \geq 1 \right\}$$

is dense in $\mathcal{F}_{\mathcal{H}}$, where $\mathbf{L}$ denotes the linear hull of the vectors in $\{...\}$. The annihilation and the creation operators in the Boson Fock space satisfy the following canonical commutation relations on $\mathcal{F}^{\infty}_{\mathcal{H}}$:

$$\begin{aligned} [a_{\mathcal{H}}(f), a^{\dagger}_{\mathcal{H}}(g)] &= <\bar{f}, g>_{\mathcal{H}}, \\ [a^{\sharp}_{\mathcal{H}}(f), a^{\sharp}_{\mathcal{H}}(g)] &= 0, \end{aligned}$$

where $a^{\sharp}_{\mathcal{H}}$ means $a_{\mathcal{H}}$ or $a^{\dagger}_{\mathcal{H}}$. Let h be a self-adjoint operator in $\mathcal{H}$. Define $d\Gamma_{\mathcal{H}}(h)$ by

$$\begin{aligned} & d\Gamma_{\mathcal{H}}(h)\Omega = 0, \\ & d\Gamma_{\mathcal{H}}(h)a^{\dagger}_{\mathcal{H}}(f_1)...a^{\dagger}_{\mathcal{H}}(f_n)\Omega_{\mathcal{H}} = \sum_{j=1}^{n} a^{\dagger}_{\mathcal{H}}(f_1)...a^{\dagger}_{\mathcal{H}}(hf_j)...a^{\dagger}_{\mathcal{H}}(f_n)\Omega_{\mathcal{H}}, \, f_j \in D(h). \end{aligned}$$

Then $d\Gamma_{\mathcal{H}}(h)$ is essentially self-adjoint. Let us use the same notation as $d\Gamma_{\mathcal{H}}(h)$ for its self-adjoint extension.

2.2 An abstract theory of a scaling limit

We overview an abstract theory of a scaling limit of self-adjoint operators acting in a tensor product Hilbert space established in [1] with a little modification. Let $\mathcal{K}$ be a Hilbert space and put $\mathcal{X} = \mathcal{H} \otimes \mathcal{K}$. Suppose that an operator, A (resp.B), is a nonnegative self-adjoint operator in $\mathcal{H}$ (resp.$\mathcal{K}$) and Ker$B = \{kG | k \in \mathbb{C}, ||G||_{\mathcal{X}} = 1\}$. Set the projection operator onto Ker$B$ by P_B. We suppose that a family of self-adjoint operators, $\{C_\Lambda\}_{\Lambda>0}$, in $\mathcal{X}$ admits the following conditions:

(1) For any $\epsilon > 0$, there exists Λ_0 so that, for all $\Lambda > \Lambda_0$, $D(C_\Lambda) \supset D(A \otimes I + \Lambda I \otimes B)$ with

$$||C_\Lambda \Phi||_{\mathcal{X}} \leq \epsilon \, ||(A \otimes I + \Lambda I \otimes B)\Phi|| + b(\epsilon) \, ||\Phi||_{\mathcal{X}} \, , \Phi \in D(A \otimes I) \cap D(I \otimes B),$$

where $b(\epsilon) > 0$ is a constant independent of $\Lambda > \Lambda_0$.

(2) There exists a symmetric operator C in $\mathcal{X}$ so that $D(C) \supset D(A) \otimes \text{Ker}B$ and, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s - \lim_{\Lambda \to \infty} C_\Lambda (A \otimes I + \Lambda I \otimes B - z)^{-1} = C \left\{ (A - z)^{-1} \otimes P_B \right\}.$$

We define an operator $E_G(C)$ with the domain $D(E_G(C)) = D(A)$ by

$$\langle f, E_G(C)g\rangle_{\mathcal{H}} = \langle f \otimes G, C(g \otimes G)\rangle_{\mathcal{X}}, \quad f \in \mathcal{H}, g \in D(A).$$

We call $E_G(C)$ "the partial expectation of C with respect to G". Set

$$K_{eff} = A + E_G(C).$$

The following proposition is fundamental in this paper.

Proposition 2.1 ([1,Theorem 2.1]) *Let operators* A,B,C_Λ, *and* C *be as above. Then*

(1) *For* $\Lambda > \Lambda_0$, $K_\Lambda = A \otimes I + \Lambda I \otimes B + C_\Lambda$ *is self-adjoint on* $D(A \otimes I) \cap D(I \otimes B)$ *and* uniformly *bounded from below. Moreover* $E_G(C)$ *is infinitesimally small with respect to* A, *i.e.,* K_{eff} *is self-adjoint on* $D(A)$.

(2) *For* $z \in \mathbb{C} \setminus \mathbb{R}$

$$s - \lim_{\Lambda \to \infty} (K_\Lambda - z)^{-1} = (K_{eff} - z)^{-1} \otimes P_B. \qquad (2.\ 1)$$

Finally we note a fundamental fact.

Proposition 2.2 *Let* K_Λ *and* K_{eff} *satisfy (2.1). Then*

$$s - \lim_{\Lambda \to \infty} e^{-itK_\Lambda} = e^{-itK_\infty} \otimes P_B.$$

Proof: See [1,Theorem 2.2] □

By Proposition 2.2, it is enough to show strong resolvent limits of β-coupling Hamiltonian to investigate (1.2).

3 THE NELSON MODEL

3.1 The Nelson model

In this section, we consider the Nelson Hamiltonian with an ultraviolet cut-off function $\hat{\varrho}$ and with a finite number of nonrelativistic particles. Fix the number of the nonrelativistic particles N. For the mathematical generality, suppose that the dimension of the space in which the nonrelativistic particles move is $d \geq 1$. (This assumption remains throughout this paper.) We use the following identification

$$\mathcal{F}_N \equiv L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_{L^2(\mathbb{R}^d)} \cong L^2(\mathbb{R}^{dN}; \mathcal{F}_{L^2(\mathbb{R}^d)}).$$

For notational simplicity, we write the annihilation or creation operators by $a^\sharp(f)$ instead of $a^\sharp_{L^2(\mathbb{R}^d)}(f)$ in sections 3 and 5. We define a time-zero scalar field $\phi(\hat{f})$ by

$$\phi(\hat{f}) \quad = \quad \frac{1}{\sqrt{2}} \left\{ a^\dagger \left(\frac{\hat{f}}{\sqrt{\omega}} \right) + a \left(\frac{\overline{\hat{f}}}{\sqrt{\omega}} \right) \right\}.$$

Here $\omega = \omega(k) = \sqrt{k^2 + \mu^2}$, $\mu \geq 0$. In this section we require that ϱ is a real valued even function, $\varrho(k) = \varrho(-k)$, with

$$\frac{\hat{\varrho}}{\omega\sqrt{\omega}}, \frac{\hat{\varrho}}{\omega}, \frac{\hat{\varrho}}{\sqrt{\omega}} \in L^2(\mathbb{R}^d). \tag{3. 1}$$

For each $x = (x^1, ..., x^N) \in \mathbb{R}^{dN}$, $x^j \in \mathbb{R}^d$, $j = 1, ..., N$, we set

$$\tilde{\hat{\varrho}}(x) = \frac{1}{\sqrt{(2\pi)^d}} \sum_{j=1}^N \hat{\varrho}(k) e^{-ikx^j}.$$

We define

$$H_I(\hat{\varrho}) \equiv \phi\left(\tilde{\hat{\varrho}}(\cdot)\right).$$

For the multiplication operator ω in $L^2(\mathbb{R}^d)$ with the maximal domain, we set $d\Gamma_{L^2(\mathbb{R}^d)}(\omega) \equiv H_b$. Define an operator in $\mathcal{F}_N$ by

$$H_N^\beta(\hat{\varrho}, \Lambda) = -\frac{1}{2m}\Delta_N \otimes I - \Lambda g H_I(\hat{\varrho}) + \Lambda^\beta I \otimes H_b, \quad 1 \leq \beta < \infty,$$

where $g \in \mathbb{R}$ is a coupling constant, $m > 0$ a mass of the nonrelativistic particles, Δ_N the Laplacian in $L^2(\mathbb{R}^{dN})$ and $\Lambda > 0$ a scaling parameter. Moreover we put a decoupled Hamiltonian $H_{\beta,N}(\Lambda)$ by

$$H_N^\beta(\Lambda) = -\frac{1}{2m}\Delta_N \otimes I + \Lambda^\beta I \otimes H_b.$$

We define a class of the set of multiplication operators in $L^2(\mathbb{R}^{dN})$. A multiplication operator V is in a class, $\mathcal{M}_\pm(N)$, if and only if V is infinitesimally small with respect to $-\Delta_N$.

Proposition 3.1 ([2]) *For $\Lambda > 0$ and $V \in \mathcal{M}_{\pm}(N)$, $H_N^{\beta}(\hat{\varrho},\Lambda) + V \otimes I$ is self-adjoint on $D(H_N^{\beta}(\Lambda))$ and bounded from below. Moreover it is essentially self-adjoint on any core for $H_N^{\beta}(\Lambda)$.*

In the case of $\beta = 2$, following proposition is well known.

Proposition 3.2 ([2,3],$\beta = 2$) *Let $V \in \mathcal{M}_{\pm}(N)$. Then*

$$s - \lim_{\Lambda\to\infty} e^{-it(H_N^{\beta}(\hat{\varrho},\Lambda)+V\otimes I)} = e^{-it(-\frac{1}{2m}\Delta_N + V + g^2 V(\hat{\varrho}))} \otimes P_N.$$

Here P_N is the projection operator onto the subspace in $\mathcal{F}$, spanned by the vector $\Omega_{L^2(\mathbb{R}^d)}$.

3.2 The case of $\beta = 1$

Put $C_0^{\infty}(\mathbb{R}^{dN})\hat{\otimes}\mathcal{F}^{\infty}_{L^2(\mathbb{R}^d)} \equiv \mathcal{F}_N^{\infty}$, where $\hat{\otimes}$ denotes the algebraic tensor product. We perform a unitary transformation

$$\mathcal{U}(g) = \exp\left(\frac{g}{\sqrt{2}}\left\{a^{\dagger}\left(\frac{\tilde{\hat{\varrho}}(\cdot)}{\omega\sqrt{\omega}}\right) - a\left(\frac{\overline{\tilde{\hat{\varrho}}(\cdot)}}{\omega\sqrt{\omega}}\right)\right\}\right)$$

with the following result:

Proposition 3.3 *The unitary operator $\mathcal{U}(\Lambda^{1-\beta}g)$ maps $\mathcal{F}_N^{\infty}$ into $D(H_N^{\beta}(\hat{\varrho},\Lambda)))$ with*

$$\begin{aligned}&\mathcal{U}(\Lambda^{1-\beta}g)^{-1}(H_N^{\beta}(\hat{\varrho},\Lambda) + V\otimes I)\mathcal{U}(\Lambda^{1-\beta}g)\\ &= \frac{1}{2m}\sum_{j=1}^{N}\left(\mathbf{p}^j \otimes I - g\Lambda^{1-\beta}\phi_j\right)^2 + g^2\Lambda^{2-\beta}V(\hat{\varrho})\otimes I + \Lambda^{\beta} I \otimes H_b + V\otimes I, \qquad (3.\ 2)\end{aligned}$$

on $\mathcal{F}_N^{\infty}$, where $\mathbf{p}^j = (-i\frac{\partial}{\partial x_1^j},...,-i\frac{\partial}{\partial x_d^j})$, $\phi_j = (\phi\left(\hat{\varrho}_1^j(\cdot)\right),...,\phi\left(\hat{\varrho}_d^j(\cdot)\right))$, $j = 1,...,N$, and

$$\begin{aligned}\hat{\varrho}_{\mu}^j(x) &= \hat{\varrho}_{\mu}^j(x,k) = \frac{1}{\sqrt{(2\pi)^d}}\frac{\hat{\varrho}(k)e^{-ikx^j}k_{\mu}}{\omega(k)}, \mu = 1,...,d,\\ V(\hat{\varrho}) &= V(\hat{\varrho},x) = -\frac{1}{2(2\pi)^d}\left\|\sum_{j=1}^{N}\frac{\hat{\varrho}e^{-ikx^j}}{\omega}\right\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Moreover, for sufficiently large $\Lambda > 0$, the right hand side (R.H.S.) of (3.2) is self-adjoint on $D(H_N^{\beta}(\Lambda))$ and the equation (3.2) can be extended to the equation on $D(H_N^{\beta}(\Lambda))$.

Proposition 3.3 implies that the following equation holds, for $V \in \mathcal{M}_\pm(N)$ and sufficiently large $\Lambda > 0$;

$$\mathcal{U}(\Lambda^{1-\beta}g)^{-1}\left(H_N^\beta(\hat{\varrho},\Lambda) - g^2\Lambda^{2-\beta}V(\hat{\varrho})\otimes I + V\otimes I\right)\mathcal{U}(\Lambda^{1-\beta}g)$$
$$= \frac{1}{2m}\sum_{j=1}^N \left(\mathbf{p}^j \otimes I - g\Lambda^{1-\beta}\phi_j\right)^2 + V\otimes I + \Lambda^\beta I \otimes H_b. \tag{3. 3}$$

In this subsection we set $\beta = 1$. Then we define a symmetric operator $Q(\hat{\varrho})$, which is independent of Λ, by

$$R.H.S.\ of (3.3) = H_N^\beta(\Lambda) + Q(\hat{\varrho}).$$

Lemma 3.4 *Let $V \in \mathcal{M}_\pm(N)$. Then, for any $\epsilon > 0$, there exists Λ_0 and $b(\epsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(Q(\hat{\varrho})) \supset D(H_N^\beta(\Lambda))$ with*

$$||Q(\hat{\varrho})\Phi||_{\mathcal{F}_N} \le \epsilon \left|\left|H_N^\beta(\Lambda)\Phi\right|\right|_{\mathcal{F}_N} + b(\epsilon)\,||\Phi||_{\mathcal{F}_N},\ \Phi \in D(H_F). \tag{3. 4}$$

Moreover $D(Q(\hat{\varrho})) \supset D(-\Delta_N)\widehat{\otimes}\mathrm{Ker}H_b$ with, for $z \in \mathbb{C}\setminus[0,\infty)$,

$$s - \lim_{\Lambda\to\infty} Q(\hat{\varrho})\left(H_N^\beta(\Lambda) - z\right)^{-1} = Q(\hat{\varrho})\left[\left(-\frac{1}{2m}\Delta_N - z\right)^{-1} \otimes P_N\right]. \tag{3. 5}$$

Proof: The proof of (3.4) follows from fundamental estimates with respect to $a^\sharp$ and H_b. By (3.4), for any $\epsilon > 0$, taking sufficiently large $\Lambda > 0$, we see that

$$\left|\left|Q(\hat{\varrho})\left\{\left(H_N^\beta(\Lambda) - z\right)^{-1} - \left(-\frac{1}{2m}\Delta_N - z\right)^{-1}\otimes P_N\right\}\Phi\right|\right|_{\mathcal{F}_N}$$
$$\le \epsilon \left|\left|H_N^\beta(\Lambda)\left\{\left(H_N^\beta(\Lambda) - z\right)^{-1} - \left(-\frac{1}{2m}\Delta_N - z\right)^{-1}\otimes P_N\right\}\Phi\right|\right|_{\mathcal{F}_N}$$
$$+ b(\epsilon)\left|\left|\left\{\left(H_N^\beta(\Lambda) - z\right)^{-1} - \left(-\frac{1}{2m}\Delta_N - z\right)^{-1}\otimes P_N\right\}\Phi\right|\right|_{\mathcal{F}_N}.$$

Taking $\Lambda \to \infty$ on the both sides above, we have

$$\lim_{\Lambda\to\infty}\left|\left|Q(\hat{\varrho})\left\{\left(H_N^\beta(\Lambda) - z\right)^{-1} - \left(-\frac{1}{2m}\Delta_N - z\right)^{-1}\otimes P_N\right\}\Phi\right|\right|_{\mathcal{F}_N}$$
$$\le \epsilon\, ||\left(I\otimes I - I\otimes P_N\right)\Phi||_{\mathcal{F}_N}.$$

Since $\epsilon > 0$ is arbitrary, (3.5) follows. □

Theorem 3.5 **(**$\beta = 1$**)** *Let* $V \in \mathcal{M}_{\pm}(N)$ *and* $z \in \mathbb{C} \setminus \mathbb{R}$. *Put* $\omega_0 = \omega_0(k) = |k|$ *and*

$$\delta(\hat{\varrho}) = \frac{1}{2(2\pi)^d} \left\| \frac{\hat{\varrho}\omega_0}{\omega\sqrt{\omega}} \right\|^2_{L^2(\mathbb{R}^d)}.$$

Then

$$\begin{aligned} & s - \lim_{\Lambda\to\infty} \left(H_N^{\beta}(\hat{\varrho}, \Lambda) - g^2\Lambda^{2-\beta} V(\hat{\varrho}) \otimes I + V \otimes I - z \right)^{-1} \\ & = \mathcal{U}(g) \left\{ \left(-\frac{1}{2m}\Delta_N + g^2 N \delta(\hat{\varrho}) + V - z \right)^{-1} \otimes P_N \right\} \mathcal{U}(g)^{-1}. \end{aligned} \qquad (3.\ 6)$$

Proof: From (3.3) it follows that

$$\begin{aligned} & \left(H_N^{\beta}(\hat{\varrho}, \Lambda) - g^2\Lambda^{2-\beta} V(\hat{\varrho}) \otimes I + V \otimes I - z \right)^{-1} \\ & = \mathcal{U}(\Lambda^{1-\beta} g) \left(H_N^{\beta}(\Lambda) + Q(\hat{\varrho}) - z \right)^{-1} \mathcal{U}(\Lambda^{1-\beta} g)^{-1}. \end{aligned}$$

By the fact that $\mathcal{U}(\Lambda^{1-\beta} g)$ is independent of Λ, it is enough to show that

$$s - \lim_{\Lambda\to\infty} \left(H_N^{\beta}(\Lambda) + Q(\hat{\rho}) - z \right)^{-1} = \left(-\frac{1}{2m}\Delta_N + g^2 N \delta(\hat{\varrho}) + V - z \right)^{-1} \otimes P_N.$$

Since the partial expectation of $Q(\hat{\varrho})$ with respect to $\Omega_{L^2(\mathbb{R}^d)}$ is

$$\begin{aligned} E_{\Omega_{L^2(\mathbb{R}^d)}}(Q(\hat{\varrho})) &= g^2 \sum_{j=1}^{N} \sum_{\mu=1}^{3} \frac{1}{2} \left\| \frac{\hat{\varrho}^j_{\mu}(\cdot)}{\sqrt{\omega}} \right\|^2_{L^2(\mathbb{R}^d)} + V \\ &= g^2 N \delta(\hat{\varrho}) + V, \end{aligned}$$

it follows (3.6) from Lemma 3.4 and Proposition 2.1 with the following correspondence :

$$A = -\frac{1}{2m}\Delta_N, \quad B = H_b, \quad C_{\Lambda} = C = Q(\hat{\varrho}), \quad G = \Omega_{L^2(\mathbb{R}^d)}.$$

□

3.3 The case of $1 < \beta < 2$, $2 < \beta$

First we study the case of $1 < \beta < 2$. We put the R.H.S. of (3.3) by

$$R.H.S.of\ (3.3) = H_N^{\beta}(\Lambda) + Q^1(\hat{\varrho}, \Lambda). \qquad (3.\ 7)$$

Similar to (3.4) and (3.5), one can see that, for $V \in \mathcal{M}_{\pm}(N)$ and any $\epsilon > 0$, there exists Λ_0 and $b(\epsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(Q^1(\hat{\varrho}, \Lambda)) \supset D(H_N^{\beta}(\Lambda))$ with

$$\left|\left|Q^1(\hat{\varrho}, \Lambda)\Phi\right|\right|_{\mathcal{F}_N} \leq \epsilon \left|\left|H_N^{\beta}(\Lambda)\Phi\right|\right|_{\mathcal{F}_N} + b(\epsilon)\left|\left|\Phi\right|\right|_{\mathcal{F}_N}, \Phi \in D(H_N^{\beta}(\Lambda)). \tag{3. 8}$$

Moreover $D(Q^1(\hat{\varrho}, \Lambda)) \supset D(-\Delta_N)\hat{\otimes}\mathrm{Ker}H_b$ and, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s - \lim_{\Lambda \to \infty} Q^1(\hat{\varrho}, \Lambda)\left(H_N^{\beta}(\Lambda) - z\right)^{-1} = \left[V\left(-\frac{1}{2m}\Delta_N - z\right)^{-1}\right] \otimes P_N. \tag{3. 9}$$

Note that, for $\beta > 1$,

$$s - \lim_{\Lambda \to \infty} \mathcal{U}\left(\frac{g}{\Lambda^{\beta-1}}\right) = I. \tag{3. 10}$$

Hence we prove the following theorem

Theorem 3.6 ($1 < \beta < 2$) *Let $V \in \mathcal{M}_{\pm}(N)$, $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$s - \lim_{\Lambda \to \infty}\left(H_N^{\beta}(\hat{\varrho}, \Lambda) - g^2\Lambda^{2-\beta}V(\hat{\varrho}) \otimes I + V \otimes I - z\right)^{-1} = \left(-\frac{1}{2m}\Delta_N + V - z\right)^{-1} \otimes P_N.$$

Proof: Since the partial expectation of $V \otimes I$ with respect to $\Omega_{L^2(\mathbb{R}^d)}$ is $E_{\Omega_{L^2(\mathbb{R}^d)}}(V \otimes I) = V$, from (3.8), (3.9), (3.10) and Proposition 2.1, theorem follows with the following correspondence:

$$A = -\frac{1}{2m}\Delta_N, \quad B = H_b, \quad C_\Lambda = Q^1(\hat{\varrho}, \Lambda), \quad C = V \otimes I \quad G = \Omega_{L^2(\mathbb{R}^d)}.$$

□

Secondly we study the case of $2 < \beta$. In this case, note that we do not need to subtract the renormalization $\Lambda^{2-\beta}V(\hat{\varrho}) \otimes I$. Put the R.H.S. of (3.2) by

$$R.H.S.of\ (3.2) = H_N^{\beta}(\Lambda) + Q^2(\hat{\varrho}, \Lambda).$$

By the same argument as that of the case of $1 < \beta < 2$ with $Q^1(\Lambda, \hat{\varrho})$ replaced by $Q^2(\Lambda, \hat{\varrho})$, one can easily prove the following theorem.

Theorem 3.7 ($2 < \beta$) *Let $V \in \mathcal{M}_{\pm}(N)$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$s - \lim_{\Lambda \to \infty}\left(H_N^{\beta}(\hat{\varrho}, \Lambda) + V \otimes I - z\right)^{-1} = \left(-\frac{1}{2m}\Delta_N + V - z\right)^{-1} \otimes P_N.$$

4 THE PAULI-FIERZ MODEL

4.1 The Pauli-Fierz model

In this section, we study the Pauli-Fierz model in quantum electrodynamics with the dipole approximation. Let

$$\mathcal{W} = \underbrace{L^2(\mathbb{R}^d) \oplus ... \oplus L^2(\mathbb{R}^d)}_{d-1}.$$

For $(0 \oplus .... \underbrace{f}_{the\ r-th} ... \oplus 0) \in \mathcal{W}$, we set $a^\sharp_{\mathcal{W}}(0 \oplus .... \underbrace{f}_{the\ r-th} ... \oplus 0) = a^{\sharp(r)}(f)$. We write

$$a^{\sharp(r)}_{\mathcal{W}}(g) = \int a^{\sharp(r)}_{\mathcal{W}}(k) g(k) dk, \quad r = 1, ..., d-1.$$

Let $e^r : \mathbb{R}^d \to \mathbb{R}^d, r = 1, ..., d-1$, be measurable functions so that

$$(1) e^r(k) \cdot k = 0, \quad r = 1, ..., d-1, \quad (2) e^r(k) \cdot e^s(k) = \delta_{rs}.$$

We denote the μ-th component of e^r by $e^r_\mu, \mu = 1, ..., d$. The quantized smeared radiation field $A_\mu(\hat{f}, x)$ with f in the Coulomb gauge, and the conjugate momentum $\Pi_\mu(\hat{f}, x)$, $\mu = 1, ..., d, x \in \mathbb{R}^d$, are defined by

$$A_\mu(\hat{f}, x) = \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \int \left\{ a^{\dagger(r)}_{\mathcal{W}}(k) \frac{e^r_\mu(k) \tilde{\hat{f}}(k) e^{-ikx}}{\sqrt{\omega(k)}} + a^{(r)}_{\mathcal{W}}(k) \frac{e^r_\mu(k) \hat{f}(k) e^{ikx}}{\sqrt{\omega(k)}} \right\} dk,$$
$$\Pi_\mu(\hat{f}, x) = \frac{i}{\sqrt{2}} \sum_{r=1}^{d-1} \int \left\{ a^{\dagger(r)}_{\mathcal{W}}(k) \sqrt{\omega(k)} e^r_\mu(k) \tilde{\hat{f}}(k) e^{-ikx} - a^{(r)}_{\mathcal{W}}(k) \sqrt{\omega(k)} e^r_\mu(k) \hat{f}(k) e^{ikx} \right\} dk.$$

Here $\tilde{g}(k) = g(-k)$. We define the free Hamiltonian in $\mathcal{F}_{\mathcal{W}}$ by

$$d\Gamma_{\mathcal{W}}(\underbrace{\omega \oplus ... \oplus \omega}_{d-1}) = H_{EM}.$$

We require that $\hat{\varrho}$ satisfies (3.1) and ϱ is real-valued rotation invariant function throughout this section. Then the Pauli-Fierz Hamiltonian with the ultraviolet cut-off function $\hat{\varrho}$ and with N-nonrelativistic particles is defined as an operator acting in

$$\mathcal{L}_N = L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_{\mathcal{W}} \cong L^2(\mathbb{R}^{dN}; \mathcal{F}_{\mathcal{W}}),$$

by

$$\frac{1}{2m}\sum_{j=1}^{N}\Big(\mathbf{p}^j \otimes I - eA(\hat{\varrho},\cdot)\Big)^2 + I \otimes H_{EM}, \tag{4. 1}$$

where $e \in \mathbb{R}$ is a coupling constant and $A(\hat{\varrho},\cdot) = (A_1(\hat{\varrho},\cdot),...,A_d(\hat{\varrho},\cdot))$. Introducing the polarization vector e^r, which corresponds to taking the Coulomb gauge, we see that, on a suitable dense domain,

$$[\mathbf{p}^j \otimes I, A(\hat{\varrho},\cdot)] = 0.$$

Then formally we may rewrite (4.1) by

$$-\frac{1}{2m}\Delta_N \otimes I - \frac{e}{m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}(\mathbf{p}^j_\mu \otimes I)A_\mu(\hat{\varrho},\cdot) + \frac{e^2 N}{2m}A^2(\hat{\varrho},\cdot) + I \otimes H_{EM}.$$

Here, for simplicity, we introduce the following assumptions to the Pauli-Fierz Hamiltonian:

(1) The self-interaction term $A^2(\hat{\varrho},\cdot)$ is neglected.

(2) We introduce the dipole approximation,i.e.,$A(\hat{\varrho},x)$ is replaced by $A(\hat{\varrho},0)$.

Then, putting $A(\hat{\varrho},0) = I \otimes A(\hat{\varrho})$, our Hamiltonian is as follows:

$$H^\beta_{EM}(\hat{\varrho},\Lambda) = -\frac{1}{2m}\Delta_N \otimes I - \Lambda e H^{EM}_I(\hat{\varrho}) + \Lambda^\beta I \otimes H_{EM},$$

where

$$H^{EM}_I(\hat{\varrho}) = \frac{1}{m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\mathbf{p}^j_\mu \otimes A_\mu(\hat{\varrho}).$$

Put

$$H^\beta_{EM}(\Lambda) = -\frac{1}{2m}\Delta_N \otimes I + \Lambda^\beta I \otimes H_{EM}.$$

Theorem 4.1 ([1]) *Let $V \in \mathcal{M}_\pm(N)$. Then the operator $H^\beta_{EM}(\hat{\varrho},\Lambda) + V \otimes I$ is self-adjoint on $D(H^\beta_{EM}(\Lambda))$ and bounded from below. Moreover it is essentially self-adjoint on any core for $H^\beta_{EM}(\Lambda)$.*

We define a unitary operator by

$$S(e) = \exp\left(-ie\left(\sum_{j=1}^{N}\sum_{\mu=1}^{d}\frac{1}{m}\mathbf{p}_{\mu}^{j}\otimes\Pi_{\mu}\left(\frac{\hat{\varrho}}{\omega^2}\right)\right)\right),$$

where we put $\Pi_\mu(0,f) = I\otimes\Pi_\mu(f)$.

Lemma 4.2 ([1]) *Let $V\in\mathcal{M}_\pm(N)$. Then the unitary operator $S(e)$ maps $D(H_{EM}^{\beta}(\hat{\varrho},\Lambda))$ onto itself with*

$$\begin{aligned}&S(\Lambda^{1-\beta}e)^{-1}(H_{EM}^{\beta}(\hat{\varrho},\Lambda)+V\otimes I)S(\Lambda^{1-\beta}e)\\ &=-\left(\frac{1}{2m}+\Lambda^{2-\beta}\frac{e^2}{2M}\right)\Delta_N\otimes I+\Lambda^{\beta}I\otimes H_{EM}+V_\beta(\hat{\varrho},\Lambda),\end{aligned} \tag{4. 2}$$

where

$$\frac{1}{2M}=\frac{d-1}{d}\left(\frac{1}{m}\right)^2\left\|\frac{\hat{\varrho}}{\omega}\right\|_{L^2(\mathbb{R}^d)}^2,\quad V_\beta(\hat{\varrho},\Lambda)=S(\Lambda^{1-\beta}e)^{-1}(V\otimes I)S(\Lambda^{1-\beta}e).$$

To obtain the scaling limit of the case of $\beta=1$, we need to fix a $dN\times dN$-matrix $\mathbf{T}$ so that

$$\mathbf{T}\begin{pmatrix}1&1&\dots&1\\1&1&\dots&1\\ \vdots&\vdots&1&\vdots\\1&1&\dots&1\end{pmatrix}\mathbf{T}^{-1}=\begin{pmatrix}N&0&\dots&0\\0&0&\dots&0\\ \vdots&\vdots&0&\vdots\\0&0&\dots&0\end{pmatrix}.$$

Then, for a multiplication operator V in $L^2(\mathbb{R}^{dN})$, we put

$$V_{\mathrm{eff}}^{\hat{\varrho}}(x)=(2\pi C_N(\hat{\varrho}))^{-\frac{d}{2}}\int_{\mathbb{R}^d}dyV\left(\mathbf{T}^{-1}(y,(\mathbf{T}x)_2,\dots,(\mathbf{T}x)_N)\right)e^{-\frac{|(\mathbf{T}x)_1-y|^2}{2C_N(\hat{\varrho})}},$$

where

$$C_N(\hat{\varrho})=\frac{d-1}{2d}\left(\frac{e}{m}\right)^2\int_{\mathbb{R}^{dN}}dk\frac{|\hat{\varrho}(k)|^2}{\omega(k)^3},$$

and $(\mathbf{T}x)_j\in\mathbb{R}^d, j=1,\dots,N$, denotes the j-th element of $\mathbf{T}x\in\mathbb{R}^{dN}$. In the case of $\beta=1$, the following proposition is well known.

Proposition 4.3 ([1,6,7],$\beta=1$) *Let $V\in\mathcal{M}_\pm(N)$ with*

$$|V|_{eff}^{\hat{\varrho}}(x)<\infty, a.e. x\in\mathbb{R}^{dN},\quad |V|_{eff}^{\hat{\varrho}}\in L_{loc}^1(\mathbb{R}^{dN}).$$

Then $-\frac{1}{2m}\Delta_N + V_{eff}^{\hat{\varrho}}$ *is self-adjoint on* $D(-\Delta_N)$ *with, for* $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned} & s - \lim_{\Lambda \to \infty} \left(H_{EM}^{\beta}(\hat{\varrho}, \Lambda) + V \otimes I + \Lambda^{2-\beta} \frac{e^2}{2M} \Delta_N \otimes I - z \right)^{-1} \\ & = S(e) \left\{ \left(-\frac{1}{2m}\Delta_N + V_{eff}^{\hat{\varrho}} - z \right)^{-1} \otimes P_{EM} \right\} S(e)^{-1}, \end{aligned}$$

where P_{EM} *is the projection operator onto the subspace* $\{k\Omega_{\mathcal{W}} | k \in \mathbb{C}\} \subset \mathcal{F}_{\mathcal{W}}$.

4.2 The case of $\beta = 2$

Put

$$\widetilde{H_{EM}^{\beta}}(\Lambda) = -\left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N \otimes I + \Lambda^{\beta} I \otimes H_{EM}.$$

Lemma 4.4 *Let* $V \in \mathcal{M}_{\pm}(N)$. *Then, for any* $\epsilon > 0$, *there exists* Λ_0 *and* $b(\epsilon) > 0$ *so that, for all* $\Lambda > \Lambda_0$, $D\left(V_\beta\left(\hat{\varrho}, \Lambda\right)\right) \supset D(\widetilde{H_{EM}^{\beta}}(\Lambda))$ *with*

$$||V_\beta\left(\hat{\varrho}\right)\Phi||_{\mathcal{L}_N} \le \epsilon \left|\left| \widetilde{H_{EM}^{\beta}}(\Lambda)\Phi \right|\right|_{\mathcal{L}_N} + b(\epsilon)\, ||\Phi||_{\mathcal{L}_N},\, \Phi \in D(\widetilde{H_{EM}^{\beta}}(\Lambda)). \tag{4. 3}$$

Moreover, for $z \in \mathbb{C} \setminus [0, \infty)$,

$$s - \lim_{\Lambda \to \infty} V_\beta(\hat{\varrho}, \Lambda) \left(\widetilde{H_{EM}^{\beta}}(\Lambda) - z \right)^{-1} = \left[V \left(-\left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N - z \right)^{-1} \right] \otimes P_{EM}. \tag{4. 4}$$

Proof: Since V is infinitesimally small with respect to $-\Delta_N$ and $-\Delta_N$ commutes $S(e)$, one can derive (4.3). Put $\left(\widetilde{H_{EM}^{\beta}}(\Lambda) - z \right)^{-1} = K_\Lambda$, $\left(-\left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N - z \right)^{-1} \otimes P_{EM} = K_\infty$ and $S(\Lambda^{1-\beta} e) = S_\Lambda$. Note that, for $\beta > 1$,

$$s - \lim_{\Lambda \to \infty} S\left(\Lambda^{1-\beta} e\right) = I,$$

By (4.3), for any $\epsilon > 0$, taking sufficiently large $\Lambda > 0$, we have

$$\begin{aligned} & ||V_\beta(\hat{\varrho}, \Lambda) K_\Lambda \Phi - (V \otimes I) K_\infty \Phi|| \\ & \le \epsilon\, ||-\Delta_N (K_\Lambda - K_\infty)\Phi||_{\mathcal{L}_N} + \epsilon\, ||-\Delta_N (S_\Lambda K_\infty - K_\infty)\Phi||_{\mathcal{L}_N} \\ & + b(\epsilon)\, ||(K_\Lambda - K_\infty)\Phi||_{\mathcal{L}_N} + b(\epsilon)\, ||(S_\Lambda K_\infty - K_\infty)\Phi||_{\mathcal{L}_N} + \left|\left|(S_\Lambda^{-1} - I)(V \otimes I) K_\infty \Phi\right|\right|_{\mathcal{L}_N}. \end{aligned}$$

Taking $\Lambda \to \infty$ on the both sides above, we have

$$\lim_{\Lambda \to \infty} ||V_\beta(\hat{\varrho}, \Lambda) K_\Lambda \Phi - (V \otimes I) K_\infty \Phi||_{\mathcal{L}_N} \le \epsilon\, ||(I \otimes I - I \otimes P_{EM})\Phi||_{\mathcal{L}_N}.$$

Since $\epsilon > 0$ is arbitrary, (4.4) follows. □

Theorem 4.5 **($\beta = 2$)** *Let $V \in \mathcal{M}_{\pm}(N)$. Then, for $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$s - \lim_{\Lambda \to \infty} \left(H_{EM}^{\beta}(\hat{\varrho}, \Lambda) + V \otimes I - z \right\}^{-1} = \left\{ - \left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N + V - z \right\}^{-1} \otimes P_{EM}. \tag{4. 5}$$

Proof: By virtue of (4.2), we see that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\left(H_{EM}^{\beta}(\hat{\varrho}, \Lambda) + V \otimes I - z \right)^{-1} = S\left(\Lambda^{1-\beta} e\right) \left(\widetilde{H_{EM}^{\beta}}(\Lambda) + V_{\beta}(\hat{\varrho}, \Lambda) - z \right)^{-1} S\left(\Lambda^{1-\beta} e\right)^{-1},$$

and the partial expectation of $V \otimes I$ with respect to $\Omega_{\mathcal{W}}$ is $E_{\Omega_{\mathcal{W}}}(V \otimes I) = V$. Hence, it follows (4.5) from Lemma 4.4 and Proposition 2.1 with the following correspondence:

$$A = - \left(\frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N, B = H_{EM}, C(\Lambda) = V_{\beta}(\hat{\varrho}, \Lambda), C = V \otimes I, G = \Omega_{\mathcal{W}}.$$

□

4.3 The case of $1 < \beta < 2$, $2 < \beta$

For the case of $1 < \beta < 2$, by (4.2), we should subtract the term $-\Lambda^{2-\beta} \frac{e^2}{2M} \Delta_N \otimes I$ from the original Hamiltonian $H_{EM}^{\beta}(\hat{\varrho}, \Lambda)$, and for the case of $\beta > 2$, we do not need any renormalization. Hence,the similar argument of the cases of $\beta = 2$ and $\beta = 1$ gives an asymptotic behaviors of $H_{EM}^{\beta}(\hat{\varrho}, \Lambda)$. See Fig. 6.2.

5 THE SPIN-BOSON MODEL

5.1 The spin-boson model

In this section we study the spin-boson model. The total Hamiltonian of the spin-boson model is defined as an operator acting in the Hilbert space

$$\mathcal{L}_{SB} = \mathbb{C}^2 \otimes \mathcal{F}_{L^2(\mathbb{R}^d)} \cong \mathcal{F}_{L^2(\mathbb{R}^d)} \oplus \mathcal{F}_{L^2(\mathbb{R}^d)},$$

by

$$H_{SB}^{\beta}(\lambda, \Lambda) \quad = \quad \nu \sigma_1 + \Lambda \sigma_3 \otimes \left(a^{\dagger}(\bar{\lambda}) + a(\lambda) \right) + \Lambda^{\beta} I \otimes H_{SB}.$$

Here $H_{SB}=d\Gamma_{L^2(\mathbb{R}^d)}(\omega)$, $\nu>0$, $\lambda\in L^2(\mathbb{R}^d)$ and

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

In what follows, we assume that

$$\lambda,\frac{\lambda}{\sqrt{\omega}},\frac{\lambda}{\omega}\in L^2(\mathbb{R}^d).$$

Theorem 5.1 **([1])** *The operator $H_{SB}(\lambda,\Lambda)$ is self-adjoint on $D(I\otimes H_{SB})$ and bounded from below. Moreover essentially self-adjoint on any core for $I\otimes H_{SB}$.*

We define a unitary operator by

$$\mathbf{T}(\lambda)=\begin{pmatrix}e^{+\left\{a^\dagger\left(\frac{\bar{\lambda}}{\omega}\right)-a\left(\frac{\lambda}{\omega}\right)\right\}}&0\\0&e^{-\left\{a^\dagger\left(\frac{\bar{\lambda}}{\omega}\right)-a\left(\frac{\lambda}{\omega}\right)\right\}}\end{pmatrix}\equiv\begin{pmatrix}T_+(\lambda)&0\\0&T_-(\lambda)\end{pmatrix}.$$

In the case of $\beta=1$, following proposition is well known.

Proposition 5.2 **([1],$\beta=1$)** *Let $F(\lambda)=\left\langle\Omega_{L^2(\mathbb{R}^d)},T_+(\lambda)\Omega_{L^2(\mathbb{R}^d)}\right\rangle_{\mathcal{F}_{L^2(\mathbb{R}^d)}}$ and $z\in\mathbb{C}\backslash\mathbb{R}$. Then*

$$s-\lim_{\Lambda\to\infty}\left(H^\beta_{SB}(\lambda,\Lambda)-\Lambda^{2-\beta}E_{SB}-z\right)^{-1}=\mathbf{T}(\lambda)\left\{(\nu F(\lambda)\sigma_1-z)^{-1}\otimes P_{SB}\right\}\mathbf{T}(\lambda)^{-1}.$$

Here P_{SB} is the projection operator onto $\{k\Omega|k\in\mathbb{C}\}\subset\mathcal{F}_{L^2(\mathbb{R}^d)}$ and $E_{SB}=-\left\|\frac{\lambda}{\sqrt{\omega}}\right\|^2_{L^2(\mathbb{R}^d)}$.

5.2 the case of $\beta=2$

It is well known and easily checked that $\mathbf{T}(\Lambda^{1-\beta}\lambda)$ maps $D(I\otimes H_{SB})$ onto itself with

$$\begin{aligned}&\mathbf{T}^{-1}(\Lambda^{1-\beta}\lambda)H^\beta_{SB}(\lambda)\mathbf{T}(\Lambda^{1-\beta}\lambda)\\&=\nu\begin{pmatrix}0&T^2_-(\Lambda^{1-\beta}\lambda)\\T^2_+(\Lambda^{1-\beta}\lambda)&0\end{pmatrix}+\Lambda^\beta I\otimes H_{SB}+\Lambda^{2-\beta}E_{SB}.\end{aligned}\tag{5. 1}$$

Theorem 5.3 **($\beta=2$)** *Let $z\in\mathbb{C}\setminus\mathbb{R}$. Then*

$$s-\lim_{\Lambda\to\infty}\left(H^\beta_{SB}(\lambda,\Lambda)-z\right)^{-1}=(\nu\sigma_1+E_{SB}-z)^{-1}\otimes P_{SB}.\tag{5. 2}$$

Proof: We see that, by (5.1),

$$\begin{aligned}&\mathbf{T}^{-1}\left(\Lambda^{1-\beta}\lambda\right)\left(H_{SB}^{\beta}(\lambda,\Lambda)-z\right)^{-1}\mathbf{T}\left(\Lambda^{1-\beta}\lambda\right)\\&=\left\{\nu\begin{pmatrix}0 & T_-^2\left(\Lambda^{1-\beta}\lambda\right)\\ T_+^2\left(\Lambda^{1-\beta}\lambda\right) & 0\end{pmatrix}+\Lambda^{\beta}I\otimes H_{SB}+\Lambda^{2-\beta}E_{SB}-z\right\}^{-1}.\end{aligned}$$

It is easily seen that $s-\lim_{\Lambda\to\infty}\mathbf{T}\left(\Lambda^{1-\beta}\lambda\right)=I$ and

$$\begin{aligned}&s-\lim_{\Lambda\to\infty}\nu\begin{pmatrix}0 & T_-^2\left(\Lambda^{1-\beta}\lambda\right)\\ T_+^2\left(\Lambda^{1-\beta}\lambda\right) & 0\end{pmatrix}\left(\Lambda^{2-\beta}E_{SB}+\Lambda^{\beta}I\otimes H_{SB}-z\right)^{-1}\\&=\left[\nu\sigma_1\left(E_{SB}-z\right)^{-1}\right]\otimes P_{SB}.\end{aligned}$$

Hence, with the following correspondence:

$$A=E_{SB},B=H_{SB},C(\Lambda)=\nu\begin{pmatrix}0 & T_-^2\left(\Lambda^{1-\beta}\lambda\right)\\ T_+^2\left(\Lambda^{1-\beta}\lambda\right) & 0\end{pmatrix},C=\nu\sigma_1,G=\Omega_{L^2(\mathbb{R}^d)},$$

one can easily check the conditions with respect to $C(\Lambda)$ and C in section 2. Since the partial expectation of $\nu\sigma_1\otimes I$ with respect to $\Omega\ L^2(\mathbb{R}^d)$ is $E_{\Omega_{L^2(\mathbb{R}^d)}}(\nu\sigma_1\otimes I)=\nu\sigma_1$, we get (5.2) by Proposition 2.1. □

5.3 The case of $1<\beta<2, 2<\beta$

For the case of $1<\beta<2$, by (5.1), we should subtract the term $\Lambda^{2-\beta}E_{SB}$ from the original Hamiltonian $H_{SB}^{\beta}(\lambda,\Lambda)$, and for the case of $\beta>2$, we do not need any renormalization. Hence,the similar argument of the cases of $\beta=2$ and $\beta=1$ gives an asymptotic behaviors of $H_{SB}^{\beta}(\lambda,\Lambda)$. See Fig 6.3.

6 CONCLUDING REMARKS

(**1**) In section 4, we studied the Pauli-Fierz model neglected the terms $A^2(\hat{\varrho},\cdot)$. In [6,7], we studied the Pauli-Fierz Hamiltonian with the terms $A^2(\hat{\varrho},\cdot)$. By the same method developed in [6,7], we can investigate the following scaling Hamiltonians:

$$-\frac{1}{2m}\Delta_N\otimes I-\Lambda eH_I^{EM}(\hat{\varrho})+\Lambda I\otimes H_{EM}+\frac{e^2N}{2m}A^2(\hat{\varrho},\cdot)+V\otimes I, \tag{6. 1}$$

$$-\frac{1}{2m}\Delta_N\otimes I-\Lambda eH_I^{EM}(\hat{\varrho})+\Lambda^2 I\otimes H_{EM}+\Lambda^2\frac{e^2N}{2m}A^2(\hat{\varrho},\cdot)+V\otimes I. \tag{6. 2}$$

Introducing different renormalizations from those given in this paper, we can get effective Hamiltonians of (6.1) and (6.2).

(**2**) In the case of $0 < \beta < 1$, we need delicate discussions of asymptotic behaviors of unitary operators $\mathcal{U}(\Lambda^{1-\beta}g)$, $S(\Lambda^{1-\beta}e)$ and $\mathbf{T}(\Lambda^{1-\beta}\lambda)$ as $\Lambda \to \infty$. We omit the discussions.

	Effective Hamiltonian	Unitary operator	Renormalization
$\beta > 2$	$-\frac{1}{2m}\Delta_N + V$	I	0
$\beta = 2$	$-\frac{1}{2m}\Delta_N + g^2 V(\hat{\varrho}) + V$	I	0
$1 < \beta < 2$	$-\frac{1}{2m}\Delta_N + V$	I	$g^2\Lambda^{2-\beta}V(\hat{\varrho})$
$\beta = 1$	$-\frac{1}{2m}\Delta_N + g^2 N\delta(\hat{\varrho}) + V$	$\mathcal{U}(g)$	$g^2\Lambda V(\hat{\varrho})$

Fig 6.1 β-coupling Nelson model $H_N^\beta(\hat{\varrho}, \Lambda)$

	Effective Hamiltonian	Unitary operator	Renormalization
$\beta > 2$	$-\frac{1}{2m}\Delta_N + V$	I	0
$\beta = 2$	$-\left(\frac{1}{2m} + \frac{e^2}{2M}\right)\Delta_N + V$	I	0
$1 < \beta < 2$	$-\frac{1}{2m}\Delta_N + V$	I	$-\Lambda^{2-\beta}\frac{e^2}{2M}\Delta_N$
$\beta = 1$	$-\frac{1}{2m}\Delta_N + V_{\rm eff}^{\hat{\varrho}}$	$S(e)$	$-\Lambda\frac{e^2}{2M}\Delta_N$

Fig.6.2 β-coupling Pauli-Fierz model $H_{EM}^\beta(\hat{\varrho}, \Lambda)$

	Effective Hamiltonian	Unitary operator	Renormalization
$\beta > 2$	$\nu\sigma_1$	I	0
$\beta = 2$	$\nu\sigma_1 + E_{SB}$	I	0
$1 < \beta < 2$	$\nu\sigma_1$	I	$\Lambda^{2-\beta} E_{SB}$
$\beta = 1$	$\nu F(\lambda)\sigma_1$	$\mathbf{T}(\lambda)$	ΛE_{SB}

Fig.6.3 β-coupling spin-boson model $H_{SB}^{\beta}(\lambda, \Lambda)$

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