Approximation of Center Manifolds on the Renormalization Group Method

Department of Applied Mathematics and Physics
Kyoto University, Kyoto, 606-8501, Japan
Hayato CHIBA [*1]

Apr/27/2008

Abstract

The renormalization group (RG) method for differential equations is one of the perturbation methods for obtaining approximate solutions. This article shows that the RG method is effectual for obtaining an approximate center manifold and an approximate flow on it, when applied to equations having a center manifold.

1 Introduction

The renormalization group (RG) method of Chen, Goldenfeld and Oono [2,3] is one of the perturbation methods for obtaining solutions which are approximate to exact solutions for a long time interval. Over the last decade, various methods and techniques for deriving RG equations and approximate solutions have been studied by many authors [2,3,4,5,6,7,9,11,12,13,15]. Kunihiro [11,12] showed that an approximate solution obtained by the RG method is an envelope of a family of curves which are constructed by the naive expansion. Ziane [15] and DeVille *et al.* [6] defined the first order RG equation by using an averaging operator and they proved that an exact solution of a given equation and an approximate solution obtained by the RG method are sufficiently close to each other for a long time interval. Chiba [4,5] defined the higher order RG equation on the idea of Kunihiro, Ziane, and DeVille *et al.* to obtain higher order approximate solutions. He also proved that if the RG equation has a normally hyperbolic invariant manifold N, the original equation also has an invariant manifold which is diffeomorphic to N.

It has been shown that the RG method covers the traditional singular perturbation methods, such as the multi-scaling method [2,3], the boundary layer theory [2,3], the averaging method [6], the normal form theory [5,6]. In particular, Chen, Goldenfeld, Oono [3] and Ei, Fujii, Kunihiro [7] applied the RG method to an equation whose linear part has eigenvalues on the left half plane and the imaginary axis to construct an approximate center manifold, while many authors had studied the case that all eigenvalues of the linear part lie on the imaginary axis.

This paper offers a rigorous proof of the fact that the RG method provides an approximate center

[*1] E mail address : chiba@amp.i.kyoto-u.ac.jp

manifold, and further the RG method is improved to raise accuracy of approximation by using the higher order RG equation. An approximate flow on an approximate center manifold is derived as well. Moreover, it is shown that if the RG equation has a normally hyperbolic invariant manifold, a given equation also has an invariant manifold on the center manifold. This method for obtaining an approximate center manifold and a flow on it is called the *restricted RG method* because a domain of the RG equation is restricted to a center subspace of an unperturbed part of a given equation.

This paper is organized as follows: Sec.2 presents basic facts and definitions in dynamical systems. In Sec.3, the restricted RG method is proposed and main theorems on the restricted RG method are proved. Sec.4 presents a few examples.

2 Basic facts

Let f be a time independent C^r vector field on a C^r manifold M and $\varphi : \mathbf{R} \times M \to M$ its flow, which satisfies $\varphi_t \circ \varphi_s = \varphi_{t+s}$, $\varphi_0 = id_M$, where id_M denotes the identity map of M. For fixed $t \in \mathbf{R}$, $\varphi_t : M \to M$ defines a diffeomorphism of M. We denote by $\varphi_t(x_0) \equiv x(t)$, $t \in \mathbf{R}$, a solution of the ODE $\dot{x} = f(x)$ through $x_0 \in M$ at $t = 0$. We assume φ_t is defined for $\forall t \in \mathbf{R}$.

For a time-dependent vector field, let $x(t, \tau, \xi)$ denote a solution of an ODE $\dot{x}(t) = f(t, x)$ through ξ at $t = \tau$, which defines a flow $\varphi : \mathbf{R} \times \mathbf{R} \times M \to M$ by $\varphi_{t,\tau}(\xi) = x(t, \tau, \xi)$. For fixed $t, \tau \in \mathbf{R}$, $\varphi_{t,\tau} : M \to M$ is a diffeomorphism of M satisfying

$$\varphi_{t,t'} \circ \varphi_{t',\tau} = \varphi_{t,\tau}, \ \ \varphi_{t,t} = id_M. \tag{2.1}$$

Conversely, a family of diffeomorphisms $\varphi_{t,\tau}$ of M, which is C^1 with respect to t and τ, satisfying the above equality for $\forall t, \tau \in \mathbf{R}$ defines a time dependent vector field on M through

$$f(t, x) = \frac{d}{d\tau}\Big|_{\tau=t} \varphi_{\tau,t}(x). \tag{2.2}$$

Theorem 2.1. (Fenichel, [8])

Let M be a C^r manifold ($r \geq 1$), and $\mathcal{X}^r(M)$ the set of C^r vector fields on M with C^1 topology. Let f be a C^r vector field on M and suppose that $N \subset M$ is a compact connected normally hyperbolic f-invariant manifold. Then, there is a neighborhood $\mathcal{U} \subset \mathcal{X}^r(M)$ of f s.t. for $\forall g \in \mathcal{U}$, there exists an unique normally hyperbolic g-invariant C^r manifold $N_g \subset M$ near N. In particular if $g \in \mathcal{U}$ is within an $O(\varepsilon)$ neighborhood of f, N_g lies within an $O(\varepsilon)$ neighborhood of N as well.

See [8],[10],[14] for the proof of Thm.2.1 and the definition of normal hyperbolicity.

Definition 2.2. Let φ_t be the flow of a vector field on a manifold M. A manifold $N \subset M$ is called a *locally invariant manifold* if there exists a number $T = T(x) > 0$ for each $x \in N$ such that $\{\varphi_t(x) \,|\, -T < t < T\} \subset N$.

Theorem 2.3. (Center Manifolds Theorem) Consider the system

$$\begin{cases} \dot{x} = Ax + f(x, y), & x \in \mathbf{R}^n, \\ \dot{y} = By + g(x, y), & y \in \mathbf{R}^m, \end{cases} \tag{2.3}$$

where A and B are constant matrices such that all eigenvalues of A lie on the imaginary axis and all eigenvalues of B lie on the left half plane. Suppose that f and g are C^2 vector fields which vanish together with their derivatives at the origin. Then, there exists an n-dimensional locally invariant manifold which is tangent to the x-plane at the origin. It is called a *local center manifold*.

See Carr [1] for the proof of Thm.2.3.

3 Restricted Renormalization Group Method

In this section, we propose the restricted RG method for obtaining a center manifold and a flow on it approximately.

Let F be an $n \times n$ matrix all of whose eigenvalues lie on the imaginary axis or the left half plane. We assume that at least one eigenvalue is on the imaginary axis because if all eigenvalues lie on the left half plane, the origin remains to be a stable fixed point of the equation under small perturbation and topological properties of the flow near the origin is trivial. Further, we suppose that the Jordan block corresponding to the eigenvalues on the imaginary axis is semisimple. Let $g(t, x, \varepsilon)$ be a time-dependent vector field on $\mathbf{R}^n$ which is C^∞ class with respect to t, x and $\varepsilon \in \mathbf{R}$. Let $g(t, x, \varepsilon)$ admit a formal power series expansion in ε, $g(t, x, \varepsilon) = g_1(t, x) + \varepsilon g_2(t, x) + \varepsilon^2 g_3(t, x) + \cdots$. We suppose that $g_i(t, x)$'s are periodic in t and polynomial in x, whose degrees are equal to or larger than 1, although the results in this section still hold even if $g_i(t, x)$'s are almost periodic in $t \in \mathbf{R}$ as long as the set of Fourier exponents of $g_i(t, x)$'s has no accumulation points (see Chiba [4]).

Consider an ODE

$$\begin{aligned} \dot{x} &= Fx + \varepsilon g(t, x, \varepsilon) \\ &= Fx + \varepsilon g_1(t, x) + \varepsilon^2 g_2(t, x) + \cdots, \ \ x \in \mathbf{R}^n, \end{aligned} \tag{3.1}$$

where $\varepsilon \in \mathbf{R}$ is a small parameter. Replacing x in (3.1) by $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$, we rewrite (3.1) as

$$\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \cdots = F(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + \sum_{i=1}^{\infty} \varepsilon^i g_i(t, x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots). \tag{3.2}$$

Expanding the right hand side of the above equation with respect to ε and equating the coefficients of each ε^i of the both sides, we obtain ODEs for $x_0, x_1, x_2, \cdots$ as

$$\dot{x}_0 = Fx_0, \tag{3.3}$$

$$\dot{x}_1 = Fx_1 + G_1(t, x_0), \tag{3.4}$$

$$\vdots$$

$$\dot{x}_i = Fx_i + G_i(t, x_0, x_1, \cdots, x_{i-1}), \tag{3.5}$$

$$\vdots$$

where the inhomogeneous term G_i is a smooth function of $t, x_0, x_1, \cdots, x_{i-1}$. For instance, G_1, G_2 and G_3 are given by

$$G_1(t, x_0) = g_1(t, x_0), \tag{3.6}$$

$$G_2(t, x_0, x_1) = \frac{\partial g_1}{\partial x}(t, x_0)x_1 + g_2(t, x_0), \tag{3.7}$$

$$G_3(t, x_0, x_1, x_2) = \frac{1}{2}\frac{\partial^2 g_1}{\partial x^2}(t, x_0)x_1^2 + \frac{\partial g_1}{\partial x}(t, x_0)x_2 + \frac{\partial g_2}{\partial x}(t, x_0)x_1 + g_3(t, x_0), \tag{3.8}$$

respectively. We denote by $X(t) = e^{Ft}$ the fundamental matrix of the unperturbed equation $\dot{x}_0 = Fx_0$. Let N_0 be the center subspace, which is a hyperplane in $\mathbf{R}^n$ spanned by the eigenvectors of F associated with eigenvalues on the imaginary axis. Note that if $A \in N_0$, then $X(t)A \in N_0$ for all $t \in \mathbf{R}$.

Define functions $R_i : N_0 \to \mathbf{R}^n$ and $h_t^{(i)} : N_0 \to \mathbf{R}^n$ with $i = 1, 2, \cdots$ by

$$R_1(A) := \lim_{t \to -\infty} \frac{1}{t} \int^t X(s)^{-1} g_1(s, X(s)A) ds, \tag{3.9}$$

$$h_t^{(1)}(A) := X(t) \int^t \Big(X(s)^{-1} g_1(s, X(s)A) - R_1(A) \Big) ds, \tag{3.10}$$

and

$$\begin{aligned} R_i(A) := \lim_{t \to -\infty} \frac{1}{t} \int^t \Big(& X(s)^{-1} G_i(s, X(s)A, h_s^{(1)}(A), \cdots, h_s^{(i-1)}(A)) \\ & - X(s)^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_A R_{i-k}(A) \Big) ds, \end{aligned} \tag{3.11}$$

$$\begin{aligned} h_t^{(i)}(A) := X(t) \int^t \Big(& X(s)^{-1} G_i(s, X(s)A, h_s^{(1)}(A), \cdots, h_s^{(i-1)}(A)) \\ & - X(s)^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_A R_{i-k}(A) - R_i(A) \Big) ds, \end{aligned} \tag{3.12}$$

for $i = 2, 3, \cdots$, where $(Dh_t^{(k)})_A$ denotes the derivative of $h_t^{(k)}(A)$ with respect to $A \in \mathbf{R}^n$. Note that $h_t^{(k)}(A)$ is differentiable as a function on $\mathbf{R}^n$. Integral constants of the indefinite integrals of the above equations are taken to be zero. What this means is as follows: Since $g_i(t, x)$ is a polynomial in x and since it can be expanded into the Fourier series with respect to t, it is easy to see that each integrand in Eqs.(3.9) to (3.12) can be written as a linear combination of functions of forms $e^{\xi s}$, $\xi \in \mathbf{C}$. In particular, the integrand in Eq.(3.10) does not have a constant term because if the Fourier series of $X(s)^{-1}g_1(s, X(s)A)$ has a constant term with respect to t, it has to be equal to the term $R_1(A)$. Therefore we can take the integral constant of the indefinite integral $\int^t (X(s)^{-1}g_1(s, X(s)A) - R_1(A))ds$ to be zero so that the indefinite integral is written as a linear combination of functions of forms $e^{\xi t}$, $\xi \neq 0$. Functions $h_t^{(i)}(A)$, $i = 2, 3, \cdots$ are defined in a similar way so that $X(t)^{-1}h_t^{(i)}(A)$ are linear combinations of functions of forms $e^{\xi t}$, $\xi \neq 0$. Note that $R_i(A)$'s are independent of integral constants in Eqs(3.9),(3.11).

Lemma 3.1. Functions $R_1(A), R_2(A), \cdots$ are well defined and
(i) each $R_i(A)$ satisfies $R_i(A) \in N_0$,
(ii) each $h_t^{(i)}(A)$ is bounded uniformly in $t \in \mathbf{R}$.

Proof. We can assume that F is of the form

$$F = \left(\begin{array}{ccc|c} \lambda_1 & & & \\ & \ddots & & 0 \\ & & \lambda_l & \\ \hline & 0 & & \mathrm{S} \end{array} \right),$$

where eigenvalues $\lambda_1, \cdots, \lambda_l$ lie on the imaginary axis and eigenvalues $\lambda_{l+1}, \cdots, \lambda_n$ of S lie on the left half plane with $\mathrm{Re}(\lambda_{l+1}) \geq \cdots \geq \mathrm{Re}(\lambda_n)$. Let π_c and π_s be the projections from $\mathbf{R}^n$ onto the center subspace $N_0 = \{(x_1, \cdots, x_l, 0, \cdots 0) \,|\, x_i \in \mathbf{R}\}$ and its complementary subspace $N_0^{\perp} := \{(0, \cdots 0, x_{l+1}, \cdots, x_n) \,|\, x_i \in \mathbf{R}\}$, respectively. Since $g_1(t, x)$ and $X(t)A$ with $A \in N_0$ are bounded in $t \in \mathbf{R}$, there exists a positive constant C such that $||g_1(t, X(t)A)|| \leq C$ for each $A \in N_0$. Since the integrand $X(s)^{-1}g_1(s, X(s)A)$ in Eq.(3.9) is bounded in $s \leq 0$, it is easy to verify that the limit in Eq.(3.9) converges and $R_1(A)$ is well-defined.

To prove the lemma (i) for $i = 1$, note that there exist positive constants D, δ such that $||\pi_s X(t)^{-1}|| \leq De^{-\mathrm{Re}(\lambda_{l+1})t - \delta t}$ for $t \leq 0$ and $-\mathrm{Re}(\lambda_{l+1}) - \delta > 0$. Then, $\pi_s R_1(A)$ proves to satisfy

$$\begin{aligned} ||\pi_s R_1(A)|| &\leq \lim_{t \to -\infty} \frac{1}{-t} \int^t ||\pi_s X(s)^{-1} g_1(s, X(s)A)|| ds \\ &\leq \lim_{t \to -\infty} \frac{1}{-t} \int^t CDe^{-\mathrm{Re}(\lambda_{l+1})s - \delta s} ds = 0. \end{aligned}$$

This means that $R_1(A) \in N_0$. Next, to prove (ii) of the lemma, we evaluate $||\pi_s h_t^{(1)}(A)||$ to get

$$||\pi_s h_t^{(1)}(A)|| \leq \int^t ||\pi_s X(t-s) g_1(s, X(s)A) - \pi_s X(t) R_1(A)|| ds \leq \int^t CDe^{\mathrm{Re}(\lambda_{l+1})(t-s) + \delta(t-s)} ds. \tag{3.13}$$

Since the integral constant is equal to zero, we obtain $||\pi_s h_t^{(1)}(A)|| \leq -CD/(\mathrm{Re}(\lambda_{l+1}) + \delta)$. On the other hand, $\pi_c h_t^{(1)}(A)$ satisfies

$$\pi_c h_t^{(1)}(A) = \pi_c X(t) \int^t \Big(\pi_c X(s)^{-1} g_1(s, X(s)A) - R_1(A) \Big) ds. \tag{3.14}$$

Then, the boundedness of the $\pi_c h_t^{(1)}(A)$ results from Prop.A.4 of Chiba[4], in which the boundedness of $h_t^{(i)}(A),\ i = 1, 2, \cdots$ is proved for the case that all eigenvalues of F lie on the imaginary axis. Therefore $h_t^{(1)}(A) = \pi_s h_t^{(1)}(A) + \pi_c h_t^{(1)}(A)$ is bounded uniformly in $t \in \mathbf{R}$.

Lemma 3.1 for $R_i(A), h_t^{(i)}(A),\ i = 2, 3, \cdots$ is also proved by induction in the same manner as above. ■

By Lemma 3.1 (i), each $R_i(A)$ defines a vector field on N_0.

Definition 3.2. Along with $R_1(A), \cdots, R_m(A)$ given by Eqs.(3.9),(3.11), we define the *m-th order restricted RG equation* for Eq.(3.1) to be

$$\dot{A} = \varepsilon R_1(A) + \varepsilon^2 R_2(A) + \cdots + \varepsilon^m R_m(A),\ A \in N_0. \tag{3.15}$$

Using $h_t^{(1)}(A), \cdots, h_t^{(m)}(A)$ given by Eqs.(3.10),(3.12), we define the *m-th order RG transformation* $\alpha_t : N_0 \to \mathbf{R}^n$ by

$$\alpha_t(A) = X(t)A + \varepsilon h_t^{(1)}(A) + \cdots + \varepsilon^m h_t^{(m)}(A). \tag{3.16}$$

Remark 3.3. A few remarks are in order. While $R_i(A) \in N_0 \subset \mathbf{R}^n$ is an n-dimensional vector, as many as $\dim N_0$ component equations in Eq.(3.15) are independent of one other. Thus we regard Eq.(3.15) as a $\dim N_0$-dimensional differential equation. Since $X(t)$ is nonsingular and $h_t^{(1)}(A), \cdots, h_t^{(m)}(A)$ are bounded uniformly in $t \in \mathbf{R}$, for sufficiently small $|\varepsilon|$, there exists an open set $U = U(\varepsilon) \subset N_0$ including the origin such that $\overline{U}$ is compact and the restriction of α_t to U is diffeomorphism from U into $\mathbf{R}^n$. It is easy to verify that $\alpha_t(0) = 0$, thus $\alpha_t(U)$ includes the origin.

Now we are in a position to state fundamental results of our restricted RG method.

Theorem 3.4. (Approximation of Orbits)

Let $A = A(t)$ be a solution to the m-th order restricted RG equation (3.15). Define the curve $\widetilde{x}(t)$ by

$$\widetilde{x}(t) = \alpha_t(A(t)) = X(t)A(t) + \varepsilon h_t^{(1)}(A(t)) + \cdots + \varepsilon^m h_t^{(m)}(A(t)). \tag{3.17}$$

Then, there exist positive constants ε_0, C, T and a compact subset $V = V(\varepsilon) \subset \alpha_0(N_0)$ including the origin such that for $\forall|\varepsilon| < \varepsilon_0$, every solution $x(t)$ of Eq.(3.1) and $\widetilde{x}(t)$ defined by Eq.(3.17) with $x(0) = \widetilde{x}(0) \in V$ satisfy the inequality

$$||x(t) - \widetilde{x}(t)|| < C\varepsilon^m, \tag{3.18}$$

for $0 \leq t \leq T/\varepsilon$.

Theorem 3.5. (Approximation of Vector Fields)

Let φ_t^{RG} be the flow of the m-th order restricted RG equation for Eq.(3.1) and α_t the m-th order RG transformation. Then, there exists a positive constant ε_0 such that the following holds for $\forall|\varepsilon| < \varepsilon_0$:

(i) A map

$$\Phi_{t,t_0} := \alpha_t \circ \varphi_{t-t_0}^{RG} \circ \alpha_{t_0}^{-1} : \alpha_{t_0}(U) \to \mathbf{R}^n \tag{3.19}$$

defines a local flow on $\alpha_{t_0}(U)$ for each $t_0 \in \mathbf{R}$, where $U = U(\varepsilon) \subset N_0$ is an open set on which α_{t_0} is a diffeomorphism (see Rem.3.3). The Φ_{t,t_0} induces a time-dependent vector field F_ε through

$$F_\varepsilon(t,x) := \frac{d}{da}\Big|_{a=t} \Phi_{a,t}(x), \quad x \in \alpha_t(U), \tag{3.20}$$

whose integral curves are the approximate solutions $\widetilde{x}(t)$ defined by Eq.(3.17).
(ii) There exists a time-dependent vector field $\widetilde{F}_\varepsilon(t,x)$ such that

$$F_\varepsilon(t,x) = Fx + \varepsilon g_1(t,x) + \cdots + \varepsilon^m g_m(t,x) + \varepsilon^{m+1}\widetilde{F}_\varepsilon(t,x), \quad x \in \alpha_t(U) \tag{3.21}$$

where $\widetilde{F}_\varepsilon(t,x)$ is C^∞ class with respect to t, x and ε. In particular, $\widetilde{F}_\varepsilon(t,x)$ and its derivatives are bounded uniformly in $t \in \mathbf{R}$.

Theorem 3.4 and Theorem 3.5 are proved in the same manner as in Thm.A.8 and Thm.A.6 of Chiba[4], respectively, in which the theorems are proved for the case that all eigenvalues of F lie on the imaginary axis.

The following two theorems are concerned with an autonomous equation

$$\dot{x} = Fx + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots, \quad x \in \mathbf{R}^n, \tag{3.22}$$

where $\varepsilon \in \mathbf{R}$ is a small parameter. Like Eq.(3.1), F is an $n \times n$ matrix all of whose eigenvalues lie on the left half plane or the imaginary axis, and the Jordan block corresponding to the eigenvalues on the imaginary axis is semisimple. Further $g_i(x)$'s are polynomial vector fields on $\mathbf{R}^n$ whose degrees are equal to or larger than 1.

Theorem 3.6. (Inheritance of the Symmetries)

(i) Suppose that a Lie group G acts on the center subspace N_0 spanned by eigenvectors of F associated with eigenvalues on the imaginary axis. If vector fields Fx and $g_1(x), g_2(x), \cdots$, are invariant under the action of G, then the m-th order restricted RG equation for Eq.(3.22) is also invariant under the action of G.

(ii) The m-th order restricted RG equation commutes with the linear vector field Fx with respect to Lie bracket product. Equivalently, each $R_i(A)$, $i = 1, 2, \cdots$ satisfies

$$X(t)R_i(A) = R_i(X(t)A), \quad A \in N_0. \tag{3.23}$$

Theorem 3.7. (Existence of Invariant Manifolds)

Let $\varepsilon^k R_k(A)$ be a first non-zero term in the RG equation (3.15). If the vector field $\varepsilon^k R_k(A)$ on N_0 has a compact normally hyperbolic invariant manifold $M_0 \subset N_0$, then the original equation (3.22) also has a normally hyperbolic invariant manifold M_ε, which is diffeomorphic to M_0, for sufficiently small $|\varepsilon|$. In particular, the stability of M_ε and of M_0 coincide.

Theorem 3.6 and Theorem 3.7 are proved in the same manner as in Thm.A.9 and Thm.A.7 of Chiba[4], respectively, in which the theorems are proved for the case that all eigenvalues of F lie on the imaginary axis.

If degrees of polynomials $g_i(x)$'s in Eq.(3.22) are equal to or larger than 2, Eq.(3.22) has a dimN_0-dimensional local center manifold tangent to N_0 at the origin. If $g_i(x)$'s have linear parts, Eq.(3.22) no longer has a dimN_0-dimensional center manifold because the linear part of right hand side of Eq.(3.22) no longer has dimN_0 eigenvalues on the imaginary axis in general. However, even in this case, there exists a locally invariant manifold which is diffeomorphic to a dimN_0-dimensional closed ball. This fact is explained as follows: Recall that N_0 is a normally hyperbolic invariant hyperplane of the vector field Fx. Fix an n-dimensional closed ball K including the origin. We can perturb the vector field Fx in a small neighborhood of the boundary of K so that $N_0 \cap K$ is a normally hyperbolic invariant manifold of the resultant perturbed vector field, say $Fx + h(x)$. If $\varepsilon > 0$ is sufficiently small, by Thm.2.1, a vector field $Fx + h(x) + \varepsilon g_1(x) + \cdots$ has a normally hyperbolic invariant manifold N_ε which is diffeomorphic to $N_0 \cap K$. Since $h(x)$ has its support in a small neighborhood of the boundary of K, Eq.(3.22) has N_ε as a locally invariant manifold. In what follows, a locally invariant manifold in the above sense is also called a center manifold.

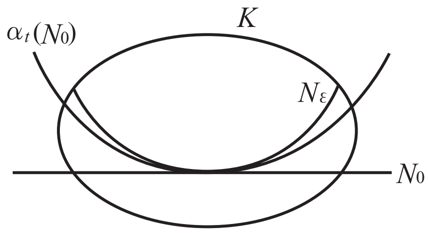

Fig. 1: A center subspace N_0, a center manifold N_ε, and an approximate center manifold $\alpha_t(N_0)$.

Now our purpose is to construct a center manifold of Eq.(3.22) approximately. Before showing the main theorem, we need a lemma.

Lemma 3.8. The set $\alpha_t(N_0) := \{\alpha_t(A) \,|\, A \in N_0\}$ is independent of $t \in \mathbf{R}$, where α_t is the m-th order RG transformation for Eq.(3.22).

Proof. At first, we prove that $h^{(i)}_{t+t'}(A) = h^{(i)}_t(X(t')A)$ hold for $i = 1, 2, \cdots$ and $\forall t' \in \mathbf{R}$. Since

$R_i(X(t)A) = X(t)R_i(A)$ holds by Thm.3.6 (ii), $h_t^{(1)}(X(t')A)$ is calculated as

$$\begin{aligned} h_t^{(1)}(X(t')A) &= X(t)\int^t \Big(X(s)^{-1}G_1(X(s)X(t')A) - R_1(X(t')A)\Big)\,ds \\ &= X(t)X(t')\int^t \Big(X(t')^{-1}X(s)^{-1}G_1(X(s)X(t')A) - R_1(A)\Big)\,ds \\ &= X(t+t')\int^t \Big(X(s+t')^{-1}G_1(X(s+t')A) - R_1(A)\Big)\,ds. \end{aligned}$$

By putting $s + t' = s'$, the above equation results in

$$h_t^{(1)}(X(t')A) = X(t+t')\int^{t+t'} \Big(X(s')^{-1}G_1(X(s')A) - R_1(A)\Big)\,ds' = h_{t+t'}^{(1)}(A). \tag{3.24}$$

The equalities $h_{t+t'}^{(i)}(A) = h_t^{(i)}(X(t')A)$ for $i = 2, 3, \cdots$ are proved in a similar manner by induction. Since $X(t)$ is a diffeomorphism on N_0, we obtain $h_{t+t'}^{(i)}(N_0) = h_t^{(i)}(N_0)$ and this proves $\alpha_{t+t'}(N_0) = \alpha_t(N_0)$. ■

Theorem 3.9. (Approximation of Center Manifolds)

Let α_t be the m-th order RG transformation for Eq.(3.22) and $W \subset U$ be a compact subset including the origin, where $U \subset N_0$ is an open set on which α_t is a diffeomorphism (see Rem.3.3). Then, the set $\alpha_t(W)$ lies within an $O(\varepsilon^{m+1})$ neighborhood of a center manifold N_ε of Eq.(3.22).

Proof. By Thm.3.5, the approximate vector field F_ε defined on $\alpha_t(U)$ is C^1 close to the restriction of the original vector field $Fx + \varepsilon g_1(x) + \cdots$ to $\alpha_t(U)$ within an $O(\varepsilon^{m+1})$. Let K be an n-dimensional closed ball including W and $\alpha_t(W)$. We can extend F_ε to a vector field F'_ε defined on K such that $F'_\varepsilon|_{K\cap\alpha_t(U)} = F_\varepsilon|_{K\cap\alpha_t(U)}$ and F'_ε is C^1 close to the original vector field $Fx + \varepsilon g_1(x) + \cdots$ on K within an $O(\varepsilon^{m+1})$. Since $\alpha_t(W)$ is a locally invariant manifold of F'_ε, our theorem immediately follows from Thm.2.1. ■

The present method for obtaining an approximate center manifold and an approximate flow on the manifold defined by Eq.(3.19) is called the *restricted RG method* because the domain of the RG equation and the RG transformation are restricted to N_0.

4 Examples

In this section, we show two simple examples of the restricted RG method.

Example 3.10. Consider the system on $\mathbf{R}^3$,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \varepsilon\begin{pmatrix} yz^2 \\ -x^3 \\ y^2 + xz \end{pmatrix}. \tag{4.1}$$

A general solution of the unperturbed part is given by

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} A\cos t + B\sin t \\ -A\sin t + B\cos t \\ Ce^{-t} \end{pmatrix} = \begin{pmatrix} pe^{it} + \overline{p}e^{-it} \\ ipe^{it} - i\overline{p}e^{-it} \\ Ce^{-t} \end{pmatrix}, \tag{4.2}$$

where $(A, B, C) \in \mathbf{R}^3$ is an initial value and p is defined by $p = A/2 + B/(2i)$. Introducing the complex variable p makes it simple to work with the RG equation. The center subspace for the unperturbed equation is expressed as $N_0 = \{(A, B, 0) \,|\, A, B \in \mathbf{R}\}$. The first order restricted RG equation and the first order RG transformation for Eq.(4.1) are given by

$$\frac{d}{dt}\begin{pmatrix} p \\ \overline{p} \end{pmatrix} = \frac{3i\varepsilon}{2}\begin{pmatrix} p|p|^2 \\ -\overline{p}|p|^2 \end{pmatrix}, \ \ p \in \mathbf{C}, \tag{4.3}$$

$$\begin{aligned} \alpha_t(A, B) = &\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}\begin{pmatrix} A \\ B \\ 0 \end{pmatrix} \\ &+\varepsilon \begin{pmatrix} \frac{p^3}{8}e^{3it} - \frac{3}{4}|p|^2 pe^{it} + \frac{\overline{p}^3}{8}e^{-3it} - \frac{3}{4}|p|^2\overline{p}e^{-it} \\ i\left(\frac{3p^3}{8}e^{3it} + \frac{3}{4}|p|^2 pe^{it} - \frac{3\overline{p}^3}{8}e^{-3it} - \frac{3}{4}|p|^2\overline{p}e^{-it}\right) \\ \frac{-p^2}{1+2i}e^{2it} + 2|p|^2 + \frac{-\overline{p}^2}{1-2i}e^{-2it} \end{pmatrix}, \end{aligned} \tag{4.4}$$

respectively. Therefore, an approximate center manifold of Eq.(4.1) is expressed as

$$\alpha_0(N_0) = \{(A, B, \varepsilon\varphi(A, B))\}, \tag{4.5}$$

$$\varphi(A, B) := \frac{-p^2}{1+2i} + 2|p|^2 + \frac{-\overline{p}^2}{1-2i} = \frac{2}{5}A^2 + \frac{3}{5}B^2 + \frac{2}{5}AB. \tag{4.6}$$

Equivalently, the approximate center manifold is expressed as the graph of the function $z = \varepsilon\frac{2}{5}x^2 + \varepsilon\frac{3}{5}y^2 + \varepsilon\frac{2}{5}xy$ in (x, y, z) space. This result coincides with an approximate center manifold obtained by a method in [1]. The restricted RG equation (4.3) is solved as

$$p(t) = \frac{1}{2}a\exp i\left(\frac{3\varepsilon}{8}a^2 t + \theta\right),$$

where a, θ are arbitrary constants. With this $p(t)$, an approximate solution defined by Eq.(3.17) on the center manifold of (4.1) is given by

$$\widetilde{x}(t) = a\cos\left(\frac{3\varepsilon}{8}a^2 t + t + \theta\right) + \frac{\varepsilon a^3}{32}\cos\left(\frac{9\varepsilon}{8}a^2 t + 3t + 3\theta\right) - \frac{3\varepsilon a^3}{16}\cos\left(\frac{3\varepsilon}{8}a^2 t + t + \theta\right),$$

$$\widetilde{y}(t) = -a\sin\left(\frac{3\varepsilon}{8}a^2 t + t + \theta\right) - \frac{3\varepsilon a^3}{32}\sin\left(\frac{9\varepsilon}{8}a^2 t + 3t + 3\theta\right) - \frac{3\varepsilon a^3}{16}\sin\left(\frac{3\varepsilon}{8}a^2 t + t + \theta\right),$$

$$\widetilde{z}(t) = \frac{\varepsilon a^2}{2} - \frac{\varepsilon a^2}{10}\cos\left(\frac{3\varepsilon}{4}a^2 t + 2t + 2\theta\right) - \frac{\varepsilon a^2}{5}\sin\left(\frac{3\varepsilon}{4}a^2 t + 2t + 2\theta\right).$$

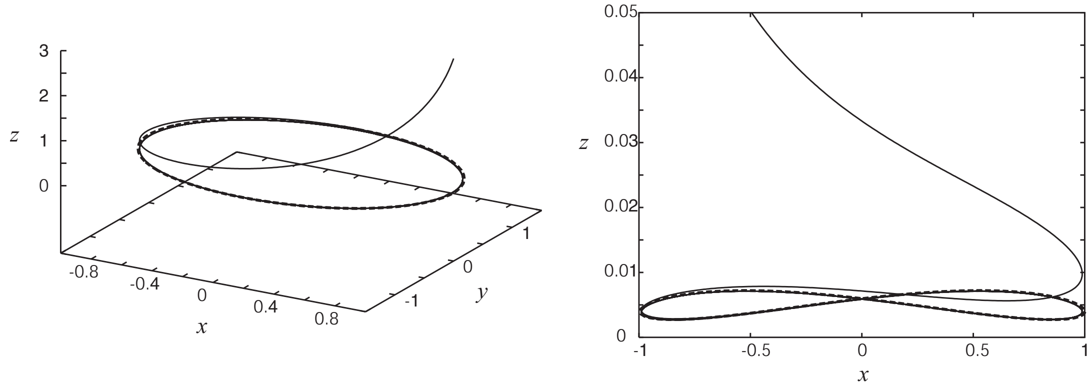

Fig. 2: An exact solution (solid line) to Eq.(4.1) and the approximate solution on the center manifold (dashed line).

Numerical observation for $\varepsilon = 0.01$ is presented in Fig.2. The solid line denotes an exact solution to Eq.(4.1) with initial value $(x, y, z) = (1, 0, 3)$, which is out of the center manifold, and dashed line denotes the above-stated approximate solution with $a = 1, \theta = 0$ on the approximate center manifold.

Example 3.11. We can show that the Hopf bifurcation occurs in the Lorenz equations by applying Thm 3.7. Consider the Lorenz equations

$$\left\{ \begin{array}{l} \dot{x} = -10x + 10y, \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = -\dfrac{8}{3}z + xy, \end{array} \right. \tag{4.7}$$

where $\rho \in \mathbf{R}$ is a parameter. This system has three fixed points:

$$(0, 0, 0), \ (a(\rho), a(\rho), \rho - 1), \ (-a(\rho), -a(\rho), \rho - 1), \tag{4.8}$$

where $a(\rho) = \sqrt{\frac{8}{3}(\rho - 1)}$. To show the presence of a bifurcation from the fixed point $(a(\rho), a(\rho), \rho - 1)$, we change the coordinate by $x \mapsto x + a(\rho)$, $y \mapsto y + a(\rho)$, $z \mapsto z + (\rho - 1)$. Then the system is rewritten as

$$\left\{ \begin{array}{l} \dot{x} = -10x + 10y, \\ \dot{y} = x - y - a(\rho)z - xz, \\ \dot{z} = a(\rho)x + a(\rho)y - \dfrac{8}{3}z + xy. \end{array} \right. \tag{4.9}$$

When $\rho = \rho_0 := 470/19$, the derivative of the right hand side of the above system at the origin has the eigenvalues

$$\alpha = -\frac{41}{3}, \ \pm\beta = \pm 4\sqrt{\frac{110}{19}}i. \tag{4.10}$$

This means that the origin is not hyperbolic fixed point, so that the Hopf bifurcation may occur. Since we are interested in the behavior of the system near the origin and near the value of the parameter $a := a(\rho_0) = \sqrt{8/3(\rho_0 - 1)}$, we change the coordinates by

$$x = \varepsilon X, \ y = \varepsilon Y, \ z = \varepsilon Z, \tag{4.11}$$

and put

$$a(\rho) = a - \varepsilon^2, \;\; a = \sqrt{8/3(\rho_0 - 1)} = \sqrt{3608/57}. \tag{4.12}$$

Substituting (4.11),(4.12) into the system (4.9), we obtain

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -a \\ a & a & -8/3 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ -XZ \\ XY \end{pmatrix} - \varepsilon^2 \begin{pmatrix} 0 \\ -Z \\ X+Y \end{pmatrix}. \tag{4.13}$$

Further, we change the coordinate so that the matrix in the right hand side of the above is brought into the diagonal matrix diag $(\beta, -\beta, \alpha)$. Then, the center subspace of the unperturbed equation of the resultant system is expressed as $N_0 = \{(X, Y, 0) \,|\, X, Y \in \mathbf{R}\}$ and the second order restricted RG equation for the system is given by

$$\begin{cases} \dot{A}/\varepsilon^2 = -\dfrac{38\sqrt{51414}A}{47779} + \dfrac{91438888520A^2B}{18481807848339} - i\dfrac{2086\sqrt{615}A}{47779} - i\dfrac{714354199417\sqrt{\frac{190}{11}}A^2B}{55445423545017}, \\ \dot{B}/\varepsilon^2 = -\dfrac{38\sqrt{51414}B}{47779} + \dfrac{91438888520AB^2}{18481807848339} + i\dfrac{2086\sqrt{615}B}{47779} + i\dfrac{714354199417\sqrt{\frac{190}{11}}AB^2}{55445423545017}. \end{cases} \tag{4.14}$$

Note that the first order restricted RG equation R_1 vanishes. On putting $A = re^{i\theta}$ and $B = \overline{A}$, Eq.(4.14) is brought into

$$\begin{cases} \dot{r} = -\varepsilon^2 p_1 r + \varepsilon^2 p_2 r^3, \\ \dot{\theta} = -\varepsilon^2 q_1 - \varepsilon^2 q_2 \theta^2, \\ p_1 = \dfrac{38\sqrt{51414}}{47779}, p_2 = \dfrac{91438888520}{18481807848339}, q_1 = \dfrac{2086\sqrt{615}}{47779}, q_2 = \dfrac{714354199417\sqrt{\frac{190}{11}}}{55445423545017}. \end{cases} \tag{4.15}$$

The equation of r has two fixed points $r = 0$ and $r = r_0 := \sqrt{p_1/p_2}$, and it is easy to show that the fixed point r_0 is unstable. This proves that the RG equation (4.14) has an unstable normally hyperbolic invariant circle. By Thm.3.7, the equation (4.9) also has an unstable periodic orbit if $a(\rho)$ is slightly smaller than $a = \sqrt{3608/57}$.

Acknowledgment

The author would like to thank Professor Toshihiro Iwai for critical reading of the manuscript and for useful comments. The author is also grateful to Assistant Professor Yoshiyuki Y. Yamaguchi for bringing his attention to the RG method.

Reference

[1] J. Carr, Applications of Centre Manifold Theory, Springer-Verlag, 1981

[2] L. Y. Chen, N. Goldenfeld, Y. Oono, Renormalization group theory for global asymptotic analysis, Phys. Rev. Lett. 73 (1994), no. 10, 1311-15

[3] L. Y. Chen, N .Goldenfeld, Y. Oono, Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory, Phys. Rev. E 54, (1996), 376-394

[4] H. Chiba, C^1 approximation of vector fields based on the renormalization group method, (to appear)

[5] H. Chiba, Simplified Renormalization Group Equations for Ordinary Differential Equations, (to appear)

[6] R. E. Lee DeVille, A. Harkin, M. Holzer, K. Josić, T. Kaper, Analysis of a renormalization group method and normal form theory for perturbed ordinary differential equations, Physica D, (preprint)

[7] S. Ei, K. Fujii, T. Kunihiro, Renormalization-group method for reduction of evolution equations; invariant manifolds and envelopes, Ann. Physics 280 (2000), no. 2, 236-298

[8] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, Indiana Univ. Math. J. 21 (1971), 193-226

[9] S. Goto, Y. Masutomi, K. Nozaki, Lie-Group Approach to Perturbative Renormalization Group Method, Prog. Theor. Phys. vol.102, No.3. (1999),pp. 471-497.

[10] M. W. Hirsch, C. C. Pugh, M. Shub, Invariant manifolds, Springer-Verlag, (1977) Lec, notes in math. 583

[11] T. Kunihiro, A geometrical formulation of the renormalization group method for global analysis, Progr. Theoret. Phys. 94 (1995), no. 4, 503-514

[12] T. Kunihiro, The renormalization-group method applied to asymptotic analysis of vector fields, Progr. Theoret. Phys. 97 (1997), no. 2, 179-200

[13] K. Nozaki, Y. Oono, Renormalization-group theoretical reduction, Phys. Rev. E, 63, 046101 (2001)

[14] S. Wiggins, Normally hyperbolic invariant manifolds in dynamical systems , Springer-Verlag , (1994)

[15] M. Ziane, On a certain renormalization group method, J. Math. Phys. 41 (2000), no. 5, 3290-3299