Reduction of weakly nonlinear parabolic partial differential equations

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Abstract

It is known that the Swift-Hohenberg equation $\partial u/\partial t = -(\partial_x^2+1)^2u+\varepsilon(u-u^3)$ can be reduced to the Ginzburg-Landau equation (amplitude equation) $\partial A/\partial t = 4\partial_x^2 A+\varepsilon(A-3A|A|^2)$ by means of the singular perturbation method. This means that if $\varepsilon > 0$ is sufficiently small, a solution of the latter equation provides an approximate solution of the former one. In this paper, a reduction of a certain class of a system of nonlinear parabolic equations $\partial u/\partial t = \mathcal{P}u + \varepsilon f(u)$ is proposed. An amplitude equation of the system is defined and an error estimate of solutions is given. Further, it is proved under certain assumptions that if the amplitude equation has a stable steady state, then a given equation has a stable periodic solution . In particular, near the periodic solution, the error estimate of solutions holds uniformly in $t > 0$.
Keywords: amplitude equation; renormalization group method; reaction diffusion equation

1 Introduction

A reduction of a certain class of nonlinear parabolic partial differential equations (PDEs)

$$\frac{\partial u}{\partial t} = \mathcal{P}u + \varepsilon f(u), \quad u = u(t, x) \in \mathbf{C}^m, (t, x) \in \mathbf{R} \times \mathbf{R}^d, \tag{1.1}$$

is considered, where $\varepsilon > 0$ is a small parameter, $\mathcal{P}$ is an elliptic differential operator with constant coefficient and f is a function on $\mathbf{C}^m$ satisfying suitable assumptions. Our study is motivated by the following three problems.

Case 1. It is well known that the Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} = -(\partial_x^2 + k^2)^2 u + \varepsilon(u - u^3), \quad u, x \in \mathbf{R}, \tag{1.2}$$

with a parameter $k \in \mathbf{R}$, can be reduced to the Ginzburg-Landau equation (amplitude equation)

$$\frac{\partial A}{\partial t} = 4k^2\partial_x^2 A + \varepsilon(A - 3A|A|^2), \tag{1.3}$$

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by means of the multiscaling method [4] or the renormalization group (RG) method [1]. Let v_0 be a function in some function space and give initial conditions

$$u(0,x) = v_0(\sqrt{\varepsilon}x)e^{ikx} + \overline{v_0(\sqrt{\varepsilon}x)}e^{-ikx}, \quad A(0,x) = v_0(\sqrt{\varepsilon}x),$$

for Eqs.(1.2) and (1.3), respectively. In [5], it is proved that there exists a positive number C such that solutions of the two initial value problems satisfy

$$\|u(t,x) - (A(t,x)e^{ikx} + \overline{A(t,x)}e^{-ikx})\| \leq C\sqrt{\varepsilon}, \tag{1.4}$$

up to the time scale $t \sim O(1/\varepsilon)$ with a certain norm. In this case, a fourth-order PDE is reduced to a second-order PDE.

Case 2. Let $\Omega = \mathbf{R} \times (0,l)$ be the strip region on $\mathbf{R}^2$. Consider the boundary value problem of a system of reaction diffusion equations on Ω

$$\begin{cases} \dfrac{\partial u}{\partial t} = d(\partial_x^2 u + \partial_y^2 u) + ku - v + \varepsilon(u - u^3), \\ \dfrac{\partial v}{\partial t} = \partial_x^2 v + \partial_y^2 v + u - v, \\ \dfrac{\partial u}{\partial y}\Big|_{y=0,l} = \dfrac{\partial v}{\partial y}\Big|_{y=0,l} = 0, \end{cases} \tag{1.5}$$

where l, d and k are positive constants. This type problem was introduced by Chen, Ei and Lin [7] to investigate a stripe pattern observed in the skin of angelfish. Under certain assumptions on parameters so that the system undergoes Turing instability, they formally derived an amplitude equation of the form

$$\frac{\partial A}{\partial t} = -\frac{2d^2}{(k+d)(1-d)}\frac{\partial^4 A}{\partial x^4} + \frac{\varepsilon}{1-d}(A - 3A|A|^2), \tag{1.6}$$

without any mathematical justification. It is remarkable that the amplitude equation is a fourth-order equation while a given system is a second-order equation because of a certain degeneracy of the dispersion relation. On the other hand, a system of equations becomes a single equation and the number of space variables is reduced.

Case 3. Let us consider a system of reaction diffusion equations on $\mathbf{R}$

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\partial_x^2 u + v + \varepsilon(u - u^3), \\ \dfrac{\partial v}{\partial t} = D\partial_x^2 v - u, \end{cases} \tag{1.7}$$

where $D > 0$ is a constant. This system can be reduced to the Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = D\partial_x^2 A + \frac{\varepsilon}{2}(A - 3A|A|^2). \tag{1.8}$$

In this case, the order of differential equations are the same, while a system is reduced to a single equation.

A purpose in this paper is to give a unified theory of such a reduction of PDEs (1.1), and give an error estimate of solutions of amplitude equations. Furthermore, we will partially prove a conjecture by Oono and Shiwa [2], which states that if a given PDE is structurally stable, its amplitude equation provides the qualitative features of the given system. For example, a stable invariant manifold of an amplitude equation implies the existence of a stable invariant manifold of a given system. For example, we will prove that Eq.(1.6) actually provides an approximate solution of the system (1.5). Further, a conjecture by [2] is solved in the following sense; if the amplitude equation (1.6) has a stable steady state, then the system (1.5) has a corresponding stable periodic solution.

Since general results for (1.1) is rather complicated, we divide main results into several steps as follows:

(1) In this Introduction, our main results are stated for one-dimensional problems $u \in \mathbf{C}$ and $x \in \mathbf{R}$ for simplicity.

(2) In Sec.3, the asymptotic behavior of linear semigroups generated by elliptic differential operators are investigated.

(2-i) Sec.3.1 deals with the case $u \in \mathbf{C}$ and $x \in \mathbf{R}^d$. The asymptotic behavior of a semigroup $e^{\mathcal{P}t}$ is given under the assumptions (B1) to (B3) (Propositions 3.1, 3.2 and 3.3).

(2-ii) In Sec.3.2, the case $u \in \mathbf{C}^m$ and $x \in \mathbf{R}^d$ is considered. The asymptotic behavior of a semigroup $e^{\mathcal{P}t}$ is given under the assumptions (C0) to (C3) (Proposition 3.6).

(3) Sec.4 is devoted to nonlinear estimates and our main results are given.

(3-i) In Sec.4.1, the case $u \in \mathbf{C}$ and $x \in \mathbf{R}^d$ is considered, which includes Case 1 above as an example. The definition of an amplitude equation (reductive equation) is given. An error estimate of solutions (Thm.4.2) is proved under the assumptions (D1) to (D3), and the existence of stable periodic solutions (Thm.4.3) is proved under the assumptions (D1) to (D4).

(3-ii) In Sec.4.2, the case $u \in \mathbf{C}^m$ and $x \in \mathbf{R}^d$ is considered, which includes Case 2 and Case 3 above. An error estimate of solutions (Thm.4.13) is proved under the assumptions (E0) to (E3), and the existence of stable periodic solutions (Thm.4.14) is proved under the assumptions (E0) to (E4). Theorems 4.13 and 4.14 include all previous results.

Since we need several integers to state our final results in Sec.4, we summarize some of them for the convenience of the reader.

- An integer m denotes the dimension of unknown function: $u \in \mathbf{C}^m$.
- An integer d denotes the dimension of space variables: $x \in \mathbf{R}^d$.
- An integer M denotes the degeneracy of the dispersion relation, which determines the order of differentiation of the amplitude equation. For Case 1 and 3, $M = 2$, while $M = 4$ for Case 2.
- An integer $D\,(1 \leq D \leq d)$ denotes a dimension of the critical direction (see Sec.3.1 for the detail), which gives the number of space variables included in the amplitude equation. For Case 2, $d = 2$ and $D = 1$.

- An integer N gives the number of critical wave numbers, at which the spectrum of the operator $\mathcal{P}$ is tangent to the imaginary axis. In other words, it gives the dimension of a center subspace (see Sec.4.1 for the detail). An amplitude equation becomes a system of N-equations.

Although our purpose is *a system* of PDEs including a degenerate case as above, it would be better to start with a one-dimensional case for the sake of simplicity. In this Introduction, we suppose $u \in \mathbf{C}$ and $x \in \mathbf{R}$ are one-dimensional variables to state our main results as simple as possible. Higher dimensional problems will be treated after Sec.2.

Let $P(x) = \sum_{\alpha=0}^{q} a_\alpha x^\alpha$ be a polynomial of $x \in \mathbf{R}$ and $\mathcal{P} := P(\partial_x)$ a corresponding differential operator on $\mathbf{R}$. For the operator $\mathcal{P}$, we suppose the following:

(A1) $\mathrm{Re}[P(i\xi)] \leq 0$ for any $\xi \in \mathbf{R}$.

(A2) There exist $\omega, k \in \mathbf{R}$ $((\omega, k) \neq (0,0))$ and an integer M such that

$$P(\pm ik) = \pm i\omega, \tag{1.9}$$
$$P'(\pm ik) = \cdots = P^{(M-1)}(\pm ik) = 0, \tag{1.10}$$
$$P^{(M)}(ik) = P^{(M)}(-ik) \neq 0. \tag{1.11}$$

(A3) $a_q i^q < 0$ and $P^{(M)}(ik)i^M < 0$.

The assumption (A1) implies that the spectrum $\sigma(\mathcal{P})$ of $\mathcal{P}$ calculated in a suitable space, which coincides with $P(i\mathbf{R})$, is included in the closed left half plane. If $\sigma(\mathcal{P})$ were included in the open left half plane, $u = 0$ is linearly stable. Since we are interested in a bifurcation occurred at $\varepsilon = 0$, we supposed in Eq.(1.9) that $\sigma(\mathcal{P})$ includes points $\pm i\omega$ on the imaginary axis. The integer M represents the degeneracy of the dispersion relation $\lambda = P(i\xi)$. Define the operator Q to be

$$Q = \frac{P^{(M)}(ik)}{M!} \frac{\partial^M}{\partial x^M}. \tag{1.12}$$

The assumption (A3) assures that $\mathcal{P}$ and Q are elliptic. In this section, we further suppose that integers j satisfying $P(ijk) = ij\omega$ are only ± 1 (this will be removed after Sec.2). For the Swift-Hohenberg equation, $\omega = 0$, $M = 2$ and k is the k in Eq.(1.2).

Let $B^r := BC^r(\mathbf{R}; \mathbf{C})$ be a vector space of complex-valued functions f on $\mathbf{R}$ such that $f(x), f'(x), \cdots, f^{(r)}(x)$ are bounded uniformly continuous. This is a Banach space with the norm defined by $||f|| = \sup\{|f(x)|, \cdots, |f^{(r)}(x)|\}$. For a function $f : B^r \to B^r$, define the function $R : (B^r \times B^r) \to B^r$ to be

$$R(A_1, A_2) = \begin{cases} \dfrac{k}{2\pi} \displaystyle\int_0^{2\pi/k} f(A_1 e^{ikx} + A_2 e^{-ikx}) e^{-ikx} dx & \text{(when } k \neq 0\text{),} \\ \dfrac{\omega}{2\pi} \displaystyle\int_0^{2\pi/\omega} f(A_1 e^{i\omega t} + A_2 e^{-i\omega t}) e^{-i\omega t} dt & \text{(when } \omega \neq 0\text{).} \end{cases} \tag{1.13}$$

One can verify that these two expressions coincide with one another if $k \neq 0$ and $\omega \neq 0$. For example if $f(u) = u - u^3$, then $R(A_1, A_2) = A_1 - 3A_1^2 A_2$.

Now we consider two initial value problems

$$\frac{\partial u}{\partial t} = \mathcal{P}u + \varepsilon f(u), \quad u(0,x) = v_1(\eta x)e^{ikx} + v_2(\eta x)e^{-ikx} \tag{1.14}$$

and

$$\begin{cases} \dfrac{\partial A_1}{\partial t} = \mathcal{Q}A_1 + \varepsilon R(A_1, A_2), \quad A_1(0,x) = v_1(\eta x), \\ \dfrac{\partial A_2}{\partial t} = \mathcal{Q}A_2 + \varepsilon R(A_2, A_1), \quad A_2(0,x) = v_2(\eta x), \end{cases} \tag{1.15}$$

where $\eta := \varepsilon^{1/M}$. We call the latter system the *amplitude equation*. When $\mathcal{P} = -(\partial_x^2 + k^2)^2$ and $f(u) = u - u^3$, the Ginzburg-Landau equation (1.3) is obtained as a special case $A_2 = \overline{A}_1$.

Theorem 1.1. Suppose $f : BC^r(\mathbf{R};\mathbf{C}) \to BC^r(\mathbf{R};\mathbf{C})\,(r \geq 1)$ is C^1 and $\varepsilon > 0$ is sufficiently small. For any $v_1, v_2 \in BC^r(\mathbf{R};\mathbf{C})$, there exist positive numbers C, T_0 and t_0 such that mild solutions of the two initial value problems satisfy

$$||u(t,x) - (A_1(t,x)e^{ikx+i\omega t} + A_2(t,x)e^{-ikx-i\omega t})|| \leq C\eta = C\varepsilon^{1/M}, \tag{1.16}$$

for $t_0 \leq t \leq T_0/\varepsilon$.

If we suppose $A_1 = A_2 := A$, the system (1.15) is reduced to a single equation $\partial A/\partial t = \mathcal{Q}A + \varepsilon R(A, A)$; the set $\{A_1 = A_2\}$ is an invariant set of (1.15). Thus we put $S(A) = R(A, A)$ and consider two initial value problems

$$\frac{\partial u}{\partial t} = \mathcal{P}u + \varepsilon f(u), \quad u(0,x) = v_1(\eta x)e^{ikx} + v_1(\eta x)e^{-ikx} \tag{1.17}$$

and

$$\frac{\partial A}{\partial t} = \mathcal{Q}A + \varepsilon S(A), \quad A(0,x) = v_1(\eta x). \tag{1.18}$$

For example if $f(u) = u - u^3$, then $S(A) = A - 3A^3$. For the equation (1.17), we further suppose that

(A4) $P(i\xi) = \overline{P(-i\xi)}$ and $f(\overline{u}) = \overline{f(u)}$.

That is, $P(x)$ and $f(u)$ are real-valued for $x, u \in \mathbf{R}$. In particular, if $v_1(\eta x) \in \mathbf{R}$ so that $u(0,x)$ is real-valued, then a solution $u(t,x)$ is also real-valued. In the next theorem, $BC^r(\mathbf{R};\mathbf{R})$ denotes the set of *real*-valued functions f on $\mathbf{R}$ such that $f(x), f'(x), \cdots, f^{(r)}(x)$ are bounded uniformly continuous.

Theorem 1.2. Suppose $f : BC^r(\mathbf{R};\mathbf{R}) \to BC^r(\mathbf{R};\mathbf{R})\,(r \geq 1)$ is C^2 such that the second derivative is locally Lipschitz continuous. Suppose that there exists a constant $\phi \in \mathbf{R}$ such that $S(\phi) = 0$ and $S'(\phi) < 0$ (that is, $A(t,x) \equiv \phi$ is an asymptotically stable steady state of Eq.(1.18)). If $\varepsilon > 0$ is sufficiently small, Eq.(1.17) has a solution of the form

$$u_p(t,x) = \Big(\phi + \eta\psi(t,x,\eta)\Big) \cdot 2\cos(kx + \omega t). \tag{1.19}$$

The functions ψ and u_p are bounded as $\eta \to 0$ and satisfy

$$\begin{cases} 2\pi/k\text{-periodic in } x \text{ (when } k \neq 0) \\ 2\pi/\omega\text{-periodic in } t \text{ (when } \omega \neq 0), \\ \text{constant in } t, x \text{ (when } \omega = 0, k = 0, \text{ respectively}), \end{cases}$$

This u_p is stable in the following sense: there is a neighborhood $U \subset BC^r(\mathbf{R};\mathbf{R})$ of ϕ in $BC^r(\mathbf{R};\mathbf{R})$ such that if $v_1 \in U$, then a mild solution u of the initial value problem (1.17) satisfies $\|u(t,\cdot) - u_p(t,\cdot)\| \to 0$ as $t \to \infty$.

The assumption (A4) and a space $BC^r(\mathbf{R};\mathbf{R})$ means that this theorem holds when every data are real numbers. The periodic solution u_p is not asymptotically stable toward a complex direction. These theorems are obtained as special cases of Theorems 4.2 and 4.3 proved in Sec.4.

Example 1.3. For the Swift-Hohenberg equation, the estimate (1.4) immediately follows from Thm.1.1 by putting $A_2 = \overline{A}_1$ and $v_2 = \overline{v}_1$. Note that the assumption for the initial value v_1 is more relaxed than that given in [5] because we use a mild solution. To prove the existence of a spatially periodic solution, note that the function $S(A)$ is given as $S(A) = A - 3A^3$, so that $A = \phi = 1/\sqrt{3}$ satisfies the assumptions for Thm.1.2. Then, it turns out that Eq.(1.2) has a stable solution of the form

$$u_p(t,x) = \frac{2}{\sqrt{3}}\cos(kx) + O(\eta), \tag{1.20}$$

which can be obtained directly without using the amplitude equation [4].

The above theorems will be extended to more higher dimensional problems in Sec.4.

Example 1.4. Consider the boundary value problem (1.5) with constants l, d and k. Let

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} d(\partial_x^2 u + \partial_y^2 u) + ku - v \\ \partial_x^2 v + \partial_y^2 v + u - v \end{pmatrix} \tag{1.21}$$

be the linear operator which defines the unperturbed part. We suppose that d and k satisfy $(k+d)^2 = 4d$. Then, the spectrum $\sigma(L)$ is the negative real axis and the origin (See Sec.2), so that the system (1.5) undergoes the Turing instability when $\varepsilon = 0$. The eigenfunction for 0-eigenvalue satisfying the boundary condition is given by

$$\begin{pmatrix} 1 \\ (k+d)/2 \end{pmatrix} e^{icy} + \begin{pmatrix} 1 \\ (k+d)/2 \end{pmatrix} e^{-icy}, \quad c := \sqrt{\frac{k-d}{2d}},\ l := \frac{\pi}{c}. \tag{1.22}$$

We will show that the corresponding amplitude equation is given by

$$\frac{\partial A}{\partial t} = -\frac{2d^2}{(k+d)(1-d)}\frac{\partial^4 A}{\partial x^4} + \frac{\varepsilon}{1-d}(A - 3A|A|^2), \quad A(0,x) = v_0(\eta x), \tag{1.23}$$

where $\eta = \varepsilon^{1/4}$. Let us consider a solution of (1.5) with the initial condition

$$\begin{pmatrix} u(0,x,y) \\ v(0,x,y) \end{pmatrix} = A(0,x)\begin{pmatrix} 1 \\ (k+d)/2 \end{pmatrix} e^{icy} + \overline{A(0,x)}\begin{pmatrix} 1 \\ (k+d)/2 \end{pmatrix} e^{-icy}. \tag{1.24}$$

From theorems shown in Sec.4, it turns out that u is approximately given by

$$u(t, x, y) = A(t, x)e^{icy} + \overline{A(t, x)}e^{-icy} + O(\eta), \tag{1.25}$$

for $t_0 \leq t \leq T/\varepsilon$. Further, the system (1.5) proves to have a steady solution of the form

$$\begin{pmatrix} u_p(t, x, y) \\ v_p(t, x, y) \end{pmatrix} = \frac{2}{\sqrt{3}} \cos(cy) \begin{pmatrix} 1 \\ (k + d)/2 \end{pmatrix} + O(\eta), \tag{1.26}$$

which is periodic in y and constant in x, t. The fourth-order amplitude equation (1.23) was also derived by [7] in a certain formal way without error estimates of solutions. The results in this paper assure that (1.23) indeed provides approximate solutions and a steady state.

For both examples, the spectra of the unperturbed linear operators are continuous spectra including the origin. Thus it is expected that when ε becomes positive from 0, the spectra get across the imaginary axis and bifurcations occur. Unfortunately, there are no systematic ways to detect such bifurcations because spectrum on the imaginary axis is not discrete. The reduction proposed in this paper provides a systematic way to detect bifurcations; a bifurcation problem is reduced to that of the amplitude equation, although to investigate the amplitude equation is still difficult in general.

In Sec.2, we will demonstrate that how the amplitude equation is derived by means of the RG method. The RG method here is one of the singular perturbation methods for differential equations proposed by Chen, Goldenfeld and Oono [1]. In [3], it is proved for ODEs that the RG method unifies classical perturbation methods such as the multiscaling method, the averaging method, normal forms and so on. In particular, when the spectrum of an unperturbed linear part is discrete and a center manifold exists, the RG method is equivalent to the center manifold reduction; the amplitude equation gives the dynamics on the center manifold. This paper shows that the RG method and the amplitude equation are still valid even when a spectrum is not discrete and a center manifold does not exist. Even if there are no center manifolds, the amplitude equation provides the dynamics near the center subspace and it is useful to study bifurcations of a given PDE. In particular, Thm.1.2 means that a bifurcation may occur at $\varepsilon = 0$; when $\omega = 0$, it is a bifurcation of a steady state and when $\omega \neq 0$, a t-periodic solution appears like as a Hopf bifurcation.

Although results in this paper are partially obtained by many authors for specific problems [4, 5, 6], our proof is systematic which is applicable to a wide class of PDEs. From our proofs, it turns out that reductions of linear operators (reduction of a given differential operator $\mathcal{P}$ to Q) and that of nonlinearities (reduction of $f(u)$ to $S(u)$) can be done independently. The reduction of linear operators is described in Prop.3.1 and 3.6, which have the following significant meaning: the semigroup $e^{\mathcal{P}t}$ generated by $\mathcal{P}$ is approximated by its self-similar part. Note that the evolution equation $\dot{u} = Qu$ has the self-similar structure in the sense that it is invariant under the transformation

$$(t, x) \mapsto (c^M t, cx), \quad c \in \mathbf{R}, \tag{1.27}$$

see (1.12). Then, Prop.3.1 implies that a *non*-self-similar part of $e^{\mathcal{P}t}$ decays to zero as $t \to \infty$. In other words, if we apply the above transformation repeatedly, then a non-self-similar part decays to zero, while a self-similar part survives because it is invariant under the transformation. This self-similar part defines the linear operator Q. Such a technique to obtain a self-similar structure is also known as the renormalization group method in statistical mechanics.

2 The renormalization group method

In this section, we demonstrate how the amplitude equation for (1.1) is obtained by the RG method with examples. The RG method is a formal way to find amplitude equations and the results in this section will not be used in later sections. Although we only consider parabolic-type PDEs, the RG method is applicable to a more large class of PDEs and it has advantages over the multiscaling method [1]. See [3] for the RG method for ODEs.

Let us consider the Swift-Hohenberg equation (1.2). We expand a solution as $u = u_0 + \varepsilon u_1 + O(\varepsilon^2)$. The zero-th order term u_0 satisfies the linear equation

$$\frac{\partial u_0}{\partial t} = -(\partial_x^2 + k^2)^2 u_0. \tag{2.1}$$

We are interested in the dynamics near the center subspace. The spectrum of $-(\partial_x^2 + k^2)^2$ intersects with the imaginary axis at the origin, and the corresponding eigenfunctions are e^{ikx} and e^{-ikx}. Thus we consider the solution of the form

$$u_0(x) = Ae^{ikx} + Be^{-ikx}, \tag{2.2}$$

where $A, B \in \mathbf{C}$ are constants. Then, the first order term u_1 satisfies the inhomogeneous linear equation

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= -(\partial_x^2 + k^2)^2 u_1 + u_0 - u_0^3 \\ &= -(\partial_x^2 + k^2)^2 u_1 + Ae^{ikx} + Be^{-ikx} - (A^3 e^{3ikx} + 3A^2 B e^{ikx} + 3AB^2 e^{-ikx} + B^3 e^{-3ikx}). \end{aligned} \tag{2.3}$$

Since the factors $e^{\pm ikx}$ in the inhomogeneous terms are eigenfunctions of the operator $-(\partial_x^2 + k^2)^2$, it is expected that a solution of this equation includes secular terms which diverge in t and x. To find secular terms arising from the factor e^{ikx}, we consider the equation

$$\frac{\partial u_1}{\partial t} = -(\partial_x^2 + k^2)^2 u_1 + (A - 3A^2 B)e^{ikx} \tag{2.4}$$

instead of Eq.(2.3). We assume a special solution of the form

$$u_1 = (\mu_1 t^{\beta_1} + \mu_2 x^{\beta_2})e^{ikx}.$$

Substituting this into Eq.(2.4), we obtain $\beta_1 = 1$, $\beta_2 = 2$, and μ_1, μ_2 prove to satisfy the relation

$$\mu_1 = 8k^2 \mu_2 + A - 3A^2 B. \tag{2.5}$$

Secular terms corresponding to the factor e^{-ikx} are obtained in the same way. Therefore, a special solution of Eq.(2.3) including secular terms are given by

$$u_1 = (\mu_1 t + \mu_2 x^2)e^{ikx} + (\tilde{\mu}_1 t + \tilde{\mu}_2 x^2)e^{-ikx} - \frac{A^3}{64}e^{3ikx} - \frac{B^3}{64}e^{-3ikx},$$

where $\tilde{\mu}_1$ and $\tilde{\mu}_2$ satisfy the same relation as (2.5). Thus we obtain

$$\begin{aligned} u &= Ae^{ikx} + Be^{-ikx} \\ &+ \varepsilon\Big((\mu_1 t + \mu_2 x^2)e^{ikx} + (\tilde{\mu}_1 t + \tilde{\mu}_2 x^2)e^{-ikx} + (\text{nonsecular})\Big) + O(\varepsilon^2). \end{aligned}$$

In what follows, we omit to write down nonsecular terms and $O(\varepsilon^2)$-terms which will not be used later. To remove the secular terms, we introduce dummy parameters τ and X, and rewrite the above u as

$$\begin{aligned} u &= Ae^{ikx} + Be^{-ikx} + (\varepsilon\mu_1\tau + \varepsilon\mu_2 X^2)e^{ikx} + (\varepsilon\tilde{\mu}_1\tau + \varepsilon\tilde{\mu}_2 X^2)e^{-ikx} \\ &+ \varepsilon\Big(\mu_1(t-\tau) + \mu_2(x^2 - X^2)\Big)e^{ikx} + \varepsilon\Big(\tilde{\mu}_1(t-\tau) + \tilde{\mu}_2(x^2 - X^2)\Big)e^{-ikx}. \end{aligned}$$

Now terms $\varepsilon\mu_1\tau + \varepsilon\mu_2 X^2$ and $\varepsilon\tilde{\mu}_1\tau + \varepsilon\tilde{\mu}_2 X^2$ are *renormalized* into the constants A and B, respectively. Thus we rewrite u as

$$\begin{aligned} u &= A(\tau, X)e^{ikx} + B(\tau, X)e^{-ikx} \\ &+ \varepsilon\Big(\mu_1(t-\tau) + \mu_2(x^2 - X^2)\Big)e^{ikx} + \varepsilon\Big(\tilde{\mu}_1(t-\tau) + \tilde{\mu}_2(x^2 - X^2)\Big)e^{-ikx}. \end{aligned}$$

Putting $\tau = t$ and $X = x$ provides

$$u(t, x) = A(t, x)e^{ikx} + B(t, x)e^{-ikx},$$

which seems to give an approximate solution if $A(t, x)$ and $B(t, x)$ are appropriately defined. Since u is independent of dummy parameters τ and X, we require that the equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial X^2} = 0$$

holds, which is called the *RG equation*. This yields

$$\begin{cases} \dfrac{\partial u}{\partial \tau}\Big|_{\tau=t, X=x} = \left(\dfrac{\partial A}{\partial t} - \varepsilon\mu_1\right)e^{ikx} + \left(\dfrac{\partial B}{\partial t} - \varepsilon\tilde{\mu}_1\right)e^{-ikx} = 0, \\ \dfrac{\partial^2 u}{\partial X^2}\Big|_{\tau=t, X=x} = \left(\dfrac{\partial^2 A}{\partial x^2} - 2\varepsilon\mu_2\right)e^{ikx} + \left(\dfrac{\partial^2 B}{\partial x^2} - 2\varepsilon\tilde{\mu}_2\right)e^{-ikx} = 0. \end{cases} \tag{2.6}$$

Since μ_1 and μ_2 satisfy (2.5), we obtain

$$\frac{\partial A}{\partial t} = \varepsilon\mu_1 = 8k^2\varepsilon\mu_2 + \varepsilon(A - 3A^2B) = 4k^2\frac{\partial^2 A}{\partial x^2} + \varepsilon(A - 3A^2B). \tag{2.7}$$

Similarly, B satisfies $\partial B/\partial t = 4k^2\partial_x^2 B + \varepsilon(B - 3AB^2)$. If we suppose that $A = \overline{B}$ to obtain a real-valued solution, the Ginzburg-Landau equation (1.3) is obtained.

Next, let us derive the amplitude equation of the system (1.5). The dispersion relation of the unperturbed operator L (1.21) is

$$\begin{aligned}
&\det\begin{pmatrix} \lambda + d(\xi_1^2+\xi_2^2) - k & 1 \\ -1 & \lambda + \xi_1^2 + \xi_2^2 + 1 \end{pmatrix} \\
= \;& \lambda^2 + (d\xi^2 + \xi^2 + 1 - k)\lambda + (d\xi^2 - k)(\xi^2+1) + 1 = 0,
\end{aligned} \tag{2.8}$$

where we put $\xi^2 = \xi_1^2 + \xi_2^2$. Let $\lambda_\pm(\xi)$ be two roots of (2.8). Then, the spectrum of L is given by $\sigma(L) = \lambda_+(\mathbf{R}) \cup \lambda_-(\mathbf{R})$. Suppose that Eq.(1.5) undergoes the Turing instability at $\varepsilon = 0$, so that $\sigma(L) = \mathbf{R}_{\leq 0}$. It is easy to verify that this is true if and only if

$$0 < d < k < 1, \;\; (k+d)^2 = 4d.$$

In particular, one of $\lambda_\pm(\xi)$ satisfies $\lambda_\pm(c) = 0$, where $c^2 = (k-d)/2d$. For any (ξ_1, ξ_2) satisfying $\xi_1^2 + \xi_2^2 = c^2$,

$$\begin{pmatrix} 1 \\ (k+d)/2 \end{pmatrix} e^{i\xi_1 x + i\xi_2 y}$$

is an eigenfunction of L associated with $\lambda = 0$. Because of the boundary condition in (1.5), we choose e^{icy} and e^{-icy}. Thus we expand a solution of (1.5) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} 1 \\ (k+d)/2 \end{pmatrix} e^{icy} + B\begin{pmatrix} 1 \\ (k+d)/2 \end{pmatrix} e^{-icy} + \varepsilon\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + O(\varepsilon^2). \tag{2.9}$$

Put $A = \overline{B}$ for simplicity. Then, (u_1, v_1) satisfies the equation

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = L\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} Ae^{icy} - A^3 e^{3icy} - 3A|A|^2 e^{icy} + c.c. \\ 0 \end{pmatrix}, \tag{2.10}$$

where $c.c.$ denotes the complex conjugate. We find secular terms of the form

$$u_1 = (\mu_1 t + \mu_2 x^2 + \mu_3 x^4)e^{icy}, \quad v_1 = \frac{k+d}{2}(\tilde{\mu}_1 t + \tilde{\mu}_2 x^2 + \tilde{\mu}_3 x^4)e^{icy}. \tag{2.11}$$

Substituting them into (2.10), we obtain

$$\left\{\begin{aligned} \tilde{\mu}_1 &= \mu_1, \\ \tilde{\mu}_3 &= \mu_3, \\ \tilde{\mu}_2 &= \tfrac{1}{2}\mu_1, \end{aligned}\right. \qquad \left\{\begin{aligned} \mu_1 &= 2d\mu_2 + A - 3A|A|^2, \\ 0 &= 12d\mu_3 + k\mu_2 - \frac{k+d}{2}\tilde{\mu}_2 - dc^2\mu_2. \end{aligned}\right. \tag{2.12}$$

Then, a formal solution is given as

$$u = Ae^{icy} + \varepsilon(\mu_1 t + \mu_2 x^2 + \mu_3 x^4)e^{icy} + c.c. + (\text{nonsecular}) + O(\varepsilon^2).$$

Introducing dummy parameters τ, X and renormalizing, we rewrite this equation as

$$u = A(\tau, X)e^{icy} + \varepsilon\left(\mu_1(t-\tau) + \mu_2(x^2 - X^2) + \mu_3(x^4 - X^4)\right)e^{icy} + c.c. + (\text{nonsecular}) + O(\varepsilon^2).$$

Since u is independent of τ and X, we require

$$\begin{cases} \dfrac{\partial u}{\partial \tau}\Big|_{\tau=t, X=x} = \left(\dfrac{\partial A}{\partial t} - \varepsilon\mu_1\right) e^{icy} + c.c. = 0, \\ \dfrac{\partial^4 u}{\partial X^4}\Big|_{\tau=t, X=x} = \left(\dfrac{\partial^4 A}{\partial X^4} - 24\varepsilon\mu_3\right) e^{icy} + c.c. = 0. \end{cases} \tag{2.13}$$

Finally, Eqs.(2.12) and (2.13) provide the amplitude equation (1.23) by eliminating μ_1, μ_2 and μ_3.

3 Reduction of a linear semigroup

For Eq.(1.1), reductions of the linear unperturbed part $\mathcal{P}u$ and the perturbation term $f(u)$ can be done independently. In this section, we give a reduction of the linear part.

3.1 One dimensional case

We start with the simple case $u \in \mathbf{C}$ and $x \in \mathbf{R}^d$. Put $x = (x_1, \cdots, x_d)$ and $\alpha = (\alpha_1, \cdots, \alpha_d)$, where α denotes a multi-index as usual: $x^\alpha = (x_1^{\alpha_1}, \cdots, x_d^{\alpha_d})$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Let $P(x) = \sum_{|\alpha|=0}^{q} a_\alpha x^\alpha$ be a polynomial of degree q and $\mathcal{P} := P(\partial_1, \cdots, \partial_d)$ a differential operator on $\mathbf{R}^d$, where ∂_j denotes the derivative with respect to x_j. We make the following assumptions.

(B1) $\mathrm{Re}[P(i\xi)] \leq 0$ for any $\xi \in \mathbf{R}^d$.
(B2) There exist $\omega \in \mathbf{R}$, $k \in \mathbf{R}^d$ and an integer M such that

$$\begin{aligned} & P(ik) = i\omega, \\ & \frac{\partial^\alpha P}{\partial x^\alpha}(ik) = 0, \ \text{for any } \alpha \text{ such that } |\alpha| = 1, \cdots, M-1, \\ & \frac{\partial^\alpha P}{\partial x^\alpha}(ik) \neq 0, \ \text{for some } \alpha \text{ such that } |\alpha| = M. \end{aligned}$$

(B3) Define $Q(x)$ and $\mathcal{Q}$ by

$$Q(x) = \sum_{|\alpha|=M} \frac{1}{(\alpha_1!)\cdots(\alpha_d!)} \frac{\partial^\alpha P}{\partial x^\alpha}(ik) x^\alpha, \quad \mathcal{Q} = Q(\partial_1, \cdots, \partial_d). \tag{3.1}$$

Then, both of $\mathcal{P}$ and $\mathcal{Q}$ are elliptic in the sense that there exist $c_1, c_2 > 0$ such that $\mathrm{Re}[P(i\xi)] < -c_2|\xi|^2$ and $\mathrm{Re}[Q(i\xi)] < -c_2|\xi|^2$ hold for $|\xi| \geq c_1$.

Put $B^r = BC^r(\mathbf{R}^d; \mathbf{C})$, a Banach space of complex-valued bounded uniformly continuous functions on $\mathbf{R}^d$ up to the r-th derivative. In the next propositions, $\|\cdot\| = \|\cdot\|_r$ denotes the standard supremum norm on B^r. Consider two initial value problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u, \quad u(0, x) = v_0(x) e^{ikx}, & \text{(3.2a)} \\ \dfrac{\partial A}{\partial t} = \mathcal{Q}A. \quad A(0, x) = v_0(x), & \text{(3.2b)} \end{cases}$$

where $kx = k_1x_1 + \cdots + k_dx_d$, and a similar notation will be used in the sequel. Because of (B3), $\mathcal{P}$ and $\mathcal{Q}$ generate C^0-semigroups $e^{\mathcal{P}t}$ and $e^{\mathcal{Q}t}$ on B^r, respectively. Thus solutions of the above problems are written as $e^{\mathcal{P}t}(e^{ikx}v_0)$ and $e^{\mathcal{Q}t}v_0$, respectively.

Proposition 3.1. Suppose (B1) to (B3) and $r \geq 0$. There exists a constant $C_1 > 0$ such that the inequality

$$||e^{\mathcal{P}t}(e^{ikx}v_0) - e^{i\omega t+ikx}e^{\mathcal{Q}t}v_0||_r \leq C_1 t^{-1/M}||v_0||_r \tag{3.3}$$

holds for any $t > 0$ and $v_0 \in BC^r(\mathbf{R}^d; \mathbf{C})$.

For the main theorems in this paper, we need the following perturbative problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u, \quad u(0, x) = v_0(\eta x)e^{ikx}, & (3.4\text{a}) \\ \dfrac{\partial A}{\partial t} = \mathcal{Q}A, \quad A(0, x) = v_0(\eta x), & (3.4\text{b}) \end{cases}$$

where $\eta = \varepsilon^{1/M}$ and $\varepsilon > 0$ is a small parameter.

Proposition 3.2. Suppose (B1) to (B3) and $r \geq 1$. For any $\varepsilon > 0$ and $t_0 > 0$, there exists a positive number $C_1 = C_1(t_0)$ such that the inequality

$$||e^{\mathcal{P}t}(e^{ikx}\hat{v}_0) - e^{i\omega t+ikx}e^{\mathcal{Q}t}\hat{v}_0||_r \leq \eta C_1||v_0||_r \tag{3.5}$$

holds for $t \geq t_0$ and $v_0 \in BC^r(\mathbf{R}^d; \mathbf{C})$, where $\hat{v}_0(\cdot) := v_0(\eta\,\cdot)$.

Proof of Prop.3.1. By putting $u = e^{i\omega t}w$, Eq.(3.2a) is rewritten as $\partial w/\partial t = (\mathcal{P} - i\omega)w$. Then, the operator $\mathcal{P} - i\omega$ satisfies (B1) to (B3) with $\omega = 0$. Hence, it is sufficient to prove the proposition for $\omega = 0$.

Two solutions are given by

$$A(t, x) = \frac{1}{(2\pi)^d}\int v_0(y + x)\int e^{-iy\xi}e^{Q(i\xi)t}d\xi dy$$

and

$$\begin{aligned} u(t, x) &= \frac{1}{(2\pi)^d}\int v_0(y + x)e^{ik(y+x)}\int e^{-iy\xi}e^{P(i\xi)t}d\xi dy \\ &= \frac{e^{ikx}}{(2\pi)^d}\int v_0(y + x)\int e^{-iy\xi}e^{P(i\xi+ik)t}d\xi dy, \end{aligned} \tag{3.6}$$

respectively. Thus we obtain

$$u(t, x) - e^{ikx}A(t, x) = \frac{e^{ikx}}{(2\pi)^d}\int v_0(y + x)\int e^{-iy\xi}e^{Q(i\xi)t}\left(e^{P(i\xi+ik)t-Q(i\xi)t} - 1\right)d\xi dy.$$

Put $\tau = t^{-1/M}$. Changing variables $\xi \mapsto \tau\xi$, $y \mapsto y/\tau$ yields

$$u(t, x) - e^{ikx}A(t, x) = \frac{e^{ikx}}{(2\pi)^d}\int v_0(y/\tau + x)\int e^{-iy\xi}e^{Q(i\xi)}\left(e^{P(i\tau\xi+ik)/\tau^M-Q(i\xi)} - 1\right)d\xi dy.$$

Due to the assumption (B2), we have

$$\begin{aligned} g(\xi,\tau) &:= P(i\tau\xi+ik)/\tau^M - Q(i\xi) \\ &= \frac{1}{\tau^M}\sum_{|\alpha|=0}^{q}\frac{1}{(\alpha_1!)\cdots(\alpha_d!)}\frac{\partial^\alpha P}{\partial x^\alpha}(ik)i^{|\alpha|}\tau^{|\alpha|}\xi^\alpha - \sum_{|\alpha|=M}\frac{1}{(\alpha_1!)\cdots(\alpha_d!)}\frac{\partial^\alpha P}{\partial x^\alpha}(ik)i^M\xi^\alpha \\ &= \sum_{|\alpha|=M+1}^{q}\frac{1}{(\alpha_1!)\cdots(\alpha_d!)}\frac{\partial^\alpha P}{\partial x^\alpha}(ik)i^{|\alpha|}\xi^\alpha\cdot\tau^{|\alpha|-M}. \end{aligned}$$

Note that $g \sim O(\tau)$ as $\tau \to 0$. In particular, there exists $0<\theta<1$ such that

$$e^{g(\xi,\tau)}-1 = \tau\frac{\partial g}{\partial\tau}(\xi,\theta\tau)e^{g(\xi,\theta\tau)}.$$

This provides

$$\begin{cases} u(t,x)-e^{ikx}A(t,x) = \tau\dfrac{e^{ikx}}{(2\pi)^d}\displaystyle\int v_0(y/\tau+x)G(y,\tau)dy, \\ G(y,\tau) := \displaystyle\int e^{-iy\xi}e^{Q(i\xi)}\frac{\partial g}{\partial\tau}(\xi,\theta\tau)e^{g(\xi,\theta\tau)}d\xi. \end{cases} \tag{3.7}$$

Because of (B3), $G(y,\tau)$ exists for each $\tau \geq 0$ and $y \in \mathbf{R}$. Since g is polynomial in τ, there exist τ_0 and $D_1 = D_1(\tau_0)$ such that $|G(y,\tau)| \leq D_1$ holds for $0 \leq \tau \leq \tau_0$ and $y \in [-1,1]^d$. Next, since the integrand in the definition of $G(y,\tau)$ is smooth in ξ, $G(y,\tau)$ is rapidly decreasing in y due to the property of the Fourier transform. Indeed, by using integration by parts, it is easy to verify that there exists $D_2 = D_2(\tau_0)$ such that $|G(y,\tau)| \leq D_2(y_1\cdots y_d)^{-2}$ holds for $0 \leq \tau \leq \tau_0$ and $y \notin [-1,1]^d$. This provides

$$\begin{aligned} |u(t,x)-e^{ikx}A(t,x)| &\leq \frac{\tau}{(2\pi)^d}\int |v_0(y/\tau+x)|\cdot|G(y,\tau)|dy \\ &\leq \frac{\tau}{(2\pi)^d}D_1\|v_0\| + \frac{\tau}{(2\pi)^d}\int_{y\notin[-1,1]^d}\frac{D_2}{y_1^2\cdots y_d^2}dy\cdot\|v_0\|. \end{aligned}$$

This proves that

$$\sup_{x\in\mathbf{R}^d}|u(t,x)-e^{ikx}A(t,x)| \leq \tau D_3\|v_0\| \tag{3.8}$$

for some $D_3 > 0$ when $0 \leq \tau \leq \tau_0$. To estimate the derivatives, note that Eq.(3.6) is rewritten as

$$e^{-ikx}u(t,x) = \frac{1}{(2\pi)^d}\int v_0(y)\int e^{-i(y-x)\xi}e^{P(i\xi+ik)t}d\xi dy,$$

and similarly for $A(t,x)$. Hence, the derivative is given as

$$\begin{cases} \dfrac{\partial^\alpha}{\partial x^\alpha}\left(e^{-ikx}u(t,x)-A(t,x)\right) = \tau\dfrac{1}{(2\pi)^d}\displaystyle\int v_0(y/\tau+x)G_\alpha(y,\tau)dy, \\ G_\alpha(y,\tau) := \displaystyle\int (i\tau\xi)^\alpha e^{-iy\xi}e^{Q(i\xi)}\frac{\partial g}{\partial\tau}(\xi,\theta\tau)e^{g(\xi,\theta\tau)}d\xi. \end{cases}$$

By the same way as above, we can show that this derivative is of $O(\tau)$ uniformly in x. Hence, the inequality

$$||u(t,x) - e^{ikx}A(t,x)|| \leq \tau D_3||v_0|| = t^{-1/M}D_3||v_0|| \tag{3.9}$$

holds with respect to the norm of B^r for some $D_3 > 0$ and any $t \geq \tau_0^{-M}$. On the other hand, since $\mathcal{P}$ and $\mathcal{Q}$ generate C^0-semigroups on B^r, there exists $D_4 > 0$ such that $||u(t,x) - e^{ikx}A(t,x)|| \leq D_4||v_0||$ for $0 \leq t \leq \tau_0^{-M}$. This and Eq.(3.9) prove Prop.3.1 (for $\omega = 0$). ■

Proof of Prop.3.2. In this case, solutions satisfy

$$u(t,x) - e^{ikx}A(t,x) = \tau\frac{e^{ikx}}{(2\pi)^d}\int v_0(\eta y/\tau + \eta x)G(y,\tau)dy,$$

where $\tau = t^{-1/M}$ and G is defined by (3.7) as before. Since $v_0 \in B^r$ ($r \geq 1$), there exists $0 < \theta_1 < 1$ such that it is expanded as

$$\begin{aligned} u(t,x) - e^{ikx}A(t,x) &= \tau\frac{e^{ikx}}{(2\pi)^d}\int\left(v_0(\eta x) + \sum_{j=1}^{d}\frac{\partial v_0}{\partial x_j}(\eta x + \theta_1\eta y/\tau)\frac{\eta}{\tau}y_j\right)G(y,\tau)dy \\ &= \eta\frac{e^{ikx}}{(2\pi)^d}\int\sum_{j=1}^{d}\frac{\partial v_0}{\partial x_j}(\eta x + \theta_1\eta y/\tau)y_jG(y,\tau)dy, \end{aligned}$$

where we used the fact $\int G(y,\tau)dy = 0$. The rest of the proof is the same as that of Prop.3.1. ■

If the polynomial $P(x)$ has no symmetries, the assumptions (B2),(B3) seem to be strong; for example, if $d = 2$ and

$$\frac{\partial^2 P}{\partial x_1^2}(ik) \neq 0, \quad \frac{\partial^2 P}{\partial x_1 x_2}(ik) = \frac{\partial^2 P}{\partial x_2^2}(ik) = 0,$$

then Q is not elliptic. To relax the assumptions, fix an integer D such that $1 \leq D \leq d$. We denote $x \in \mathbf{R}^d$ as $x = (\hat{x}_1, \hat{x}_2)$ with $\hat{x}_1 = (x_1, \cdots, x_D)$ and $\hat{x}_2 = (x_{D+1}, \cdots, x_d)$. Accordingly, a multi-index α is also denoted as $\alpha = (\beta, \gamma)$. Instead of (B2) and (B3), we suppose that

(B2)$_D$ there exist $\omega \in \mathbf{R}$, $k \in \mathbf{R}^d$ and an integer M such that

$$\begin{aligned} &P(ik) = i\omega, \\ &\frac{\partial^\beta P}{\partial \hat{x}_1^\beta}(ik) = 0, \text{ for any } \beta \text{ such that } |\beta| = 1, \cdots, M-1, \\ &\frac{\partial^\beta P}{\partial \hat{x}_1^\beta}(ik) \neq 0, \text{ for some } \beta \text{ such that } |\beta| = M. \end{aligned}$$

(B3)$_D$ Define $Q(x)$ and $\mathcal{Q}$ by

$$Q(x) = Q(\hat{x}_1, 0) = \sum_{|\beta|=M}\frac{1}{(\beta_1!)\cdots(\beta_D!)}\frac{\partial^\beta P}{\partial \hat{x}_1^\beta}(ik)\hat{x}_1^\beta, \quad \mathcal{Q} = Q(\partial_1, \cdots, \partial_D, 0, \cdots, 0). \tag{3.10}$$

Then, both of $\mathcal{P}$ and Q are elliptic in the sense that there exist $c_1, c_2 > 0$ such that $\mathrm{Re}[P(i\xi)] < -c_2|\xi|^2$ and $\mathrm{Re}[Q(i\hat{\xi}_1)] < -c_2|\hat{\xi}_1|^2$ hold for $|\xi|, |\hat{\xi}_1| \geq c_1$, where $\hat{\xi}_1 = (\xi_1, \cdots, \xi_D)$.

When $D = d$, these assumptions are reduced to (B2) and (B3) before. The assumption $(\mathrm{B3})_D$ implies that Q is an elliptic operator on $\mathbf{R}^D$ although it is not on $\mathbf{R}^d$. Consider two initial value problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u, \quad u(0,x) = v_0(\hat{x}_1)e^{ikx}, & (3.11\mathrm{a}) \\ \dfrac{\partial A}{\partial t} = QA. \quad A(0,x) = v_0(\hat{x}_1). & (3.11\mathrm{b}) \end{cases}$$

Note that v_0 depends only on $\hat{x}_1 = (x_1, \cdots, x_D)$. In particular, Eq.(3.11b) can be regarded as a parabolic equation on $BC^r(\mathbf{R}^D, \mathbf{C})$, while Eq.(3.11a) is a parabolic equation on $BC^r(\mathbf{R}^d, \mathbf{C})$. We also consider the perturbative problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u, \quad u(0,x) = v_0(\eta\hat{x}_1)e^{ikx}, & (3.12\mathrm{a}) \\ \dfrac{\partial A}{\partial t} = QA, \quad A(0,x) = v_0(\eta\hat{x}_1), & (3.12\mathrm{b}) \end{cases}$$

where $\eta = \varepsilon^{1/M}$ and $\varepsilon > 0$ is a small parameter.

Proposition 3.3. Under the assumptions (B1), $(\mathrm{B2})_D$ and $(\mathrm{B3})_D$, Prop.3.1 holds for Eqs.(3.11a),(3.11b), and Prop.3.2 holds for Eqs.(3.12a),(3.12b).

Proof. For Eq.(3.11a), $u(t,x)$ is given as

$$\begin{aligned} u(t,x) &= \frac{e^{ikx}}{(2\pi)^d}\int v_0(\hat{y}_1 + \hat{x}_1)\int e^{-iy\xi}e^{P(i\xi+ik)t}d\xi dy \\ &= \frac{e^{ikx}}{(2\pi)^d}\int v_0(\hat{y}_1 + \hat{x}_1)\int e^{-i\hat{y}_1\hat{\xi}_1 - i\hat{y}_2\hat{\xi}_2}e^{P(i\hat{\xi}_1+i\hat{k}_1, i\hat{\xi}_2+i\hat{k}_2)t}d\hat{\xi}_1 d\hat{\xi}_2 d\hat{y}_1 d\hat{y}_2. \end{aligned}$$

To calculate this, we need the next lemma.

Lemma 3.4. Let $\mathcal{S}$ be a space of C^∞ rapidly decreasing functions on $\mathbf{R}^{d-D}$ (Schwartz space). For any $f \in \mathcal{S}$, we have

$$\int\int e^{-i\hat{y}_2\hat{\xi}_2}f(\hat{\xi}_2)d\hat{\xi}_2 d\hat{y}_2 = (2\pi)^{d-D}f(0). \tag{3.13}$$

Proof. Let $\mathcal{S}'$ be a dual space of $\mathcal{S}$. For the pairing of $\mathcal{S}'$ and $\mathcal{S}$, we use a bracket $\langle\ ,\ \rangle$. Let $\mathcal{F}$ be the Fourier transform. Then,

$$\begin{aligned} \int\int e^{-i\hat{y}_2\hat{\xi}_2}f(\hat{\xi}_2)d\hat{\xi}_2 d\hat{y}_2 &= (2\pi)^{(d-D)/2}\int \mathcal{F}[f](\hat{y}_2)d\hat{y}_2 \\ &= (2\pi)^{(d-D)/2}\langle 1\,,\, \mathcal{F}[f]\rangle \\ &= (2\pi)^{(d-D)/2}\langle \mathcal{F}[1]\,,\, f\rangle \\ &= (2\pi)^{d-D}\langle \delta\,,\, f\rangle = (2\pi)^{d-D}f(0), \end{aligned}$$

where δ is the Dirac delta. ■

Due to this lemma, we obtain

$$u(t,x) = \frac{e^{ikx}}{(2\pi)^D} \int v_0(\hat{y}_1 + \hat{x}_1) \int e^{-i\hat{y}_1\hat{\xi}_1} e^{P(i\hat{\xi}_1 + i\hat{k}_1, i\hat{k}_2)t} d\hat{\xi}_1 d\hat{y}_1.$$

Since Eq.(3.11b) is an equation on $BC^r(\mathbf{R}^D, \mathbf{C})$, $A(t,x)$ is given as

$$A(t,x) = \frac{1}{(2\pi)^D} \int v_0(\hat{y}_1 + \hat{x}_1) \int e^{-i\hat{y}_1\hat{\xi}_1} e^{Q(i\hat{\xi}_1, 0)t} d\hat{\xi}_1 d\hat{y}_1.$$

The rest of the proof is the same as those of Prop.3.1 and 3.2. ■

3.2 Higher dimensional case

Suppose $u = (u_1, \cdots, u_m) \in \mathbf{C}^m$ and $x = (x_1, \cdots, x_d) \in \mathbf{R}^d$. For fixed $1 \leq D \leq d$, we use the same notation $x = (\hat{x}_1, \hat{x}_2)$ as in Sec.3.1. Let $\{P_{ij}(x)\}_{i,j=1}^m$ be the set of polynomials of x. Define the matrix $P(x)$ by

$$P(x) = P(x_1, \cdots, x_d) = \begin{pmatrix} P_{11}(x) & \cdots & P_{1m}(x) \\ \vdots & \ddots & \vdots \\ P_{m1}(x) & \cdots & P_{mm}(x) \end{pmatrix}. \tag{3.14}$$

The differential operator $\mathcal{P}$ is defined to be $\mathcal{P} = P(\partial_1, \cdots, \partial_d)$. The algebraic equation

$$\det(\lambda - P(i\xi)) = \det \begin{pmatrix} \lambda - P_{11}(i\xi) & \cdots & -P_{1m}(i\xi) \\ \vdots & \ddots & \vdots \\ -P_{m1}(i\xi) & \cdots & \lambda - P_{mm}(i\xi) \end{pmatrix} = 0 \tag{3.15}$$

is called the dispersion relation. Let $\lambda_1(\xi), \cdots, \lambda_m(\xi)$ be roots of this equation. Then, $\lambda_1(\mathbf{R}) \cup \cdots \cup \lambda_m(\mathbf{R})$ gives the spectrum of $\mathcal{P}$. We suppose for simplicity that only $\lambda_1(\xi)$ contributes to the center subspace of $\mathcal{P}$ (see (C1) below). Extending to more general situations is not difficult (see Remark 3.7 below).

(C0) The matrix $P(i\xi)$ is diagonalizable for any $\xi \in \mathbf{R}^d$.
(C1) $\mathrm{Re}[\lambda_1(\xi)] \leq 0$ and $\mathrm{Re}[\lambda_j(\xi)] < 0$ for any $\xi \in \mathbf{R}^d$ and $j = 2, \cdots, m$.
(C2) There exist $\omega \in \mathbf{R}$, $k \in \mathbf{R}^d$ and an integer M such that

$$\begin{aligned} &\lambda_1(k) = i\omega, \\ &\frac{\partial^\beta \lambda_1}{\partial \hat{x}_1^\beta}(k) = 0, \ \text{ for any } \beta \text{ such that } |\beta| = 1, \cdots, M-1, \\ &\frac{\partial^\beta \lambda_1}{\partial \hat{x}_1^\beta}(k) \neq 0, \ \text{ for some } \beta \text{ such that } |\beta| = M. \end{aligned}$$

(C3) Define $Q(x)$ and $\mathcal{Q}$ by

$$Q(x) = Q(\hat{x}_1, 0) = \sum_{|\beta|=M} \frac{1}{(\beta_1!)\cdots(\beta_D!)} \frac{\partial^\beta \lambda_1}{\partial \hat{x}_1^\beta}(k)(\hat{x}_1/i)^\beta, \quad \mathcal{Q} = Q(\partial_1, \cdots, \partial_D, 0, \cdots, 0). \tag{3.16}$$

Then, both of $\mathcal{P}$ and $\mathcal{Q}$ are elliptic in the sense that there exist $c_1, c_2 > 0$ such that $\mathrm{Re}[\lambda_j(\xi)] < -c_2|\xi|^2$ $(j = 1, \cdots, m)$ and $\mathrm{Re}[Q(i\hat{\xi}_1)] < -c_2|\hat{\xi}_1|^2$ hold for $|\xi|, |\hat{\xi}_1| \geq c_1$, where $\hat{\xi}_1 = (\xi_1, \cdots, \xi_D)$.

Put $B^r = BC^r(\mathbf{R}^r; \mathbf{C})$ and let $(B^r)^m$ be the product space. The norm on $(B^r)^m$ is defined by $||u|| = \max_{1\leq j\leq m} ||u_j||$ for $u = (u_1, \cdots, u_m)$. Note that $\mathcal{P}$ is an operator densely defined on $(B^r)^m$ while $\mathcal{Q}$ is an operator densely defined on B^r. When $m = 1$, $\lambda_1(\xi) = P(i\xi)$, so that the above assumptions and Q are reduced to those in Sec.3.1.

Example 3.5. Suppose $m = d = 2$ and consider the operator L defined by (1.21) with the condition $0 < d < k < 1$, $(k + d)^2 = 4d$. The dispersion relation is given by (2.8), whose roots are denoted as $\lambda_2(\xi) < \lambda_1(\xi)$. It is easy to verify that $\lambda_1(\xi) = 0$ if and only if $\xi = (\xi_1, \xi_2)$ satisfies $\xi_1^2 + \xi_2^2 = c^2 := (k - d)/2d$. Thus there are infinitely many points ξ satisfying $\lambda_1(\xi) = 0$. We choose $(\xi_1, \xi_2) = (0, c)$. Then, we can show that

$$\lambda_1(0, c) = \frac{\partial \lambda_1}{\partial \xi_1}(0, c) = \frac{\partial^2 \lambda_1}{\partial \xi_1^2}(0, c) = \frac{\partial^3 \lambda_1}{\partial \xi_1^3}(0, c) = 0, \quad \frac{\partial^4 \lambda_1}{\partial \xi_1^4}(0, c) \neq 0,$$

while

$$\frac{\partial \lambda_1}{\partial \xi_2}(0, c) = 0, \quad \frac{\partial^2 \lambda_1}{\partial \xi_2^2}(0, c) \neq 0.$$

Hence, (C2) is satisfied with $k = (0, c)$, $\omega = 0$, $D = 1$ (i.e. $\hat{x}_1 = x$ and $\hat{x}_2 = y$), and $M = 4$. In this case,

$$\mathcal{Q} = \frac{1}{4!} \frac{\partial^4 \lambda_1}{\partial \xi_1^4}(0, c) \frac{\partial^4}{\partial x^4} = -\frac{2d^2}{(k + d)(1 - d)} \frac{\partial^4}{\partial x^4}, \tag{3.17}$$

see Eq.(1.23).

Let $\boldsymbol{w} = (w_1, \cdots, w_m)$ be an eigenvector of $P(ik)$ associated with $\lambda_1(k) = i\omega$. Note that $e^{ikx}\boldsymbol{w}$ is an eigenfunction of $\mathcal{P}$ included in the center subspace. Consider two systems of PDEs:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u, \quad u(0, x) = v_0(\hat{x}_1)e^{ikx}\boldsymbol{w}, & (3.18\text{a}) \\[2ex] \dfrac{\partial A}{\partial t} = \mathcal{Q}A. \quad A(0, x) = v_0(\hat{x}_1). & (3.18\text{b}) \end{cases}$$

Note that v_0 depends only on $\hat{x}_1 = (x_1, \cdots, x_D)$. The former is a system of m-equations on $\mathbf{R}^d$, while the latter is a single equation on $\mathbf{R}^D$. We also consider the perturbative problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u, \quad u(0, x) = v_0(\eta\hat{x}_1)e^{ikx}\boldsymbol{w}, & (3.19\text{a}) \\[2ex] \dfrac{\partial A}{\partial t} = \mathcal{Q}A, \quad A(0, x) = v_0(\eta\hat{x}_1), & (3.19\text{b}) \end{cases}$$

where $\eta = \varepsilon^{1/M}$ and $\varepsilon > 0$ is a small parameter. Solutions of them satisfy the next proposition.

Proposition 3.6. Suppose $r \geq 0$. Under the assumptions (C0) to (C3), there exists a constant $C_1 > 0$ such that

$$\|e^{\mathcal{P}t}(e^{ikx}v_0 \cdot \boldsymbol{w}) - e^{i\omega t+ikx}(e^{Qt}v_0)\cdot \boldsymbol{w}\| \leq C_1 t^{-1/M}\|v_0\| \tag{3.20}$$

holds for any $t > 0$ and $v_0 \in BC^r(\mathbf{R}^r;\mathbf{C})$. Next, suppose $r \geq 1$. For any $\varepsilon > 0$ and $t_0 > 0$, there exists a positive number $C_1 = C_1(t_0)$ such that the inequality

$$\|e^{\mathcal{P}t}(e^{ikx}\hat{v}_0 \cdot \boldsymbol{w}) - e^{i\omega t+ikx}(e^{Qt}\hat{v}_0)\cdot \boldsymbol{w}\| \leq \eta C_1\|v_0\| \tag{3.21}$$

holds for $t \geq t_0$ and $v_0 \in BC^r(\mathbf{R}^r;\mathbf{C})$, where $\hat{v}_0(\cdot) := v_0(\eta\,\cdot)$.

Proof. We suppose $D = d$ for simplicity; that is, $\hat{x}_1 = x$ and $\beta = \alpha$. The case $D < d$ is easily reduced to the case $D = d$ as in the proof of Prop.3.3. We also suppose $\omega = 0$ without loss of generality.

Like as the proof of Prop.3.1, a solution of (3.18a) is written as

$$u(t,x) = \frac{e^{ikx}}{(2\pi)^d}\int v_0(y+x)\int e^{-iy\xi}e^{P(i\xi+ik)t}\boldsymbol{w}d\xi dy.$$

Note that $e^{P(i\xi+ik)t}$ is an exponential of a matrix. Let $S(\xi)$ be a matrix such that

$$S(\xi)^{-1}P(i\xi)S(\xi) := \Lambda(\xi) = \begin{pmatrix} \lambda_1(\xi) & & \\ & \ddots & \\ & & \lambda_m(\xi)\end{pmatrix}. \tag{3.22}$$

Because of the assumption (C0), we can assume that $S(\xi), S(\xi)^{-1}$ and $\lambda_j(\xi)$'s are smooth in ξ. Then,

$$u(t,x) = \frac{e^{ikx}}{(2\pi)^d}\int v_0(y+x)\int e^{-iy\xi}S(k+\xi)e^{\Lambda(k+\xi)t}S(k+\xi)^{-1}\boldsymbol{w}d\xi dy.$$

Put $\tau = t^{-1/M}$. Changing variables $\xi \mapsto \tau\xi$, $y \mapsto y/\tau$ yields

$$u(t,x) = \frac{e^{ikx}}{(2\pi)^d}\int v_0(y/\tau+x)\int e^{-iy\xi}S(k+\tau\xi)e^{\Lambda(k+\tau\xi)/\tau^M}S(k+\tau\xi)^{-1}\boldsymbol{w}d\xi dy.$$

Expanding $S(k+\tau\xi)$ and $S(k+\tau\xi)^{-1}$, it turns out that there is a function G_1 such that

$$u(t,x) = \frac{e^{ikx}}{(2\pi)^d}\int v_0(y/\tau+x)\int e^{-iy\xi}S(k)e^{\Lambda(k+\tau\xi)/\tau^M}S(k)^{-1}\boldsymbol{w}d\xi dy + \tau\int v_0(y/\tau+x)G_1(y,\tau)dy.$$

By a similar estimate used in the proof of Prop.3.1, we can show that there exists $D_1 > 0$ such that the norm of the second term above has an upper bound $\tau D_1\|v_0\|$ for any $\tau > 0$. Since $\boldsymbol{w}$ is an eigenvector associated with $\lambda_1(\xi)$, we obtain

$$u(t,x) = \frac{e^{ikx}}{(2\pi)^d}\int v_0(y/\tau+x)\int e^{-iy\xi}e^{\lambda_1(k+\tau\xi)/\tau^M}\boldsymbol{w}d\xi dy + \tau\int v_0(y/\tau+x)G_1(y,\tau)dy.$$

Therefore, we have

$$\begin{aligned} & u(t,x) - e^{ikx}A(t,x)\boldsymbol{w} \\ = \; & \frac{e^{ikx}}{(2\pi)^d}\int v_0(y/\tau + x)\int e^{-iy\xi}e^{Q(i\xi)}\left(e^{g(\xi,\tau)}-1\right)\boldsymbol{w}d\xi dy + \tau\int v_0(y/\tau+x)G_1(y,\tau)dy, \end{aligned}$$

where $g(\xi,\tau) = \lambda_1(k+\tau\xi)/\tau^M - Q(i\xi)$. The rest of the proof is the same as those of Prop.3.1 and 3.2. ■

Remark 3.7. Let $\lambda_1(\xi),\cdots,\lambda_m(\xi)$ be eigenvalues of $P(i\xi)$ as before. Even if several eigenvalues lie on the imaginary axis and (C1) is violated, to modify Prop.3.6 is very easy; since equations are linear, the superposition principle is applicable. A typical problem is that $P(i\xi)$ is real-valued and eigenvalues occur in complex conjugate pairs. For example, suppose (C0), (C2), (C3) and the following (C1)' instead of (C1):

(C1)' $\lambda_2(\xi) = \overline{\lambda_1(\xi)}$. $\mathrm{Re}[\lambda_{1,2}(\xi)] \leq 0$ and $\mathrm{Re}[\lambda_j(\xi)] < 0$ for any $\xi \in \mathbf{R}^d$ and $j = 3,\cdots,m$.

Put $\overline{Q} := \overline{Q(\partial_1,\cdots,\partial_D,0,\cdots,0)}$. Let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ be eigenvectors of $P(ik)$ associated with $\lambda_1(k) = i\omega$ and $\lambda_2(k) = -i\omega$, respectively. In this case, instead of Eq.(3.20), the inequality

$$\|e^{\mathcal{P}t}(e^{ikx}v_1\cdot\boldsymbol{w}_1 + e^{ikx}v_2\cdot\boldsymbol{w}_2) - e^{i\omega t+ikx}(e^{Qt}v_1)\cdot\boldsymbol{w}_1 - e^{-i\omega t+ikx}(e^{\overline{Q}t}v_2)\cdot\boldsymbol{w}_2\| \leq C_1t^{-1/M}(\|v_1\|+\|v_2\|) \tag{3.23}$$

holds for any $t > 0$ and $v_1, v_2 \in BC^r(\mathbf{R}^r;\mathbf{C})$, and similarly for Eq.(3.21).

Example 3.8. Suppose $m = 2$ and $d = 1$. Define a linear operator

$$\mathcal{P}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} D\partial^2 & 1 \\ -1 & D\partial^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.24}$$

where $D > 0$ is a diffusion constant. This operator arises from Eq.(1.7). Eigenvalues of $P(i\xi)$ are $\lambda_1(\xi) = -D\xi^2 + i$ and $\lambda_2(\xi) = -D\xi^2 - i$. Hence, the assumptions (C0), (C1)', (C2) and (C3) are satisfied with

$$\lambda_1(0) = i,\; \lambda_2(0) = -i,\; M = 2,\; Q = \overline{Q} = D\frac{\partial^2}{\partial x^2},$$
$$\boldsymbol{w}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix},\quad \boldsymbol{w}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \overline{\boldsymbol{w}}_1.$$

Eq.(3.23) is given as

$$\|e^{\mathcal{P}t}(v_1\cdot\boldsymbol{w}_1 + v_2\cdot\boldsymbol{w}_2) - e^{i\omega t}(e^{Qt}v_1)\cdot\boldsymbol{w}_1 - e^{-i\omega t}(e^{Qt}v_2)\cdot\boldsymbol{w}_2\| \leq C_1t^{-1/M}(\|v_1\|+\|v_2\|). \tag{3.25}$$

In most applications, we take $v_2(x) = \overline{v_1(x)}$ to obtain a real-valued solution of $\dot{\boldsymbol{u}} = \mathcal{P}\boldsymbol{u}$. The above inequality implies that an approximate solution of $\dot{\boldsymbol{u}} = \mathcal{P}\boldsymbol{u}$ is constructed through the complex heat equation $\dot{A} = QA$.

4 Main theorems

In this section, a reduction of a perturbation term $f(u)$ is given. Combined with the reduction of linear semigroups, a reduction of Eq.(1.1) is performed.

4.1 One dimensional case

We start with the case $u \in \mathbf{C}$ and $x \in \mathbf{R}^d$. Put $x = (x_1, \cdots, x_d)$ and $\alpha = (\alpha_1, \cdots, \alpha_d)$, where α denotes a multi-index. For a fixed integer $1 \leq D \leq d$, we denote $x \in \mathbf{R}^d$ as $x = (\hat{x}_1, \hat{x}_2)$ with $\hat{x}_1 = (x_1, \cdots, x_D)$ and $\hat{x}_2 = (x_{D+1}, \cdots, x_d)$. Accordingly, a multi-index α is also denoted as $\alpha = (\beta, \gamma)$. Let $P(x) = \sum_{|\alpha|=0}^{q} a_\alpha x^\alpha$ be a polynomial and $\mathcal{P} := P(\partial_1, \cdots, \partial_d)$ a differential operator on $\mathbf{R}^d$, where ∂_j denotes the derivative with respect to x_j. For the main theorems, we make the following assumptions.

(D1) $\mathrm{Re}[P(i\xi)] \leq 0$ for any $\xi \in \mathbf{R}^d$.
(D2) There exist $\omega \in \mathbf{R}$, $k \in \mathbf{R}^d$ ($(\omega, k) \neq (0,0)$), a finite set of integers $J = \{j_1, \cdots, j_N\}$ and $\{M_1, \cdots, M_N\}$ such that

$$\begin{aligned}
&P(ij_nk) = ij_n\omega, \ \ (n = 1, \cdots, N),\\
&\frac{\partial^\beta P}{\partial \hat{x}_1^\beta}(ij_nk) = 0, \ \ \text{for any } \beta \text{ such that } |\beta| = 1, \cdots, M_n - 1, \ \ (n = 1, \cdots, N),\\
&\frac{\partial^\beta P}{\partial \hat{x}_1^\beta}(ij_nk) \neq 0, \ \ \text{for some } \beta_n \text{ such that } |\beta_n| = M_n, \ \ (n = 1, \cdots, N).
\end{aligned}$$

The set J consists of *all* integers satisfying $P(ijk) = ij\omega$.
(D3) For $n = 1, \cdots, N$, define $Q_n(x)$ and $\mathcal{Q}_n$ by

$$Q_n(x) = Q_n(\hat{x}_1, 0) = \sum_{|\beta|=M_n} \frac{1}{(\beta_1!)\cdots(\beta_D!)} \frac{\partial^\beta P}{\partial \hat{x}_1^\beta}(ij_nk)\hat{x}_1^\beta, \quad \mathcal{Q}_n = Q_n(\partial_1, \cdots, \partial_D, 0, \cdots, 0). \tag{4.1}$$

Then, both of $\mathcal{P}$ and $\mathcal{Q}_n$ are elliptic in the sense that there exist $c_1, c_2 > 0$ such that $\mathrm{Re}[P(i\xi)] < -c_2|\xi|^2$ and $\mathrm{Re}[Q_n(i\hat{\xi}_1, 0)] < -c_2|\hat{\xi}_1|^2$ hold for $|\xi|, |\hat{\xi}_1| \geq c_1$ and $n = 1, \cdots, N$, where $\hat{\xi}_1 = (\xi_1, \cdots, \xi_D)$.

In addition to (B2) before, a new assumption $(\omega, k) \neq (0,0)$ and the set J are introduced. In most examples, J consists of $J = \{+1, -1\}$ as Sec.1, see also an example below.

Put $B^r = BC^r(\mathbf{R}^d; \mathbf{C})$. For a given function $f : B^r \to B^r$, let us consider the Fourier series of the quantity $f(\sum_{n=1}^N A_n e^{ij_n\omega t + ij_nkx})$, where $A = (A_1, \cdots, A_N) \in \mathbf{C}^N$. Since $(\omega, k) \neq (0,0)$, the Fourier series is well-defined and it is easy to verify that the series is of the form

$$f(\sum_{n=1}^N A_n e^{ij_n\omega t + ij_nkx}) = \sum_{j=-\infty}^{\infty} C_j(A) e^{ij\omega t + ijkx}. \tag{4.2}$$

For example, when $k_1 \neq 0$, $C_j(A)$ is given by

$$C_j(A) = C_j(A_1, \cdots, A_N) := \frac{k_1}{2\pi}\int_0^{2\pi/k_1} f(\sum_{n=1}^N A_n e^{ij_nk_1x_1})e^{-ijk_1x_1}dx_1. \tag{4.3}$$

When $\omega \neq 0$, it is also written as

$$C_j(A) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f(\sum_{n=1}^N A_n e^{ij_n\omega t})e^{-ij\omega t}dt. \tag{4.4}$$

In particular, $C_{j_n}(A)$ is denoted by $R_n(A)$ if $j_n \in J$. For any $\theta \in \mathbf{R}$ and $j \in \mathbf{Z}$, $C_j(A)$ satisfies the equality

$$C_j(e^{ij_1\theta}A_1, \cdots, e^{ij_N\theta}A_N) = e^{ij\theta}C_j(A_1, \cdots, A_N). \tag{4.5}$$

Let $\varepsilon > 0$ be a small parameter. We will consider the two initial value problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u + \varepsilon f(u), \quad u(0,x) = \displaystyle\sum_{n=1}^{N} e^{ij_nkx}v_n(\eta\hat{x}_1), & \text{(4.6a)} \\[2ex] \dfrac{\partial A_n}{\partial t} = \mathcal{Q}_n A_n + \varepsilon R_n(A), \quad A_n(0,x) = v_n(\eta\hat{x}_1), \ (n = 1, \cdots, N), & \text{(4.6b)} \end{cases}$$

where $\eta = \varepsilon^{1/M}$ and $M := \min\{M_1, \cdots, M_N\}$. Note that the former is a single equation while the latter is a system of PDEs.

Example 4.1. Let us consider the Swift-Hohenberg equation (1.2). For this equation, $D = d = 1$ and $P(x) = -(x^2 + k^2)^2$. Since $P(i\xi) = 0$ if and only if $\xi = \pm k$, the set J consists of $j_1 = +1,\ j_2 = -1$. We have

$$\frac{\partial P}{\partial x}(\pm ik) = 0, \quad \frac{\partial^2 P}{\partial x^2}(\pm ik) = 8k^2.$$

Thus $M_1 = M_2 = 2$, and both of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are given by

$$\mathcal{Q}_{1,2} = \frac{1}{2}\frac{\partial^2 P}{\partial x^2}(\pm ik)\frac{\partial^2}{\partial x^2} = 4k^2\frac{\partial^2}{\partial x^2}.$$

Since $f(u) = u - u^3$, the expansion of $f(A_1e^{ikx} + A_2e^{-ikx})$ is

$$f(A_1e^{ikx} + A_2e^{-ikx}) = A_1e^{ikx} + A_2e^{-ikx} - (A_1^3e^{3ikx} + 3A_1^2A_2e^{ikx} + 3A_1A_2^2e^{-ikx} + A_2^3e^{-3ikx}).$$

This provides

$$R_1(A) = C_1(A) = A_1 - 3A_1^2A_2, \quad R_2(A) = C_{-1}(A) = A_2 - 3A_1A_2^2.$$

Therefore, the amplitude equation (4.6b) is given by

$$\frac{\partial A_1}{\partial t} = 4k^2\frac{\partial^2 A_1}{\partial x^2} + \varepsilon(A_1 - 3A_1^2A_2), \quad \frac{\partial A_2}{\partial t} = 4k^2\frac{\partial^2 A_2}{\partial x^2} + \varepsilon(A_2 - 3A_1A_2^2). \tag{4.7}$$

Usually, we assume $A_1 = \overline{A_2}$, which gives the Ginzburg-Landau equation (1.3).

Put $\hat{v}_n(x) = v_n(\eta x)$. The above equations are rewritten as integral equations of the form

$$\begin{cases} u = e^{\mathcal{P}t}(\displaystyle\sum_{n=1}^{N} e^{ij_nkx}\hat{v}_n) + \varepsilon\int_0^t e^{\mathcal{P}(t-s)}f(u(s))ds, & \text{(4.8a)} \\[2ex] A_n = e^{\mathcal{Q}_nt}\hat{v}_n + \varepsilon\displaystyle\int_0^t e^{\mathcal{Q}_n(t-s)}R_n(A(s))ds, \ (n = 1, \cdots, N), & \text{(4.8b)} \end{cases}$$

whose solutions are called mild solutions. When $f : B^r \to B^r$ is C^1, then $R_n : B^r \to B^r$ is also C^1, and due to the standard existence theorem (see Pazy[8]), there exists a positive number $T_0 > 0$ such that the above integral equations have mild solutions $u(t, \cdot)$, $A_n(t, \cdot) \in B^r$ for $0 \le t \le T_0/\varepsilon$. In particular, when the initial condition $\{v_n\}$ is included in the domain of $\mathcal{P}$ and Q_n, then a mild solution is a classical solution which is differentiable in $t > 0$. In this paper, we only consider mild solutions. The main theorems for a one-dimensional case are stated as follows:

Theorem 4.2. Suppose $f : BC^r(\mathbf{R}^d; \mathbf{C}) \to BC^r(\mathbf{R}^d; \mathbf{C})$ $(r \ge 1)$ is C^1 and $\varepsilon > 0$ is sufficiently small. For any $\{v_n\}_{n=1}^N \subset BC^r(\mathbf{R}^d; \mathbf{C})$, there exist positive numbers C, T_0 and t_0 such that mild solutions of the two initial value problems (4.5) satisfy

$$||u(t, x) - \sum_{n=1}^{N} A_n(t, x) e^{i j_n \omega t + i j_n k x}|| \le C\eta = C\varepsilon^{1/M}, \tag{4.9}$$

for $t_0 \le t \le T_0/\varepsilon$.

Next, let us show that the error estimate above holds for any $t > t_0$ under a suitable condition. For ordinary differential equations, it is proved in [3] that if the amplitude equation has a stable hyperbolic invariant manifold, then a given equation has a stable invariant manifold of the same type and approximate solutions are valid for any $t > 0$ near the manifold. For our situation, suppose that there is a constant vector $\phi \in \mathbf{R}^N$ such that $R(\phi) = 0$, $R = (R_1, \cdots, R_N)$. Then, ϕ is a steady state of the amplitude equation. Unfortunately, ϕ is not hyperbolic because of the symmetry (4.5); the Jacobi matrix of R at ϕ has a zero-eigenvalue in general. For example, although the amplitude equation (4.7) for the Swift-Hohenberg equation has a steady state $(A_1, A_2) = (1/\sqrt{3}, 1/\sqrt{3})$, the Jacobi matrix of R at $(1/\sqrt{3}, 1/\sqrt{3})$ has a zero-eigenvalue. However, if we restrict solutions to the invariant set $\{A_1 = A_2\}$, (4.7) is reduced to

$$\frac{\partial A}{\partial t} = 4k^2 \frac{\partial^2 A}{\partial x^2} + \varepsilon(A - 3A^3), \tag{4.10}$$

and the derivative of the function $A - 3A^3$ at $A = 1/\sqrt{3}$ is negative. This implies that $A = 1/\sqrt{3}$ is a hyperbolically stable steady state of (4.10), and we expect that the Swift-Hohenberg equation also has a corresponding stable solution. For more general situations, we make the following assumption.

(D4) For $|\beta| = M_n$ and $n = 1, \cdots, N$,

$$P(i\xi) = \overline{P(-i\xi)}, \ \ f(\overline{u}) = \overline{f(u)} \quad \text{and} \quad \frac{\partial^\beta P}{\partial \hat{x}_1^\beta}(i j_n k) = \frac{\partial^\beta P}{\partial \hat{x}_1^\beta}(-i j_n k).$$

The first two equalities imply that $P(x)$ and $f(u)$ are real-valued when $x, u \in \mathbf{R}$. Due to this assumption, $P(-i j_n k) = -i j_n \omega$ when $P(i j_n k) = i j_n \omega$. Hence, the set J of integers satisfying $P(ijk) = ij\omega$ is given by $J = \{j_1, \cdots, j_N\} \cup \{-j_1, \cdots, -j_N\}$. We denote $-j_n$ by j_{-n}. Then, M_n and Q_n are defined for $n = \pm 1, \cdots, \pm N$ as in (D2), (D3). It follows from (D4) that $M_n = M_{-n}$ and $Q_n = Q_{-n}$. For many examples, J consists of $J = \{\pm 1\}$ and this assumption is satisfied. In the present notation, the function $C_j(A)$ defined by (4.3) is given by

$$C_j(A_1,\cdots,A_N,A_{-1},\cdots,A_{-N}) = \frac{k_1}{2\pi}\int_0^{2\pi/k_1} f(\sum_{n=1}^N A_n e^{ij_nk_1x_1} + \sum_{n=1}^N A_{-n}e^{-ij_nk_1x_1})e^{-ijk_1x_1}dx_1. \quad (4.11)$$

In particular, $C_{j_n}(A)$ is denoted by $R_n(A)$ for $n = \pm 1,\cdots,\pm N$. Hence, the amplitude equation is given as a system of $2N$-equations of the form

$$\frac{\partial A_n}{\partial t} = Q_n A_n + \varepsilon R_n(A), \quad (n = \pm 1,\cdots,\pm N). \quad (4.12)$$

This system can be reduced as follows; It is easy to verify that C_j satisfies

$$C_j(A_1,\cdots,A_N,A_{-1},\cdots,A_{-N}) = C_{-j}(A_{-1},\cdots,A_{-N},A_1,\cdots,A_N). \quad (4.13)$$

Thus putting $A_n = A_{-n}$ yields $R_n(A) = R_{-n}(A)$. Since $Q_n = Q_{-n}$, putting $A_n = A_{-n}$ shows that Eq.(4.12) is reduced to the system of N-equations

$$\frac{\partial A_n}{\partial t} = Q_n A_n + \varepsilon R_n(A_1,\cdots,A_N,A_1,\cdots,A_N), \quad (n = 1,\cdots,N). \quad (4.14)$$

Define the function S_n to be

$$S_n(A) = S_n(A_1,\cdots,A_N) = R_n(A_1,\cdots,A_N,A_1,\cdots,A_N). \quad (4.15)$$

We consider the two initial value problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u + \varepsilon f(u), \quad u(0,x) = \displaystyle\sum_{n=1}^N \left(e^{ij_nkx} + e^{-ij_nkx}\right) v_n(\eta\hat{x}_1), & (4.16\text{a}) \\ \dfrac{\partial A_n}{\partial t} = Q_n A_n + \varepsilon S_n(A), \quad A_n(0,x) = v_n(\eta\hat{x}_1),\ (n = 1,\cdots,N), & (4.16\text{b}) \end{cases}$$

where $\eta = \varepsilon^{1/M}$ and $M := \min\{M_1,\cdots,M_N\}$. Due to Thm.4.2, solutions of them satisfy

$$||u(t,x) - \sum_{n=1}^N A_n(t,x)(e^{ij_n\omega t + ij_nkx} + e^{-ij_n\omega t - ij_nkx})|| \leq C\eta,$$

for $t_0 \leq t \leq T_0/\varepsilon$. Further, we can show the next theorem, in which $B^r = BC^r(\mathbf{R}^d;\mathbf{R})$ denotes the set of real-valued functions.

Theorem 4.3. Suppose (D1) to (D4) and $f : BC^r(\mathbf{R}^d;\mathbf{R}) \to BC^r(\mathbf{R}^d;\mathbf{R})$ $(r \geq 1)$ is C^2 such that the second derivatives are locally Lipschitz continuous. Suppose that there exists a constant vector $\phi = (\phi_1,\cdots,\phi_N) \in \mathbf{R}^N$ such that
(i) $S_n(\phi) = 0$ for $n = 1,\cdots,N$,
(ii) the Jacobi matrix of $(S_1,\cdots,S_N)$ at ϕ is diagonalizable and all eigenvalues of the

matrix have negative real parts.
If $\varepsilon > 0$ is sufficiently small, Eq.(4.16a) has a solution of the form

$$u_p(t,x) = \sum_{n=1}^{N} \Big(\phi_n + \eta\psi_n(t,x,\eta)\Big) \cdot (e^{ij_n\omega t + ij_n kx} + e^{-ij_n\omega t - ij_n kx}). \tag{4.17}$$

The functions ψ_n and u_p are bounded as $\eta \to 0$ and satisfy

$$\begin{cases} 2\pi/\omega\text{-periodic in } t \text{ (when } \omega \neq 0), \\ \text{constant in } t \text{ (when } \omega = 0), \end{cases} \qquad \begin{cases} 2\pi/k_j\text{-periodic in } x_j \text{ (when } k_j \neq 0), \\ \text{constant in } x_j \text{ (when } k_j = 0), \end{cases}$$

for $j = 1, \cdots, d$. This u_p is stable in the following sense: For any $n = 1, \cdots, N$, there is a neighborhood $U_n \subset BC^r(\mathbf{R}^d; \mathbf{R})$ of ϕ_n in $BC^r(\mathbf{R}^d; \mathbf{R})$ such that if $v_n \in U_n$, then a mild solution u of the initial value problem (4.16a) satisfies $\|u(t,\cdot) - u_p(t,\cdot)\| \to 0$ as $t \to \infty$.

The above conditions (i),(ii) show that $A(t,x) \equiv \phi$ is an asymptotically stable steady state of Eq.(4.16b). Thus this theorem implies that a stable steady state of Eq.(4.16b) induces a periodic solution of Eq.(4.16a). Due to the symmetry (4.5), Eq.(4.12) has a steady state $A_n = e^{ij_n\theta}\phi_n$ $(n = \pm 1, \cdots, \pm N)$ for any $\theta \in [0, 2\pi)$. Accordingly, we can prove that Eq.(4.16a) has a stable periodic solution

$$u_p(t,x) = \sum_{n=1}^{N} \Big(\phi_n + \eta\psi_n(t,x,\eta)\Big) \cdot (e^{ij_n\omega t + ij_n kx + ij_n\theta} + e^{-ij_n\omega t - ij_n kx - ij_n\theta}),$$

for any $\theta \in [0, 2\pi)$, the proof of which is reduced to (4.17) by the translation of t or x.

Proof of Thm.4.2. We prove the theorem for the case $D = d$ (thus $\hat{x}_1 = x$ and $\alpha = \beta$) for simplicity of notation. The general case $D < d$ can be proved in the same way. A proof is divided into four steps.

Step 1. notation. It is convenient to introduce some notation: We define a new coordinate (T, X) by

$$x = X/\eta,\ \ t = T/\varepsilon,\ \ \hat{u}(T,X) = u(t,x),\ \ \hat{A}_n(T,X) = A_n(t,x),$$
$$\hat{\mathcal{P}} = \frac{1}{\varepsilon}P(\eta\partial_X),\ \ \hat{\mathcal{Q}}_n = \frac{1}{\varepsilon}Q_n(\eta\partial_X).$$

Then, Eqs.(4.6a) and (4.6b) are rewritten as

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial T} = \hat{\mathcal{P}}\hat{u} + f(\hat{u}), \quad \hat{u}(0,X) = \displaystyle\sum_{n=1}^{N} e^{ij_n kX/\eta} v_n(X), & (4.18\text{a}) \\ \dfrac{\partial \hat{A}_n}{\partial T} = \hat{\mathcal{Q}}_n \hat{A}_n + R_n(\hat{A}), \quad \hat{A}_n(0,X) = v_n(X),\ (n = 1, \cdots, N). & (4.18\text{b}) \end{cases}$$

Integrating them yields

$$\begin{cases} \hat{u} = e^{\hat{\mathcal{P}}T}(\displaystyle\sum_{n=1}^{N} e^{ij_n kX/\eta} v_n) + \int_0^T e^{\hat{\mathcal{P}}(T-s)} f(\hat{u}(s))ds, & (4.19\text{a}) \\ \hat{A}_n = e^{\hat{\mathcal{Q}}_n T} v_n + \displaystyle\int_0^T e^{\hat{\mathcal{Q}}_n(T-s)} R_n(\hat{A}(s))ds,\ (n = 1, \cdots, N), & (4.19\text{b}) \end{cases}$$

which have mild solutions in B^r for $0 \leq T \leq T_0$. For the function space B^r written in the X-variable, we introduce the norm $\|\cdot\|_\eta$ by

$$\|\varphi\|_\eta = \max_{0\leq|\alpha|\leq r} \sup_{X\in\mathbf{R}^d}\{ \eta^{|\alpha|}\, |\partial^\alpha \varphi(X)|\ \}. \tag{4.20}$$

If we put $\hat{\varphi}(X) := \varphi(X/\eta) = \varphi(x)$ for a given $\varphi(x)$, it is easy to see that $\|\varphi\| = \|\hat{\varphi}\|_\eta$, where $\|\varphi\|$ represents the standard norm on B^r. In the present notation, Prop.3.2 is restated as follows: there exist $t_0, C_1 > 0$ such that the inequality

$$\|e^{\hat{\mathcal{P}}T}(e^{ij_nkX/\eta}v_n) - e^{ij_n\omega T/\varepsilon + ij_nkX/\eta}e^{\hat{Q}_nT}v_n\|_\eta \leq \eta C_1\|v_n\|_\eta \tag{4.21}$$

holds for $T \geq \varepsilon t_0$ and for each $n = 1, \cdots, N$. Let us estimate $\hat{u} - \sum_{n=1}^N \hat{A}_n e^{ij_n\omega t + ij_nkx}$ by using the norm $\|\cdot\|_\eta$.

Step 2. Gronwall inequality. It follows from Eq.(4.18) that

$$\begin{aligned}
\hat{u} - \sum_{n=1}^N \hat{A}_n e^{ij_n\omega T/\varepsilon + ij_nkX/\eta} &= \sum_{n=1}^N e^{\hat{\mathcal{P}}T}(e^{ij_nkX/\eta}v_n) - \sum_{n=1}^N e^{ij_n\omega T/\varepsilon+ij_nkX/\eta}e^{\hat{Q}_nT}v_n \\
&\quad + \int_0^T e^{\hat{\mathcal{P}}(T-s)}f(\hat{u}(s))ds - \sum_{n=1}^N \int_0^T e^{ij_n\omega T/\varepsilon+ij_nkX/\eta}e^{\hat{Q}_n(T-s)}R_n(\hat{A}(s))ds \\
&= F(T) + \int_0^T e^{\hat{\mathcal{P}}(T-s)}\Big(f(\hat{u}(s)) - f(\textstyle\sum_{n=1}^N \hat{A}(s)e^{ij_n\omega s/\varepsilon+ij_nkX/\eta})\Big)ds,
\end{aligned}$$

where

$$\begin{aligned}
F(T) &= \sum_{n=1}^N e^{\hat{\mathcal{P}}T}(e^{ij_nkX/\eta}v_n) - \sum_{n=1}^N e^{ij_n\omega T/\varepsilon+ij_nkX/\eta}e^{\hat{Q}_nT}v_n \\
&\quad + \int_0^T e^{\hat{\mathcal{P}}(T-s)}f(\textstyle\sum_{n=1}^N \hat{A}(s)e^{ij_n\omega s/\varepsilon+ij_nkX/\eta})ds - \displaystyle\sum_{n=1}^N \int_0^T e^{ij_n\omega T/\varepsilon+ij_nkX/\eta}e^{\hat{Q}_n(T-s)}R_n(\hat{A}(s))ds.
\end{aligned}$$

Because of the existence theorem of mild solutions, there exists a positive constant D_1, which is independent of ε, such that

$$\|\hat{u}(T)\|_\eta \leq D_1, \quad \|\sum_{n=1}^N \hat{A}_n(T)e^{ij_n\omega T/\varepsilon+ij_nkX/\eta}\|_\eta \leq D_1, \quad \|e^{\hat{\mathcal{P}}T}\|_\eta \leq D_1$$

hold for $0 \leq T \leq T_0$. Let $L > 0$ be a Lipschitz constant of f in the ball $\{\varphi \in B^r \,|\, \|\varphi\|_\eta \leq D_1\}$. Then, we obtain

$$\|\hat{u} - \sum_{n=1}^N \hat{A}_n e^{ij_n\omega T/\varepsilon+ij_nkX/\eta}\|_\eta \leq \|F(T)\|_\eta + \int_0^T D_1L\,\|\hat{u}(s) - \textstyle\sum_{n=1}^N \hat{A}_n(s)e^{ij_n\omega s/\varepsilon+ij_nkX/\eta}\|_\eta ds.$$

Gronwall inequality yields

$$\|\hat{u} - \sum_{n=1}^N \hat{A}_n e^{ij_n\omega T/\varepsilon+ij_nkX/\eta}\|_\eta \leq \|F(T)\|_\eta + D_1L\int_0^T e^{D_1L(T-s)}\|F(s)\|_\eta ds.$$

To estimate $F(T)$, we rewrite it with the aid of Eq.(4.2) and $C_{j_n} = R_n$ as

$$\begin{aligned} F(T) &= \sum_{n=1}^{N} e^{\hat{\mathcal{P}}T}(e^{ij_n kX/\eta}v_n) - \sum_{n=1}^{N} e^{ij_n\omega T/\varepsilon + ij_n kX/\eta} e^{\hat{Q}_n T} v_n \\ &+ \sum_{n=1}^{N} \int_0^T e^{ij_n\omega s/\varepsilon} \left(e^{\hat{\mathcal{P}}(T-s)} e^{ij_n kX/\eta} - e^{ij_n\omega(T-s)/\varepsilon + ij_n kX/\eta} e^{\hat{Q}_n(T-s)} \right) R_n(\hat{A}(s))ds \\ &+ \sum_{j\notin J} H_j(T,X), \end{aligned} \tag{4.22}$$

where H_j is defined by

$$H_j(T,X) = \int_0^T e^{\hat{\mathcal{P}}(T-s)} e^{ij\omega s/\varepsilon + ijkX/\eta} C_j(\hat{A}(s))ds. \tag{4.23}$$

Step 3. estimate of H_j. Let $C([0,T_0];B^r)$ be a Banach space of functions $g(T,X)$ on $[0,T_0]\times \mathbf{R}^d$ such that $T\mapsto g(T,\,\cdot\,)\in B^r$ is continuous. The norm is defined by

$$||g||_{C^{0,r}} := \max_{T\in[0,T_0]} ||g(T,\,\cdot\,)||_\eta. \tag{4.24}$$

Let $C([0,T_0];B^r)^N$ be the product space with the norm defined by $||g||_{C^{0,r}} = \max_{1\leq n\leq N} ||g_n||_{C^{0,r}}$ for $g=(g_1,\cdots,g_N)$. Due to the existence theorem, a mild solution $\hat{A}(T,X)$ of (4.19b) is included in $C([0,T_0];B^r)^N$. The next lemma will be used several times.

Lemma 4.4. Suppose $f : B^r \to B^r$ is C^1. There exists a function $h : [0,T_0]\times\mathbf{R}^d\times C([0,T_0];B^r)^N \to C([0,T_0];B^r)$, which is bounded as $\eta\to 0$, such that

$$\sum_{j\notin J} H_j(T,X) = \eta h(T,X,\hat{A}(T,X)). \tag{4.25}$$

Further, if $f : B^r\to B^r$ is C^2, $h(T,X,\hat{A})$ is Lipschitz continuous in $\hat{A}\in C([0,T_0];B^r)^N$.

Proof. In the x-coordinate, we have

$$\begin{aligned} & e^{\hat{\mathcal{P}}(T-s)} e^{ij\omega s/\varepsilon + ijkX/\eta} C_j(\hat{A}(s,X)) \\ &= \frac{1}{(2\pi)^d} \int e^{ij\omega s/\varepsilon + ijk(x+y)} C_j(\hat{A}(s,\eta x+\eta y)) \int e^{-iy\xi} e^{P(i\xi)(T-s)/\varepsilon} dyd\xi. \end{aligned} \tag{4.26}$$

There exists a number $0<\theta<1$ such that

$$\begin{aligned} & e^{\hat{\mathcal{P}}(T-s)} e^{ij\omega s/\varepsilon + ijkX/\eta} C_j(\hat{A}(s,X)) \\ &= \frac{1}{(2\pi)^d} \int e^{ij\omega s/\varepsilon + ijk(x+y)} C_j(\hat{A}(s,\eta x)) \int e^{-iy\xi} e^{P(i\xi)(T-s)/\varepsilon} dyd\xi \\ &+ \frac{\eta}{(2\pi)^d} \int e^{ij\omega s/\varepsilon + ijk(x+y)} \sum_{i=1}^{d} \frac{\partial}{\partial y_i}\Big|_{y\mapsto \eta x+\theta\eta y} C_j(\hat{A}(s,y))\cdot y_i \int e^{-iy\xi} e^{P(i\xi)(T-s)/\varepsilon} dyd\xi. \end{aligned}$$

Since $\int e^{-iy\xi}e^{P(i\xi)(T-s)/\varepsilon}d\xi$ is rapidly decreasing in y, the right hand side above exists. We denote the first term and the second term above by I_1 and I_2, respectively; $H_j = \int_0^T I_1 ds + \int_0^T I_2 ds$. At first, we consider I_2. Since f is a C^1 function on B^r and $[0, T_0]$ is a finite interval, f is regarded as a C^1 function on $C([0, T_0]; B^r)$. Since the derivatives ∂C_j's are Fourier coefficients of ∂f, the series $\sum_{j\notin J} I_2$ converges and there exists a function $h_2(T, X, \hat{A})$ such that

$$\sum_{j\notin J}\int_0^T I_2 ds = \eta h_2(T, X, \hat{A}).$$

From the definition, we verify that h_2 is a mapping from $[0, T_0]\times\mathbf{R}^d\times C([0, T_0]; B^r)^N$ into $C([0, T_0]; B^0)$ (we will show later that this is a mapping into $C([0, T_0]; B^r)$). Furthermore, if f is C^2 so that C_j's are C^2, then $h_2(T, X, \hat{A})$ is C^1 in $\hat{A}\in C([0, T_0]; B^r)^N$ (in particular, Lipschitz continuous).

Next, let us calculate the first term I_1. Note that the equality

$$\int\int e^{ijky}e^{-iy\xi}e^{P(i\xi)(T-s)/\varepsilon}dyd\xi = (2\pi)^d e^{P(ijk)(T-s)/\varepsilon} \tag{4.27}$$

holds, which can be proved by the same way as Lemma 3.4. Thus we obtain

$$\int_0^T I_1 ds = e^{ijkx}e^{P(ijk)T/\varepsilon}\int_0^T C_j(\hat{A}(s, \eta x))e^{(ij\omega - P(ijk))s/\varepsilon}ds.$$

If $\hat{A}$ is differentiable in s (i.e. when the initial condition is included in the domain of Q_n), then integration by parts proves that the above quantity is of $O(\varepsilon)$. When $\hat{A}$ is not differentiable, we need further analysis.

Let J' be the set of integers j such that $j\notin J$ and $\mathrm{Re}[P(ijk)] = 0$. Due to the assumption (D3), J' is a finite set. Put $ij\omega - P(ijk) = p_j + iq_j$ with $p_j, q_j\in\mathbf{R}$. When $j\notin J\cup J'$ (i.e. $p_j\neq 0$), the mean value theorem proves that there exists $0\le\tau_j\le T$ such that

$$\begin{aligned}\int_0^T C_j(\hat{A}(s, \eta x))e^{(p_j+iq_j)s/\varepsilon}ds &= C_j(\hat{A}(\tau_j, \eta x))e^{iq_j\tau_j/\varepsilon}\int_0^T e^{p_j s/\varepsilon}ds\\ &= \frac{\varepsilon}{p_j}C_j(\hat{A}(\tau_j, \eta x))e^{iq_j\tau_j/\varepsilon}(e^{p_jT/\varepsilon} - 1).\end{aligned}$$

Since C_j's are Fourier coefficients of a C^1 function f and since $p_j\to\infty$ as $|j|\to\infty$, the following series

$$\begin{aligned}&\varepsilon\sum_{j\notin J\cup J'} e^{ijkx}e^{P(ijk)T/\varepsilon}\frac{1}{p_j}C_j(\hat{A}(\tau_j, \eta x))e^{iq_j\tau_j/\varepsilon}(e^{p_jT/\varepsilon} - 1)\\ = \ &\varepsilon\sum_{j\notin J\cup J'} e^{ijkx}e^{i\,\mathrm{Im}[P(ijk)]T/\varepsilon}\frac{1}{p_j}C_j(\hat{A}(\tau_j, \eta x))e^{iq_j\tau_j/\varepsilon}(1 - e^{-p_jT/\varepsilon})\end{aligned}$$

converges, and there exists a function $h_1(T, X, \hat{A})$ from $[0, T_0] \times \mathbf{R}^d \times C([0, T_0]; B^r)^N$ into $C([0, T_0]; B^0)$, which is C^1 in $\hat{A}$, such that

$$\sum_{j \notin J \cup J'} \int_0^T I_1 ds = \varepsilon h_1(T, X, \hat{A}).$$

To estimate the case $j \in J'$, we need the next lemma.

Lemma 4.5. For any constants $c, t_1 > 0$ that are independent of ε, a mild solution of (4.19b) satisfies $\|\hat{A}_n(T + c\varepsilon) - \hat{A}_n(T)\|_\eta \sim O(\eta)$ for $\varepsilon t_1 \leq T \leq T_0$ and $n = 1, \cdots, N$.

Proof. Recall $x = X/\eta$, $t = T/\varepsilon$. In the (t, x)-coordinates, linear semigroups satisfy

$$e^{\hat{Q}_n(T+c\varepsilon)} v_n - e^{\hat{Q}_n T} v_n = \frac{1}{(2\pi)^d} \int v_n(\eta x + \eta y) \int e^{-iy\xi} (e^{Q_n(i\xi)(t+c)} - e^{Q_n(i\xi)t}) dy d\xi.$$

Since

$$\frac{1}{(2\pi)^d} \int v_n(\eta x) \int e^{-iy\xi} e^{Q_n(i\xi)t} dy d\xi = v_n(\eta x)$$

for any t, there exists $0 < \theta < 1$ such that

$$e^{\hat{Q}_n(T+c\varepsilon)} v_n - e^{\hat{Q}_n T} v_n = \frac{\eta}{(2\pi)^d} \int \sum_{i=1}^d \frac{\partial v_n}{\partial x_i} (\eta x + \theta \eta y) y_i \int e^{-iy\xi} e^{Q_n(i\xi)t} (e^{Q_n(i\xi)c} - 1) dy d\xi,$$

which is of order $O(\eta)$ uniformly in x and $t_1 \leq t$. Next, the derivatives satisfy

$$\eta^{|\alpha|} \frac{\partial^\alpha}{\partial X^\alpha} \left(e^{\hat{Q}_n(T+c\varepsilon)} v_n - e^{\hat{Q}_n(T)} v_n \right) = \frac{1}{(2\pi)^d} \int v_n(\eta x + \eta y) \int (i\xi)^\alpha e^{-iy\xi} e^{Q_n(i\xi)t} (e^{Q_n(i\xi)c} - 1) dy d\xi.$$

By the same calculation as above, it turns out that

$$\|e^{\hat{Q}_n(T+c\varepsilon)} v_n - e^{\hat{Q}_n T} v_n\|_\eta \sim O(\eta) \tag{4.28}$$

for $t_1 \leq t$; that is, for $\varepsilon t_1 \leq T$. Then, Eq.(4.19b) yields

$$\begin{aligned} \hat{A}_n(T + c\varepsilon) - \hat{A}_n(T) &= e^{\hat{Q}_n(T+c\varepsilon)} v_n - e^{\hat{Q}_n T} v_n + \int_T^{T+c\varepsilon} e^{\hat{Q}_n(T+c\varepsilon-s)} R_n(\hat{A}(s)) ds \\ &+ \int_0^{T-\varepsilon t_1} (e^{\hat{Q}_n(T+c\varepsilon-s)} - e^{\hat{Q}_n(T-s)}) R_n(\hat{A}(s)) ds + \int_{T-\varepsilon t_1}^T (e^{\hat{Q}_n(T+c\varepsilon-s)} - e^{\hat{Q}_n(T-s)}) R_n(\hat{A}(s)) ds. \end{aligned}$$

Using (4.28), we obtain the lemma. ■

Suppose $j \in J'$, so that $ij\omega - P(ijk) = iq_j$;

$$\int_0^T I_1 ds = e^{ijkx} e^{P(ijk)T/\varepsilon} \int_0^T C_j(\hat{A}(s, \eta x)) e^{iq_j s/\varepsilon} ds.$$

Since the set J consists of all integers satisfying $P(ijk) = ij\omega$ (the assumptions (D3)), $q_j \neq 0$. Put $I_{3,j} = \int_0^T C_j(\hat{A}(s, \eta x))e^{iq_j s/\varepsilon}ds$. Changing the variable $s \mapsto s + \varepsilon\pi/q_j$ yields

$$I_{3,j} = -\int_{\varepsilon\pi/q_j}^{T+\varepsilon\pi/q_j} C_j(\hat{A}(s - \varepsilon\pi/q_j, \eta x))e^{iq_j s/\varepsilon}ds.$$

Hence, we obtain

$$\begin{aligned} 2I_{3,j} &= \int_0^{\varepsilon\pi/q_j} C_j(\hat{A}(s, \eta x))e^{iq_j s/\varepsilon}ds - \int_T^{T+\varepsilon\pi/q_j} C_j(\hat{A}(s - \varepsilon\pi/q_j, \eta x))e^{iq_j s/\varepsilon}ds \\ &\quad + \int_{\varepsilon\pi/q_j}^{T} \left(C_j(\hat{A}(s, \eta x)) - C_j(\hat{A}(s - \varepsilon\pi/q_j, \eta x)) \right) e^{iq_j s/\varepsilon}ds \\ &= \varepsilon \int_0^{\pi/q_j} C_j(\hat{A}(\varepsilon s, \eta x))e^{iq_j s}ds - \varepsilon \int_{T/\varepsilon}^{T/\varepsilon+\pi/q_j} C_j(\hat{A}(\varepsilon s - \varepsilon\pi/q_j, \eta x))e^{iq_j s}ds \\ &\quad + \eta \cdot \int_{\varepsilon\pi/q_j}^{T} \frac{C_j(\hat{A}(s, \eta x)) - C_j(\hat{A}(s - \varepsilon\pi/q_j, \eta x))}{\eta} e^{iq_j s/\varepsilon}ds. \end{aligned}$$

Lemma 4.5 shows that

$$\tilde{h}_{3,j}(\hat{A}) := \frac{C_j(\hat{A}(s, \eta x)) - C_j(\hat{A}(s - \varepsilon\pi/q_j, \eta x))}{\eta}$$

defines a function from $C([0, T_0]; B^r)^N$ into $C([0, T_0]; B^r)$, which is bounded as $\eta \to 0$. Hence, there exists a function $h_{3,j}(T, X, \hat{A})$ from $[0, T_0] \times \mathbf{R}^d \times C([0, T_0]; B^r)^N$ into $C([0, T_0]; B^0)$ such that

$$\int_0^T I_1 ds = \eta h_{3,j}(T, X, \hat{A}).$$

Further, if f and C_j are C^2, $h_{3,j}(T, X, \hat{A})$ is C^1 in $\hat{A}$. Therefore, putting $h = h_2 + \eta^{M-1}h_1 + \sum_{j\in J'} h_{3,j}$ proves Eq.(4.25) satisfying $h(T, X, \hat{A}) \in C([0, T_0]; B^0)$.

Let us estimate the derivative. Eq.(4.26) yields

$$\begin{aligned} &\eta^{|\alpha|} \frac{\partial^\alpha}{\partial X^\alpha} \left(e^{\hat{\mathcal{P}}(T-s)} e^{ij\omega s/\varepsilon + ijkX/\eta} C_j(\hat{A}(s, X)) \right) \\ = &\frac{1}{(2\pi)^d} \int e^{ij\omega s/\varepsilon + ijk(x+y)} C_j(\hat{A}(s, \eta x + \eta y)) \int (i\xi)^\alpha e^{-iy\xi} e^{P(i\xi)(T-s)/\varepsilon} dy d\xi. \end{aligned} \quad (4.29)$$

Repeating the same argument, it turns out that $h(T, X, \hat{A}) \in C([0, T_0]; B^r)$ if $\hat{A}$ is in $C([0, T_0]; B^r)^N$. This completes the proof of the lemma. ■

Because of this lemma, there exists a positive number D_2 such that

$$\| \sum_{j\notin J} H_j(T, X) \|_\eta \leq \eta D_2 \quad (4.30)$$

holds for $0 \leq T \leq T_0$.

Step 4. estimate of $F(T)$. By using Eq.(4.21) and (4.30), we can show there exists a positive constant D_3 such that $\|F(T)\|_\eta \leq \eta D_3$ for $\varepsilon t_0 \leq T \leq T_0$. Therefore, we obtain

$$
\begin{aligned}
& \|\hat{u} - \sum_{n=1}^{N} \hat{A}_n e^{ij_n\omega T/\varepsilon + ij_n kX/\eta}\|_\eta \\
\leq\ & \eta D_3 + D_1 L \int_{\varepsilon t_0}^{T} e^{D_1 L(T-s)} \eta D_3 ds + D_1 L \int_0^{\varepsilon t_0} e^{D_1 L(T-s)} \|F(s)\|_\eta ds \sim O(\eta).
\end{aligned}
$$

for $\varepsilon t_0 \leq T \leq T_0$. Changing to the (t,x)-coordinate proves Theorem 4.2. ■

Proof of Thm.4.3. Let us consider the systems (4.15). Recall that in this situation, $C_j(A_1, \cdots, A_N, A_{-1}, \cdots, A_{-N})$ is defined by Eq.(4.11). $S_n(A)$ is defined by

$$
S_n(A_1, \cdots, A_N) = R_n(A_1, \cdots, A_N, A_1, \cdots, A_N) = C_{j_n}(A_1, \cdots, A_N, A_1, \cdots, A_N)
$$

for $n = 1, \cdots, N$. Again we assume $D = d$ and use the same notation as the previous proof. A mild solution of (4.16a) written in the (T,X)-coordinate satisfies

$$
\hat{u} = e^{\hat{\mathcal{P}}T}(\hat{u}(0)) + \int_0^T e^{\hat{\mathcal{P}}(T-s)} f(\hat{u}(s)) ds, \tag{4.31}
$$

with the initial condition $\hat{u}(0,X) = \hat{u}(0)$. Let us consider the system of $2N$-integral equations of $w_+ := (w_1, \cdots, w_N)$ and $w_- := (w_{-1}, \cdots, w_{-N})$ of the form

$$
\begin{aligned}
e^{ij_n\omega T/\varepsilon + ij_n kX/\eta} w_n &= e^{\hat{\mathcal{P}}T}(e^{ij_n kX/\eta} w_n(0)) + \int_0^T e^{\hat{\mathcal{P}}(T-s)} R_n(w_+(s), w_-(s)) e^{ij_n\omega s/\varepsilon + ij_n kX/\eta} ds \\
&\quad + \frac{1}{2N} \sum_{j \notin J} \int_0^T e^{\hat{\mathcal{P}}(T-s)} C_j(w_+(s), w_-(s)) e^{ij\omega s/\varepsilon + ijkX/\eta} ds, \qquad (4.32) \\
e^{-ij_n\omega T/\varepsilon - ij_n kX/\eta} w_{-n} &= e^{\hat{\mathcal{P}}T}(e^{-ij_n kX/\eta} w_{-n}(0)) + \int_0^T e^{\hat{\mathcal{P}}(T-s)} R_{-n}(w_+(s), w_-(s)) e^{-ij_n\omega s/\varepsilon - ij_n kX/\eta} ds \\
&\quad + \frac{1}{2N} \sum_{j \notin J} \int_0^T e^{\hat{\mathcal{P}}(T-s)} C_{-j}(w_+(s), w_-(s)) e^{-ij\omega s/\varepsilon - ijkX/\eta} ds, \qquad (4.33)
\end{aligned}
$$

which has a unique solution $w_n \in C([0,T_0]; B^r)$ satisfying $w_n(0,X) = w_n(0) \in B^r$. This yields

$$
\begin{aligned}
& \sum_{n=-N}^{N} e^{ij_n\omega T/\varepsilon + ij_n kX/\eta} w_n \\
=\ & e^{\hat{\mathcal{P}}T}(\sum_{n=-N}^{N} e^{ij_n kX/\eta} w_n(0)) + \sum_{j=-\infty}^{\infty} \int_0^T e^{\hat{\mathcal{P}}(T-s)} C_j(w_+(s), w_-(s)) e^{ij\omega s/\varepsilon + ijkX/\eta} ds \\
=\ & e^{\hat{\mathcal{P}}T}(\sum_{n=-N}^{N} e^{ij_n kX/\eta} w_n(0)) + \int_0^T e^{\hat{\mathcal{P}}(T-s)} f(\textstyle\sum_{n=-N}^{N} w_n(s) e^{ij_n\omega s/\varepsilon + ij_n kX/\eta}) ds,
\end{aligned}
$$

where we used the abbreviation $\sum_{n=-N}^{N} := \sum_{n=-1}^{-N} + \sum_{n=1}^{N}$. Recall also that $j_{-n} = -j_n$. This means that $\hat{u} = \sum_{n=-N}^{N} e^{ij_n\omega T/\varepsilon + ij_nkX/\eta} w_n$ when $\hat{u}(0) = \sum_{n=-N}^{N} e^{ij_nkX/\eta} w_n(0)$.

By using (D4), we can show the equalities

$$\overline{e^{\hat{\mathcal{P}}T}(e^{ijkX/\eta}w_n)} = e^{\hat{\mathcal{P}}T}(e^{-ijkX/\eta}\overline{w_n}),$$

and

$$\overline{C_j(A_1, \cdots, A_N, A_{-1}, \cdots, A_{-N})} = C_{-j}(\overline{A}_{-1}, \cdots, \overline{A}_{-N}, \overline{A}_1, \cdots, \overline{A}_N).$$

Therefore, $\{w_n = w_{-n} \in \mathbf{R}\}_{n=1}^{N}$ is the invariant set of the $2N$-equations. By putting $w_n = w_{-n} \in \mathbf{R}$ and $w := w_+ = w_-$, the system is reduced to N-equations of the form

$$\begin{aligned} e^{ij_n\omega T/\varepsilon + ij_nkX/\eta} w_n &= e^{\hat{\mathcal{P}}T}(e^{ij_nkX/\eta} w_n(0)) + \int_0^T e^{\hat{\mathcal{P}}(T-s)} S_n(w(s)) e^{ij_n\omega s/\varepsilon + ij_nkX/\eta} ds \\ &\quad + \frac{1}{2N} \sum_{j \notin J} \int_0^T e^{\hat{\mathcal{P}}(T-s)} C_j(w(s)) e^{ij\omega s/\varepsilon + ijkX/\eta} ds, \end{aligned} \tag{4.34}$$

for $n = 1, \cdots, N$, where $C_j(w) := C_j(w, w)$.

Suppose that there exists $\phi = (\phi_1, \cdots, \phi_N) \in \mathbf{R}^N$ satisfying the assumptions of Thm.4.3. Without loss of generality, we assume that the Jacobi matrix of $(S_1, \cdots, S_N)$ at $A = \phi$ is diagonal. Thus we put

$$S_n(\phi_1, \cdots, \phi_N) = 0, \quad \frac{\partial S_n}{\partial x_m}(\phi_1, \cdots, \phi_N) = -\beta_n \cdot \delta_{n,m}, \quad \mathrm{Re}[\beta_n] > 0, \tag{4.35}$$

for $n, m = 1, \cdots, N$. Put $w_n = \phi_n + \eta W_n$. Due to the assumption of Thm.4.3, S_n is C^2 having the Lipschitz continuous second derivatives. Hence, there is a Lipschitz continuous function $\hat{S}_n(\,\cdot\,, \eta) : C([0, T_0]; B^r)^N \to C([0, T_0]; B^r)$, which is bounded as $\eta \to 0$, such that

$$S_n(\phi + \eta W) = -\eta\beta_n W_n + \eta^2 \hat{S}_n(W, \eta), \quad W = (W_1, \cdots, W_n). \tag{4.36}$$

We denote $\hat{S}_n(W, \eta)$ by $\hat{S}_n(W)$ for simplicity. Note that Eq.(4.27) gives

$$e^{\hat{\mathcal{P}}T}(e^{ij_nkX/\eta}\phi_n) = e^{P(ij_nk)T/\varepsilon} e^{ij_nkX/\eta}\phi_n = e^{ij_n\omega T/\varepsilon + ij_nkX/\eta}\phi_n.$$

Thus Eq.(4.34) is rewritten as

$$\begin{aligned} e^{ij_n\omega T/\varepsilon + ij_nkX/\eta} W_n &= e^{\hat{\mathcal{P}}T}(e^{ij_nkX/\eta} W_n(0)) - \beta_n \int_0^T e^{\hat{\mathcal{P}}(T-s)} W_n(s) e^{ij_n\omega s/\varepsilon + ij_nkX/\eta} ds \\ + \eta \int_0^T e^{\hat{\mathcal{P}}(T-s)} \hat{S}_n(W(s)) e^{ij_n\omega s/\varepsilon + ij_nkX/\eta} ds &+ \frac{1}{2\eta N} \sum_{j \notin J} \int_0^T e^{\hat{\mathcal{P}}(T-s)} C_j(\phi + \eta W(s)) e^{ij\omega s/\varepsilon + ijkX/\eta} ds. \end{aligned}$$

Remark that the second term in the right hand side is linear in W_n. Therefore, we can show that this equation is rewritten as

$$\begin{aligned} e^{ij_n\omega T/\varepsilon + ij_nkX/\eta} W_n &= e^{(\hat{\mathcal{P}}-\beta_n)T}(e^{ij_nkX/\eta} W_n(0)) + \eta \int_0^T e^{(\hat{\mathcal{P}}-\beta_n)(T-s)} \hat{S}_n(W(s)) e^{ij_n\omega s/\varepsilon + ij_nkX/\eta} ds \\ &\quad + \frac{1}{2\eta N} \sum_{j \notin J} \int_0^T e^{(\hat{\mathcal{P}}-\beta_n)(T-s)} C_j(\phi + \eta W(s)) e^{ij\omega s/\varepsilon + ijkX/\eta} ds. \end{aligned} \tag{4.37}$$

Motivated by this equation, let us consider the system of integral equations of the form

$$\begin{aligned} e^{ij_n\omega T/\varepsilon + ij_n kX/\eta} W_n^*(T) &= \eta \int_{-\infty}^T e^{(\hat{\mathcal{P}}-\beta_n)(T-s)} e^{ij_n\omega s/\varepsilon + ij_n kX/\eta} \hat{S}_n(W^*(s)) ds \\ &\quad + \frac{1}{2\eta N} \sum_{j\notin J} \int_{-\infty}^T e^{(\hat{\mathcal{P}}-\beta_n)(T-s)} e^{ij\omega s/\varepsilon + ijkX/\eta} C_j(\phi + \eta W^*(s)) ds. \quad (4.38) \end{aligned}$$

We will show later that a solution $W^* = (W_1^*, \cdots, W_N^*)$ of this system satisfies Eq.(4.37) with a suitable initial condition $W_n(0) = W_n(0, X)$.

To prove the existence of a periodic solution, let C_p^r be the set of functions $\psi(T, X)$ in $C([0, T_0]; B^r)$ such that

$$\begin{cases} \psi(T + \dfrac{2\pi\varepsilon}{\omega}, X_1, \cdots, X_d) = \psi(T, X_1, \cdots, X_d), & \text{when } \omega \neq 0, \\ \psi(T, X_1, \cdots, X_d) \text{ is constant in } T \text{ when } \omega = 0, \end{cases} \quad (4.39)$$

and

$$\begin{cases} \psi(T, X_1, \cdots, X_{j-1}, X_j + \dfrac{2\pi\eta}{k_j}, X_{j+1}, \cdots, X_d) = \psi(T, X_1, \cdots, X_d), & \text{when } k_j \neq 0, \\ \psi(T, X_1, \cdots, X_d) \text{ is constant in } X_j \text{ when } k_j = 0, \end{cases} \quad (4.40)$$

for $j = 1, \cdots, d$. By the norm $\|\psi\|_{C_p^r} := \max_{T\in\mathbf{R}} \|\psi(T, \cdot)\|_\eta$, C_p^r becomes a Banach space, which is a closed subspace of $C([0, T_0]; B^r)$. Let $(C_p^r)^N$ be the product space with the norm $\|\psi\|_{C_p^r} = \max_{1\le n\le N} \|\psi_n\|_{C_p^r}$ for $\psi = (\psi_1, \cdots, \psi_N)$. Define mappings $\Omega_{1,n}$ and $\Omega_{2,n}$ to be

$$\begin{aligned} (\Omega_{1,n}W)(T, X) &= \eta \int_{-\infty}^T e^{-ij_n\omega(T-s)/\varepsilon - ij_n kX/\eta} e^{(\hat{\mathcal{P}}-\beta_n)(T-s)} e^{ij_n kX/\eta} \hat{S}_n(W(s, X)) ds, \\ (\Omega_{2,n}W)(T, X) &= \frac{1}{2\eta N} \sum_{j\notin J} \int_{-\infty}^T e^{-ij_n\omega T/\varepsilon - ij_n kX/\eta} e^{(\hat{\mathcal{P}}-\beta_n)(T-s)} e^{ij\omega s/\varepsilon + ijkX/\eta} C_j(\phi + \eta W(s, X)) ds, \end{aligned}$$

for $n = 1, \cdots, N$.

Lemma 4.6. $\Omega_{1,n}$ and $\Omega_{2,n}$ are mappings from $(C_p^r)^N$ into C_p^r.

Proof. By using the expression (3.6) of the semigroup, we can show that if $W \in (C_p^r)^N$, then there exists a positive constant D_1 such that

$$\eta^{|\alpha|} \Big| \frac{\partial^\alpha}{\partial X^\alpha} e^{-ij_n kX/\eta} e^{\hat{\mathcal{P}}(T-s)} e^{ij_n kX/\eta} \hat{S}_n(W(s, X)) \Big| \le D_1$$

for $X \in \mathbf{R}^d$, $-\infty < s \le T$ and $|\alpha| = 1, \cdots, r$. Thus we obtain

$$\|\Omega_{1,n}W\|_{C_p^r} \le \eta \cdot \sup_{T\in\mathbf{R}} \int_{-\infty}^T D_1 e^{-\beta_n(T-s)} ds.$$

Since $\mathrm{Re}[\beta_n] > 0$, the right hand side above exists. The periodicity conditions (4.39),(4.40) immediately follow from the definition. The proof for $\Omega_{2,n}$ is done in the same way. ■

Lemma 4.7. Fix a positive number δ and let $\mathcal{D} = \{W \in (C^r_p)^N \mid \|W\|_{C^r_p} \le \delta\}$ be a closed ball in $(C^r_p)^N$. If $\eta = \eta(\delta) > 0$ is sufficiently small, $(\Omega_{1,1}, \cdots, \Omega_{1,N})$ and $(\Omega_{2,1}, \cdots, \Omega_{2,N})$ are contraction mappings on $\mathcal{D}$.

Proof. It follows from the proof of Lemma 4.6 that $\|\Omega_{1,n}W\|_{C^r_p}$ is of order $O(\eta)$. Thus it is easy to verify that $(\Omega_{1,1}, \cdots, \Omega_{1,N})$ is a contraction mapping on $\mathcal{D}$ if η is sufficiently small. Next, by the same way as the proof of Lemma 4.4, we can show that there exists a function $h_n : \mathbf{R} \times \mathbf{R}^d \times (C^r_p)^N \to C^r_p$ such that

$$(\Omega_{2,n}W)(T, X) = \frac{1}{2\eta N} \cdot \eta h_n(T, X, \phi + \eta W(T, X)), \tag{4.41}$$

where $h_n(T, X, \cdot\,)$ is Lipschitz continuous. Therefore, there exists $L_n > 0$ such that

$$\|\Omega_{2,n}W - \Omega_{2,n}V\|_{C^r_p} \le \frac{L_n}{2N}\|\eta W - \eta V\|_{C^r_p} \sim O(\eta).$$

This proves that $(\Omega_{2,1}, \cdots, \Omega_{2,N})$ is contraction if η is sufficiently small. ■

Due to this lemma, the system (4.38) of integral equations has a unique solution $W^* = (W^*_1, \cdots, W^*_N)$ in $(C^r_p)^N$, which is periodic in X and T.

Lemma 4.8. The solution W^* is a solution of (4.37) satisfying the initial condition

$$\begin{aligned} e^{ij_nkX/\eta}W_n(0) &= \eta \int_{-\infty}^{0} e^{-(\hat{\mathcal{P}}-\beta_n)s} e^{ij_n\omega s/\varepsilon + ij_nkX/\eta} \hat{S}_n(W^*(s))ds \\ &+ \frac{1}{2\eta N} \sum_{j\notin J} \int_{-\infty}^{0} e^{-(\hat{\mathcal{P}}-\beta_n)s} e^{ij\omega s/\varepsilon + ijkX/\eta} C_j(\phi + \eta W^*(s))ds. \end{aligned} \tag{4.42}$$

Proof. This follows from the substitution of (4.42) into (4.37). ■

Now we have proved that the system (4.34) has a solution $w_n(T, X) = \phi_n + \eta W^*_n(T, X)$ satisfying $W^*_n \in C^r_p$. Therefore, the equation (4.31) has a solution

$$\hat{u}(T, X) = \sum_{n=1}^{N} (\phi_n + \eta W^*_n(T, X)) \cdot (e^{ij_n\omega T/\varepsilon + ij_nkX/\eta} + e^{-ij_n\omega T/\varepsilon - ij_nkX/\eta}). \tag{4.43}$$

Changing to the (t, x)-coordinate yields a mild solution of (4.16a) of the form

$$u(t, x) = \sum_{n=1}^{N} (\phi_n + \eta W^*_n(\varepsilon t, \eta x)) \cdot (e^{ij_n\omega t + ij_nkx} + e^{-ij_n\omega t - ij_nkx}), \tag{4.44}$$

which proves the first part of Thm.4.3.

Finally, let us prove the stability part of Thm.4.3. Let $W^*(T, X)$ be the periodic solution of (4.37). There exists a positive constant $D_1 \ge 1$ such that $\|e^{-ij_nkX/\eta}e^{\hat{\mathcal{P}}T}e^{ij_nkX/\eta}\|_\eta \le D_1$ for any $T > 0$ and $n = 1, \cdots, N$. Fix a positive number M and put $\delta = M/(2D_1)$.

Let $\mathcal{D} = \{W \in (B^r)^N \,|\, \|W - W^*(0,\cdot)\|_\eta \leq \delta\}$ be a neighborhood of the periodic solution $W^*(0, X)$ at $T = 0$. Due to the existence theorem of mild solutions, there exists $T_0 > 0$ such that when $W(0) \in \mathcal{D}$, then Eq.(4.37) has a solution $W(T)$ in $(B^r)^N$ for $0 \leq T \leq T_0$. We define a time T_0 map as

$$\mathcal{T} : \mathcal{D} \to (B^r)^N, \quad W(0) \mapsto W(T_0). \tag{4.45}$$

Lemma 4.9. If $\eta > 0$ is sufficiently small, T_0 can be taken so that $\mathcal{T}$ is a mapping on $\mathcal{D}$.

Proof. Define $\Omega_{3,n}$ and $\Omega_{4,n}$ to be

$$\begin{aligned}(\Omega_{3,n}W)(T,X) &= \int_0^T e^{-ij_n\omega(T-s)/\varepsilon - ij_nkX/\eta} e^{(\hat{\mathcal{P}}-\beta_n)(T-s)} e^{ij_nkX/\eta} \hat{S}_n(W(s,X))ds,\\ (\Omega_{4,n}W)(T,X) &= \frac{1}{2N}\sum_{j\notin J}\int_0^T e^{-ij_n\omega T/\varepsilon - ij_nkX/\eta} e^{(\hat{\mathcal{P}}-\beta_n)(T-s)} e^{ij\omega s/\varepsilon + ijkX/\eta} C_j(\phi+\eta W(s,X))ds,\end{aligned}$$

for $n = 1, \cdots, N$. Eq.(4.37) gives

$$\begin{aligned}W_n - W_n^* &= e^{-ij_n\omega T/\varepsilon - ij_nkX/\eta} e^{(\hat{\mathcal{P}}-\beta_n)T} e^{ij_nkX/\eta}(W_n(0) - W_n^*(0))\\ &\quad +\eta(\Omega_{3,n}W - \Omega_{3,n}W^*) + \frac{1}{\eta}(\Omega_{4,n}W - \Omega_{4,n}W^*).\end{aligned}$$

By the same way as the proof of Lemma 4.4, we can prove that there exist Lipschitz continuous functions $h_n : [0, T_0]\times\mathbf{R}^d\times C([0, T_0], B^r)^N \to C([0, T_0], B^r)$ such that $\Omega_{4,n}W = \eta h_n(T, X, \phi + \eta W)$. This provides

$$\begin{aligned}W_n - W_n^* &= e^{-ij_n\omega T/\varepsilon - ij_nkX/\eta} e^{(\hat{\mathcal{P}}-\beta_n)T} e^{ij_nkX/\eta}(W_n(0) - W_n^*(0))\\ &\quad +\eta(\Omega_{3,n}W - \Omega_{3,n}W^*) + h_n(T, X, \phi + \eta W) - h_n(T, X, \phi + \eta W^*).\end{aligned}$$

Since $\|W(0) - W^*(0)\|_\eta \leq \delta = M/(2D_1)$ for $W(0) \in \mathcal{D}$, we can assume that T_0 is chosen so that $\|W(T) - W^*(T)\|_\eta \leq M$ for $0 \leq T \leq T_0$. Put $\beta := \min_{1\leq n\leq N}\beta_n$. Since $\hat{S}_n$ and h_n are locally Lipschitz continuous, there exist positive constants D_2, D_3 such that

$$e^{\beta T}\|W(T) - W^*(T)\|_\eta \leq M/2 + \eta D_2\int_0^T e^{\beta s}\|W(s) - W^*(s)\|_\eta ds + \eta D_3 e^{\beta T}\|W - W^*\|_{C^{0,r}}$$

for $0 \leq T \leq T_0$ (see Eq.(4.24) for the definition of the norm $\|\cdot\|_{C^{0,r}}$ on $C([0, T_0]; B^r)^N$). The Gronwall inequality gives

$$\begin{aligned}&e^{\beta T}\|W(T) - W^*(T)\|_\eta\\ \leq\ & M/2 + \eta D_3 e^{\beta T}\|W - W^*\|_{C^{0,r}} + \eta D_2\int_0^T e^{\eta D_2(T-s)}\left(M/2 + \eta D_3 e^{\beta s}\|W - W^*\|_{C^{0,r}}\right)ds\\ =\ & Me^{\eta D_2 T}/2 + \frac{\eta D_3\beta}{\beta - \eta D_2}e^{\beta T}\|W - W^*\|_{C^{0,r}} - \frac{\eta^2 D_2 D_3}{\beta - \eta D_2}e^{\eta D_2 T}\|W - W^*\|_{C^{0,r}}\end{aligned}$$

for $0 \leq T \leq T_0$. Hence, we obtain

$$\|W(T) - W^*(T)\|_\eta \leq Me^{(\eta D_2-\beta)T}/2 + \frac{\eta D_3\beta}{\beta - \eta D_2}\|W - W^*\|_{C^{0,r}}.$$

Using the standard existence theorem, we can verify that a solution $W_n(T, X)$ of (4.37) is bounded as $\eta \to 0$. Thus, $||W - W^*||_{C^{0,r}}$ is bounded as $\eta \to 0$. Therefore, if η is sufficiently small, we obtain $||W(T) - W^*(T)||_\eta \leq M/2$. This implies that T_0 can be taken arbitrarily large and $||W(T) - W^*(T)||_\eta \leq M$ holds for any $T > 0$. Hence, if T_0 is sufficiently large, we obtain $||W(T_0) - W^*(T_0)||_\eta \sim O(\eta)$. Since W^* is $2\pi\varepsilon/\omega$-periodic in T, we can choose T_0 so that $||W(T_0) - W^*(0)||_\eta \sim O(\eta)$, which proves $W(T_0) \in \mathcal{D}$. ■

Lemma 4.10. If $\eta > 0$ is sufficiently small and T_0 is sufficiently large, $\mathcal{T}$ is a contraction mapping on $\mathcal{D}$.

Proof. Let $W(T)$ and $V(T)$ be two solutions of (4.37) with the initial conditions $W(0), V(0) \in \mathcal{D}$, respectively. By the same calculation as above, we can show the inequality

$$||W(T) - V(T)||_\eta \leq D_1 ||W(0) - V(0)||_\eta e^{(\eta D_2 - \beta)T} + \eta D_4 ||W - V||_{C^{0,r}},$$

where we put $D_4 = D_3\beta/(\beta - \eta D_2)$. Since $||W - V||_{C^{0,r}} = \max_{0 \leq T \leq T_0} ||W(T) - V(T)||_\eta$, we obtain

$$||W - V||_{C^{0,r}} \leq D_1 ||W(0) - V(0)||_\eta + \eta D_4 ||W - V||_{C^{0,r}}.$$

Substituting this into the above inequality provides

$$||W(T) - V(T)||_\eta \leq \left(D_1 e^{(\eta D_2 - \beta)T} + \frac{\eta D_1 D_4}{1 - \eta D_4} \right) ||W(0) - V(0)||_\eta,$$

which proves the lemma. ■

Take $T_0 = 2\pi\varepsilon/\omega \cdot l$ so that Lemmas 4.9 and 4.10 hold, where l is a sufficiently large integer. Due to the lemma, there exists a unique function $W^{**}(X) \in \mathcal{D}$ such that any solutions $W(T, X)$ in $\mathcal{D}$ satisfy $W(nT_0, \cdot) \to W^{**}(\cdot)$ as $n \to \infty$. If $W^{**} \neq W^*$, then $W^*(nT_0, \cdot)$ converges to W^{**} as $n \to \infty$. This contradicts with the fact that $W^*(nT_0, \cdot) = W^*(0, \cdot)$ is independent of n. Hence, $W^{**} = W^*$ and $W(T, X)$ in $\mathcal{D}$ converges to W^* as $T \to \infty$. Since a mild solution of (4.16a) is written as

$$u(t, x) = \sum_{n=1}^{N} (\phi_n + \eta W_n(\varepsilon t, \eta x)) \cdot (e^{ij_n\omega t + ij_n kx} + e^{-ij_n\omega t - ij_n kx}), \tag{4.46}$$

the proof of Thm.4.3 is completed. ■

4.2 Higher dimensional case

Suppose $u = (u_1, \cdots, u_m) \in \mathbf{C}^m$ and $x = (x_1, \cdots, x_d) \in \mathbf{R}^d$. For fixed $1 \leq D \leq d$, we use the same notation $x = (\hat{x}_1, \hat{x}_2)$, $\alpha = (\beta, \gamma)$ as in Sec.4.1. Let $\{P_{ij}(x)\}_{i,j=1}^m$ be the set of polynomials of x. The $m \times m$ matrix $P(x)$, the differential operator $\mathcal{P}$ and eigenvalues $\lambda_1(\xi), \cdots, \lambda_m(\xi)$ are defined in the same way as Sec.3.2. We suppose for simplicity that only $\lambda_1(\xi)$ contributes to the center subspace of $\mathcal{P}$ (see (E1) below). Extending to more general situations is not difficult (see Remark 3.7 and Example 4.12).

(E0) The matrix $P(i\xi)$ is diagonalizable for any $\xi \in \mathbf{R}^d$.
(E1) $\mathrm{Re}[\lambda_1(\xi)] \leq 0$ and $\mathrm{Re}[\lambda_j(\xi)] < 0$ for any $\xi \in \mathbf{R}^d$ and $j = 2, \cdots, m$.
(E2) There exist $\omega \in \mathbf{R}$, $k \in \mathbf{R}^d$ ($(\omega, k) \neq (0,0)$), a finite set of integers $J = \{j_1, \cdots, j_N\}$ and $\{M_1, \cdots, M_N\}$ such that

$$
\begin{aligned}
&\lambda_1(j_n k) = i j_n \omega, \ \ (n = 1, \cdots, N), \\
&\frac{\partial^\beta \lambda_1}{\partial \hat{x}_1^\beta}(j_n k) = 0, \ \text{ for any } \beta \text{ such that } |\beta| = 1, \cdots, M_n - 1, \ \ (n = 1, \cdots, N), \\
&\frac{\partial^\beta \lambda_1}{\partial \hat{x}_1^\beta}(j_n k) \neq 0, \ \text{ for some } \beta_n \text{ such that } |\beta_n| = M_n, \ \ (n = 1, \cdots, N).
\end{aligned}
$$

The set J consists of *all* integers satisfying $\lambda_1(jk) = ij\omega$.
(E3) For $n = 1, \cdots, N$, define $Q_n(x)$ and $\mathcal{Q}_n$ by

$$
Q_n(x) = Q_n(\hat{x}_1, 0) = \sum_{|\beta| = M_n} \frac{1}{(\beta_1 !) \cdots (\beta_D !)} \frac{\partial^\beta \lambda_1}{\partial \hat{x}_1^\beta}(j_n k)(\hat{x}_1 / i)^\beta, \quad \mathcal{Q}_n = Q_n(\partial_1, \cdots, \partial_D, 0, \cdots, 0). \tag{4.47}
$$

Then, both of $\mathcal{P}$ and $\mathcal{Q}_n$ are elliptic in the sense that there exist $c_1, c_2 > 0$ such that $\mathrm{Re}[\lambda_j(\xi)] < -c_2 |\xi|^2$ ($j = 1, \cdots, m$) and $\mathrm{Re}[Q_n(i\hat{\xi}_1)] < -c_2 |\hat{\xi}_1|^2$ ($n = 1, \cdots, N$) hold for $|\xi|, |\hat{\xi}_1| \geq c_1$, where $\hat{\xi}_1 = (\xi_1, \cdots, \xi_D)$.

When $m = 1$, $\lambda_1(\xi) = P(i\xi)$, so that the above assumptions and $\mathcal{Q}_n$ are reduced to those given in Sec.4.1. Let $B^r = BC^r(\mathbf{R}^r; \mathbf{C})$ and $(B^r)^m = B^r \times \cdots \times B^r$ a product space. The norm on $(B^r)^m$ is defined by $\|u\| = \max_{1 \leq n \leq m} \|u_n\|$. Note that $\mathcal{P}$ is an operator densely defined on $(B^r)^m$, while $\mathcal{Q}_n$ is an operator densely defined on B^r.

Let $\boldsymbol{w_n} = (w_{n,1}, \cdots, w_{n,m})$ be an eigenvector of $P(ij_n k)$ associated with the eigenvalue $\lambda_1(j_n k) = ij_n \omega$ for $n = 1, \cdots, N$. The projection to the eigenspace span$\{\boldsymbol{w}_n\}$ is denoted by Π_n, and the projection to the eigenspace associated with the other eigenvalues $\lambda_2(j_n k), \cdots, \lambda_m(j_n k)$ is denoted by $\Pi_n^\perp = id - \Pi_n$. Functions $e^{ij_1 kx}\boldsymbol{w}_1, \cdots, e^{ij_N kx}\boldsymbol{w}_N$ span the center subspace of $\mathcal{P}$.

When $k = (k_1, \cdots, k_d) \neq 0$, we can assume without loss of generality that $k_1 \neq 0$ and $j_1 = 1$. For a given function $f : (B^r)^m \to (B^r)^m$, define a function $\boldsymbol{C}_j : (B^r)^N \to (B^r)^m$ by

$$
\boldsymbol{C}_j(A) = \boldsymbol{C}_j(A_1, \cdots, A_N) = \frac{k_1}{2\pi} \int_0^{2\pi/k_1} f\Big(\sum_{n=1}^N A_n e^{ij_n k_1 x_1} \boldsymbol{w}_n\Big) e^{-ijk_1 x_1} dx_1. \tag{4.48}
$$

When $k = 0$ and $\omega \neq 0$, we use

$$
\boldsymbol{C}_j(A) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} f\Big(\sum_{n=1}^N A_n e^{ij_n \omega t} \boldsymbol{w}_n\Big) e^{-ij\omega t} dt \tag{4.49}
$$

instead of (4.48). Then, we obtain the expansion

$$
f\Big(\sum_{n=1}^N A_n e^{ij_n \omega t + ij_n kx} \boldsymbol{w}_n\Big) = \sum_{j=-\infty}^{\infty} \boldsymbol{C}_j(A) e^{ij\omega t + ijkx}. \tag{4.50}
$$

Further, define the function $\boldsymbol{R}_n : (B^r)^N \to (B^r)^m$ to be

$$\boldsymbol{R}_n(A) = \Pi_n \boldsymbol{C}_{j_n}(A), \quad j_n \in J, \ (n = 1, \cdots, N), \tag{4.51}$$

and define $R_n : (B^r)^N \to B^r$ so that $\boldsymbol{R}_n(A) = R_n(A)\boldsymbol{w}_n$. Let $\varepsilon > 0$ be a small parameter. Let us consider the two initial value problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u + \varepsilon f(u), \quad u(0,x) = \displaystyle\sum_{n=1}^{N} e^{i j_n k x} v_n(\eta \hat{x}_1)\boldsymbol{w}_n, & \text{(4.52a)} \\ \dfrac{\partial A_n}{\partial t} = Q_n A_n + \varepsilon R_n(A), \quad A_n(0,x) = v_n(\eta \hat{x}_1), \ (n = 1, \cdots, N), & \text{(4.52b)} \end{cases}$$

where $\eta = \varepsilon^{1/M}$ and $M := \min\{M_1, \cdots, M_N\}$.

Example 4.11. Let us consider the system (1.5), whose perturbation term is given by

$$f(u,v) = \begin{pmatrix} u - u^3 \\ 0 \end{pmatrix}. \tag{4.53}$$

The reduction of the linear part was calculated in Example 3.5, in which it was shown that (E0) to (E3) are satisfied with $m = d = 2$, $D = 1$, $k = (0,c)$, $\omega = 0$, $J = \{j_1 = 1, \ j_2 = -1\}$ and $M_1 = M_2 = 4$. We use the same notation as Example 3.5. The matrix $P(i\xi)$ at $(\xi_1, \xi_2) = (0, \pm c)$ is given by

$$P(\pm ic) = \begin{pmatrix} (k+d)/(2d) & -1 \\ 1 & (k+d)/2 \end{pmatrix}.$$

Eigenvalues and eigenvectors of this matrix are

$$\boldsymbol{w} = \begin{pmatrix} 1 \\ (k+d)/2 \end{pmatrix} \quad \text{for } \lambda_1(0, \pm c) = 0,$$
$$\boldsymbol{v} = \begin{pmatrix} 1 \\ (k+d)/(2d) \end{pmatrix} \quad \text{for } \lambda_2(0, \pm c) = -\frac{(1-d)(k+d)}{2d} < 0.$$

Since both eigenvectors of $\lambda_1(0,c)$ and $\lambda_1(0,-c)$ are given by $\boldsymbol{w}$ above, we calculate the Fourier expansion of $f(A_1 e^{icy}\boldsymbol{w} + A_2 e^{-icy}\boldsymbol{w})$. Then, it turns out that

$$\boldsymbol{C}_1(A) = \begin{pmatrix} A_1 - 3A_1^2 A_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{C}_{-1}(A) = \begin{pmatrix} A_2 - 3A_1 A_2^2 \\ 0 \end{pmatrix}.$$

Then, it is easy to show that projections of them are

$$\boldsymbol{R}_1(A) = \Pi_1 \boldsymbol{C}_1(A) = \frac{A_1 - 3A_1^2 A_2}{1-d}\boldsymbol{w}, \quad \boldsymbol{R}_2(A) = \Pi_2 \boldsymbol{C}_{-1}(A) = \frac{A_2 - 3A_1 A_2^2}{1-d}\boldsymbol{w}.$$

Therefore, the amplitude equation (4.52b) is given by

$$\begin{cases} \dfrac{\partial A_1}{\partial t} = QA_1 + \dfrac{\varepsilon}{1-d}(A_1 - 3A_1^2 A_2), \\ \dfrac{\partial A_2}{\partial t} = QA_2 + \dfrac{\varepsilon}{1-d}(A_2 - 3A_1 A_2^2), \end{cases} \tag{4.54}$$

where Q is defined by (3.17). If we suppose $A_2 = \overline{A}_1$, the equation (1.23) is obtained.

Example 4.12. Noting Remark 3.7, to extend (E1) to the case that several eigenvalues lie on the imaginary axis is straightforward and the amplitude equation is defined in a similar manner as above. Let us consider the system (1.7), whose perturbation term is the same as the above Example. By the same calculation as Example 3.8 and 4.11, we obtain the amplitude equation

$$\begin{cases} \dfrac{\partial A_1}{\partial t} = QA_1 + \dfrac{\varepsilon}{2}(A_1 - 3A_1^2 A_2), \\ \dfrac{\partial A_2}{\partial t} = QA_2 + \dfrac{\varepsilon}{2}(A_2 - 3A_1 A_2^2), \end{cases} \tag{4.55}$$

where $Q = D\partial^2$ is obtained in Example 3.8. If we suppose $A_2 = \overline{A}_1$, the equation (1.8) is obtained.

Theorem 4.13. Suppose (E0) to (E3), $f : (B^r)^m \to (B^r)^m$ $(r \geq 1)$ is C^1 and $\varepsilon > 0$ is sufficiently small. For any $\{v_n\}_{n=1}^N \subset B^r$, there exist positive numbers C, T_0 and t_0 such that mild solutions of the two initial value problems (4.50) satisfy

$$||u(t, x) - \sum_{n=1}^{N} A_n(t, x) e^{ij_n\omega t + ij_n kx} \boldsymbol{w}_n|| \leq C\eta = C\varepsilon^{1/M}, \tag{4.56}$$

for $t_0 \leq t \leq T_0/\varepsilon$.

Further, suppose that

(E4) For $|\beta| = M_n$ and $n = 1, \cdots, N$,

$$P(i\xi) = \overline{P(-i\xi)}, \;\; f(\overline{u}) = \overline{f(u)} \quad \text{and} \quad \frac{\partial^\beta \lambda_1}{\partial \hat{x}_1^\beta}(j_n k) = \frac{\partial^\beta \lambda_1}{\partial \hat{x}_1^\beta}(-j_n k).$$

In this case, the set J consists of $J = \{j_1, \cdots, j_N\} \cup \{-j_1, \cdots, -j_N\}$ as is Sec.4.1. We consider the two initial value problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{P}u + \varepsilon f(u), \quad u(0, x) = \displaystyle\sum_{n=1}^{N} \left(e^{ij_n kx} + e^{-ij_n kx}\right) v_n(\eta \hat{x}_1) \boldsymbol{w}_n, & (4.57a) \\[2ex] \dfrac{\partial A_n}{\partial t} = Q_n A_n + \varepsilon S_n(A), \quad A_n(0, x) = v_n(\eta \hat{x}_1), \;\; (n = 1, \cdots, N), & (4.57b) \end{cases}$$

where $S_n(A)$ is defined by (4.15). In the next proposition, $B^r = BC^r(\mathbf{R}^d; \mathbf{R})$ denotes the set of real-valued functions.

Theorem 4.14. Suppose (E0) to (E4) and $f : (B^r)^m \to (B^r)^m$ $(r \geq 1)$ is C^2 such that the second derivatives are locally Lipschitz continuous. Suppose that there exists a constant vector $\phi = (\phi_1, \cdots, \phi_N) \in \mathbf{R}^N$ such that

(i) $S_n(\phi) = 0$ for $n = 1, \cdots, N$,

(ii) the Jacobi matrix of $(S_1, \cdots, S_N)$ at ϕ is diagonalizable and all eigenvalues of the

matrix have negative real parts.
If $\varepsilon > 0$ is sufficiently small, Eq.(4.57a) has a solution of the form

$$u_p(t,x) = \sum_{n=1}^{N} \Big(\phi_n \boldsymbol{w}_n + \eta \boldsymbol{\psi}_n(t,x,\eta)\Big) \cdot (e^{ij_n\omega t + ij_n kx} + e^{-ij_n\omega t - ij_n kx}). \tag{4.58}$$

The vector-valued functions $\boldsymbol{\psi}_n$ and u_p are bounded as $\eta \to 0$ and satisfy

$$\begin{cases} 2\pi/\omega\text{-periodic in } t \text{ (when } \omega \neq 0), \\ \text{constant in } t \text{ (when } \omega = 0), \end{cases} \quad \begin{cases} 2\pi/k_j\text{-periodic in } x_j \text{ (when } k_j \neq 0), \\ \text{constant in } x_j \text{ (when } k_j = 0), \end{cases}$$

for $j = 1, \cdots, d$. This u_p is stable in the following sense: For any $n = 1, \cdots, N$, there is a neighborhood $U_n \subset B^r$ of ϕ_n in B^r such that if $v_n \in U_n$, then a mild solution u of the initial value problem (4.57a) satisfies $\|u(t,\cdot) - u_p(t,\cdot)\| \to 0$ as $t \to \infty$.

Proof. We suppose $D = d$ for simplicity and use the same notation as the proof of Thm.4.2 (see Step 1). Two mild solutions of Eq.(4.50) satisfy

$$\begin{aligned} \hat{u} - \sum_{n=1}^{N} \hat{A}_n e^{ij_n\omega T/\varepsilon + ij_n kX/\eta} \boldsymbol{w}_n &= \sum_{n=1}^{N} e^{\hat{\mathcal{P}}T}(e^{ij_n kX/\eta} v_n \boldsymbol{w}_n) - \sum_{n=1}^{N} e^{ij_n\omega T/\varepsilon + ij_n kX/\eta}(e^{\hat{Q}_n T} v_n)\boldsymbol{w}_n \\ &+ \int_0^T e^{\hat{\mathcal{P}}(T-s)} f(\hat{u}(s))ds - \sum_{n=1}^{N} \int_0^T e^{ij_n\omega T/\varepsilon + ij_n kX/\eta} e^{\hat{Q}_n(T-s)} R_n(\hat{A}(s))\boldsymbol{w}_n ds \\ = \quad F(T) &+ \int_0^T e^{\hat{\mathcal{P}}(T-s)}\Big(f(\hat{u}(s)) - f(\textstyle\sum_{n=1}^{N} \hat{A}(s) e^{ij_n\omega s/\varepsilon + ij_n kX/\eta} \boldsymbol{w}_n)\Big)ds, \end{aligned}$$

where

$$\begin{aligned} F(T) &= \sum_{n=1}^{N} e^{\hat{\mathcal{P}}T}(e^{ij_n kX/\eta} v_n \boldsymbol{w}_n) - \sum_{n=1}^{N} e^{ij_n\omega T/\varepsilon + ij_n kX/\eta}(e^{\hat{Q}_n T} v_n)\boldsymbol{w}_n \\ &+ \int_0^T e^{\hat{\mathcal{P}}(T-s)} f(\textstyle\sum_{n=1}^{N} \hat{A}(s) e^{ij_n\omega s/\varepsilon + ij_n kX/\eta} \boldsymbol{w}_n)ds - \displaystyle\sum_{n=1}^{N} \int_0^T e^{ij_n\omega T/\varepsilon + ij_n kX/\eta} e^{\hat{Q}_n(T-s)} R_n(\hat{A}(s))\boldsymbol{w}_n ds. \end{aligned}$$

As before, Gronwall inequality yields

$$\|\hat{u} - \sum_{n=1}^{N} \hat{A}_n e^{ij_n\omega T/\varepsilon + ij_n kX/\eta} \boldsymbol{w}_n\|_\eta \leq \|F(T)\|_\eta + D_1 L \int_0^T e^{D_1 L(T-s)} \|F(s)\|_\eta ds$$

for some constants $D_1, L > 0$. By using definitions of $\boldsymbol{C}_j$ and R_n, we rewrite $F(T)$ as

$$\begin{aligned} F(T) &= \sum_{n=1}^{N} e^{\hat{\mathcal{P}}T}(e^{ij_nkX/\eta}v_n\boldsymbol{w}_n) - \sum_{n=1}^{N} e^{ij_n\omega T/\varepsilon + ij_nkX/\eta}(e^{\hat{Q}_nT}v_n)\boldsymbol{w}_n \\ &+ \sum_{n=1}^{N}\int_0^T e^{ij_n\omega s/\varepsilon}\left(e^{\hat{\mathcal{P}}(T-s)}e^{ij_nkX/\eta} - e^{ij_n\omega(T-s)/\varepsilon+ij_nkX/\eta}e^{\hat{Q}_n(T-s)}\right)R_n(\hat{A}(s))\boldsymbol{w}_n ds \\ &+ \sum_{j\notin J} H_j(T,X) \\ &+ \sum_{n=1}^{N}\int_0^T e^{ij_n\omega s/\varepsilon}e^{\hat{\mathcal{P}}(T-s)}e^{ij_nkX/\eta}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s))ds, \end{aligned} \tag{4.59}$$

where H_j is defined by

$$H_j(T,X) = \int_0^T e^{\hat{\mathcal{P}}(T-s)}e^{ij\omega s/\varepsilon+ijkX/\eta}\boldsymbol{C}_j(\hat{A}(s))ds. \tag{4.60}$$

Now we have arrived at the same situation as (4.22) except for the last term.

Lemma 4.15. There exists a function $g_n : [0,T_0]\times\mathbf{R}^d\times C([0,T_0];B^r)^N \to C([0,T_0];B^r)^m$ such that

$$\int_0^T e^{ij_n\omega s/\varepsilon}e^{\hat{\mathcal{P}}(T-s)}e^{ij_nkX/\eta}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s))ds = \eta g_n(T,X,\hat{A}). \tag{4.61}$$

$g(T,X,\hat{A})$ is Lipschitz continuous in $\hat{A}\in C([0,T_0];B^r)^N$ and bounded as $\eta\to 0$.

If this lemma is true, the rest of the proofs of Thm.4.13, 4.14 are completely the same as those of Thm.4.2, 4.3.

Proof. We calculate $e^{\hat{\mathcal{P}}(T-s)}e^{ij_nkX/\eta}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s))$ as

$$\begin{aligned} e^{\hat{\mathcal{P}}(T-s)}e^{ij_nkX/\eta}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s)) &= \frac{1}{(2\pi)^d}\iint e^{-iy\xi}e^{P(i\eta\xi)(T-s)/\varepsilon}e^{ij_nk(X+y)/\eta}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s,X+y))d\xi dy \\ &= \frac{e^{ij_nkX/\eta}}{(2\pi)^d}\iint e^{-iy\xi}e^{P(i\eta\xi+ij_nk)(T-s)/\varepsilon}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s,X+y))d\xi dy. \end{aligned}$$

Let us suppose that the coordinate of $u=(u_1,\cdots,u_m)$ is defined so that the $m\times m$ matrix $P(ij_nk)$ is diagonal. Define a matrix $S(\xi)$ such that Eq.(3.22) holds. By the assumption, $S(j_nk)$ is the identity matrix. Then, we obtain

$$\begin{aligned} & e^{\hat{\mathcal{P}}(T-s)}e^{ij_nkX/\eta}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s)) \\ =\ & \frac{e^{ij_nkX/\eta}}{(2\pi)^d}\iint e^{-iy\xi}S(j_nk+\eta\xi)e^{\Lambda(j_nk+\eta\xi)(T-s)/\varepsilon}S(j_nk+\eta\xi)^{-1}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s,X+y))d\xi dy. \end{aligned}$$

By expanding $S(j_nk+\eta\xi)^{\pm 1}$, it turns out that there is a function $G_n(T,X,\hat{A})$, which is Lipschitz continuous in $\hat{A}\in C([0,T_0];B^r)^N$ and bounded as $\eta\to 0$, such that

$$\begin{aligned} & e^{\hat{\mathcal{P}}(T-s)}e^{ij_nkX/\eta}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s)) \\ =\ & \frac{e^{ij_nkX/\eta}}{(2\pi)^d}\iint e^{-iy\xi}e^{\Lambda(j_nk+\eta\xi)(T-s)/\varepsilon}\Pi_n^{\perp}\boldsymbol{C}_{j_n}(\hat{A}(s,X+y))d\xi dy + \eta G_n(T-s,X,\hat{A}). \end{aligned}$$

Let $\boldsymbol{C}_{j_n}^{(l)}$ be the l-th component of the vector $\boldsymbol{C}_{j_n}$. Due to the definition of $\Pi_n^\perp$, the first component of $e^{\Lambda(j_nk+\eta\xi)(T-s)/\varepsilon}\Pi_n^\perp \boldsymbol{C}_{j_n}$ is zero, and the l-th component is given by $e^{\lambda_l(j_nk+\eta\xi)(T-s)/\varepsilon}\boldsymbol{C}_{j_n}^{(l)}$ for $l = 2, \cdots, m$. To prove the lemma, it is sufficient to estimate

$$I_{n,l} := \int_0^T e^{ij_n\omega s/\varepsilon} \iint e^{-iy\xi} e^{\lambda_l(j_nk+\eta\xi)(T-s)/\varepsilon} \boldsymbol{C}_{j_n}^{(l)}(\hat{A}(s, X+y))d\xi dy ds.$$

Because of (E1) and (E3), there exists a positive number β such that

$$\mathrm{Re}[\lambda_l(j_nk+\eta\xi)+\beta] \leq 0, \quad \mathrm{Re}[\lambda_l(j_nk+\eta\xi)+\beta] \sim O(-|\xi|^2) \tag{4.62}$$

as $|\xi| \to \infty$ for any $l = 2, \cdots, m$. Then, we obtain

$$\begin{aligned} I_{n,l} &= \int_0^T e^{ij_n\omega s/\varepsilon} e^{-\beta(T-s)/\varepsilon} \iint e^{-iy\xi} e^{(\lambda_l(j_nk+\eta\xi)+\beta)(T-s)/\varepsilon} \boldsymbol{C}_{j_n}^{(l)}(\hat{A}(s, X+y))d\xi dy ds \\ &:= \int_0^T e^{ij_n\omega s/\varepsilon} e^{-\beta(T-s)/\varepsilon} K_{n,l}(T-s, X, \hat{A})ds, \end{aligned}$$

where $K_{n,l}$ is Lipschitz continuous in $\hat{A} \in C([0, T_0]; B^r)^N$ and bounded as $\eta \to 0$. The mean value theorem proves that there exists $0 \leq \tau \leq T$ such that

$$I_{n,l} = e^{ij_n\omega\tau/\varepsilon} K_{n,l}(T-\tau, X, \hat{A}(\tau, X)) \int_0^T e^{-\beta(T-s)/\varepsilon} ds,$$

which is of order $O(\varepsilon)$. Hence, putting $(0, I_{n,2}, \cdots, I_{n,N}) = \varepsilon I_n(T, X, \hat{A})$ and

$$\eta g_n(T, X, \hat{A}) = \varepsilon \frac{e^{ij_nkX/\eta}}{(2\pi)^d} I_n(T, X, \hat{A}) + \eta \int_0^T e^{ij_n\omega s/\varepsilon} G_n(T-s, X, \hat{A})ds$$

proves the lemma. ■

Now the function $F(T)$ in Eq.(4.59) is estimated with the aid of Prop.3.6, Lemma 4.4 and Lemma 4.15 to show $||F(T)||_\eta \sim O(\eta)$. Then, the Gronwall inequality proves Thm.4.13. A proof of Thm.4.14 is also done in the same way as that of Thm.4.3. ■

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