

Simplified Renormalization Group Method for Ordinary Differential Equations

Hayato CHIBA ¹

Department of Applied Mathematics and Physics
Kyoto University, Kyoto, 606-8501, Japan

Abstract

The renormalization group (RG) method for differential equations is one of the perturbation methods which allows one to obtain invariant manifolds of a given ordinary differential equation together with approximate solutions to it. This article investigates higher order RG equations which serve to refine an error estimate of approximate solutions obtained by the first order RG equations. It is shown that the higher order RG equation maintains the similar theorems to those provided by the first order RG equation, which are theorems on well-definedness of approximate vector fields, and on inheritance of invariant manifolds from those for the RG equation to those for the original equation, for example. Since the higher order RG equation is defined by using indefinite integrals and is not unique for the reason of the undetermined integral constants, the simplest form of RG equation is available by choosing suitable integral constants. It is shown that this simplified RG equation is sufficient to determine whether the trivial solution to time-dependent linear equations is hyperbolically stable or not, and thereby a synchronous solution of a coupled oscillators is shown to be stable.

Keyword : singular perturbation, renormalization group method, normal forms

1 Introduction

The renormalization group (RG) method for differential equations is one of the perturbation technique proposed by Chen, Goldenfeld, and Oono [1,2], which provides approximate solutions of the system of the form

$$\dot{x} = Fx + \varepsilon g_1(t, x) + \varepsilon^2 g_2(t, x) + \cdots, \quad x \in \mathbf{R}^n, \quad (1.1)$$

where $\varepsilon > 0$ is a small parameter. The RG method unifies traditional singular perturbation methods, such as the multi-scaling method [1,2], the boundary layer theory [1,2], the averaging method [3,5], the normal form theory [3,5] and the center manifold theory [2,4,6]. Kunihiro [10,11] showed that an approximate solution obtained by the RG method is an envelope of a family of curves constructed by the naive expansion. Ziane [15], DeVille *et al.* [5] and Chiba [3] gave an error estimate of approximate solutions obtained by the RG method. Chiba [3] proved that a family of approximate solutions constructed by the RG method defines a vector field which is approximate to an original vector field (ODE) in C^1 topology. Further, he gave a definition of the higher order RG equation, and proved that if the RG equation has a normally hyperbolic

¹E mail address : chiba@amp.i.kyoto-u.ac.jp

invariant manifolds N , the original equation also has an invariant manifold which is diffeomorphic to N .

In this paper, properties of higher order RG equations and of RG transformations are investigated in detail, although some of them, such as an error estimate of approximate solutions, well-definedness of approximate vector fields, existence of invariant manifolds and inheritance of symmetries, are proved in Chiba [3]. It is to be noted that the higher order RG equation and the RG transformation are *not* uniquely determined because of the indefiniteness of integral constants in the integrals in the definitions of them. This non-uniqueness has already been seen in the normal form theory [13], although its origin is not integral constants. In general, for a given vector field, many kinds of normal forms are possible, and there exist many coordinate transformations which bring the original vector field into the respective normal forms. The simplest form among them is called hypernormal form or simplified normal form [12,13].

Our purpose in the present paper is to define and derive the simplified RG equation in an analogous way to the hypernormal form theory. It is known that the RG equation is easier to solve than the original equation because the RG equation has larger symmetries than the original equation (Thm.3.6). The simplified RG equation proposed in this paper enables one to obtain more simpler equations than the conventional RG equation for both nonlinear and linear equations. In particular, the simplified RG equation for time-dependent linear equations of the form

$$\dot{x} = Fx + \varepsilon G_1(t)x + \varepsilon^2 G_2(t)x + \cdots, \quad x \in \mathbf{R}^n, \quad |\varepsilon| \ll 1 \quad (1.2)$$

are investigated in detail (see Sec.5 for the assumptions for matrices F and $G_i(t)$). We show that the simplified RG equation to the extent of finite order is sufficient to determine whether the trivial solution $x(t) \equiv 0$ to Eq.(1.2) is hyperbolically stable or not. This method is also useful to investigate nonlinear equations because a variational equation for a nonlinear equation is a linear equation. In Sec.5, we prove that a synchronous solution to a coupled oscillators (5.52) is stable by analyzing the simplified variational equation for the RG equation of the original equation.

This paper is organized as follows: Sec.2 presents definitions and basic facts on dynamical systems. Sec.3 gives a brief review of and main theorems on the RG method. In Secs.4 and 5, the simplified RG equation is defined, and applied to time-dependent linear equations, respectively.

2 Notations

Let f be a time independent C^∞ vector field on \mathbf{R}^n and $\varphi : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ its flow. We denote by $\varphi_t(x_0) \equiv x(t)$, $t \in \mathbf{R}$, a solution to the ODE $\dot{x} = f(x)$ through $x_0 \in \mathbf{R}^n$, which satisfies $\varphi_t \circ \varphi_s = \varphi_{t+s}$, $\varphi_0 = id_{\mathbf{R}^n}$, where $id_{\mathbf{R}^n}$ denotes the identity map of \mathbf{R}^n . For fixed $t \in \mathbf{R}$, $\varphi_t : \mathbf{R}^n \rightarrow \mathbf{R}^n$ defines a diffeomorphism of \mathbf{R}^n . We assume that φ_t is defined for all $t \in \mathbf{R}$.

For a time-dependent vector field $f(t, x)$, let $x(t, \tau, \xi)$ denote a solution to the ODE $\dot{x}(t) = f(t, x)$ through ξ at $t = \tau$, which defines a flow $\varphi : \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ by $\varphi_{t,\tau}(\xi) = x(t, \tau, \xi)$. For fixed $t, \tau \in \mathbf{R}$, $\varphi_{t,\tau} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a diffeomorphism of \mathbf{R}^n satisfying

$$\varphi_{t,t'} \circ \varphi_{t',\tau} = \varphi_{t,\tau}, \quad \varphi_{t,t} = id_{\mathbf{R}^n}. \quad (2.1)$$

Conversely, a family of diffeomorphism $\varphi_{t,\tau}$ of \mathbf{R}^n , which are C^1 with respect to t and τ , satisfying the above equality for any $t, \tau \in \mathbf{R}$ defines a time-dependent vector field on \mathbf{R}^n through

$$f(t, x) = \left. \frac{d}{d\tau} \right|_{\tau=t} \varphi_{\tau,t}(x). \quad (2.2)$$

Next theorem will be used to prove Thm.5.3. See [7],[8],[14] for the proof of Thm 2.1 and the definition of normal hyperbolicity.

Theorem 2.1. (Fenichel, [7]) Let $\mathcal{X}(\mathbf{R}^n)$ be the set of C^∞ vector fields on \mathbf{R}^n with C^1 topology. Let $f \in \mathcal{X}(\mathbf{R}^n)$ and suppose that $N \subset \mathbf{R}^n$ is a compact connected normally hyperbolic f -invariant manifold. Then, there exists a neighborhood $\mathcal{U} \subset \mathcal{X}(\mathbf{R}^n)$ of f s.t. for $\forall g \in \mathcal{U}$, there exists a normally hyperbolic g -invariant manifold $N_g \subset \mathbf{R}^n$, which is diffeomorphic to N .

3 Review of the Renormalization Group Method

In this section, we give the definition of the higher order RG equation and show how to construct approximate solutions on the RG method. Four fundamental theorems on the RG method will be given, all of whose proofs and ideas are shown in Chiba [3].

Let F be a diagonalizable $n \times n$ matrix all of whose eigenvalues lie on the imaginary axis and $g(t, x, \varepsilon)$ a time-dependent vector field on \mathbf{R}^n which is of C^∞ class with respect to t, x and ε . Let $g(t, x, \varepsilon)$ admit a formal power series expansion in ε , $g(t, x, \varepsilon) = g_1(t, x) + \varepsilon g_2(t, x) + \varepsilon^2 g_3(t, x) + \dots$. We suppose that $g_i(t, x)$'s are periodic in $t \in \mathbf{R}$ and polynomial in x , although the results in this section still hold even if $g_i(t, x)$'s are almost periodic functions as long as the set of fourier exponents of $g_i(t, x)$'s does not have accumulation points (see Chiba [3]).

Consider an ODE

$$\begin{aligned} \dot{x} &= Fx + \varepsilon g(t, x, \varepsilon) \\ &= Fx + \varepsilon g_1(t, x) + \varepsilon^2 g_2(t, x) + \dots, \quad x \in \mathbf{R}^n, \end{aligned} \quad (3.1)$$

where $\varepsilon \in \mathbf{R}$ is a small parameter. Replacing x in (3.1) by $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$, we rewrite (3.1) as

$$\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \dots = F(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) + \sum_{i=1}^{\infty} \varepsilon^i g_i(t, x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots). \quad (3.2)$$

Expanding the right hand side of the above equation with respect to ε and equating the coefficients of each ε^i of the both sides, we obtain ODEs of x_0, x_1, x_2, \dots as

$$\dot{x}_0 = Fx_0, \quad (3.3)$$

$$\dot{x}_1 = Fx_1 + G_1(t, x_0), \quad (3.4)$$

\vdots

$$\dot{x}_i = Fx_i + G_i(t, x_0, x_1, \dots, x_{i-1}), \quad (3.5)$$

\vdots

where the inhomogeneous term G_i is a smooth function of $t, x_0, x_1, \dots, x_{i-1}$. For instance,

G_1, G_2, G_3 and G_4 are given by

$$G_1(t, x_0) = g_1(t, x_0), \quad (3.6)$$

$$G_2(t, x_0, x_1) = \frac{\partial g_1}{\partial x}(t, x_0)x_1 + g_2(t, x_0), \quad (3.7)$$

$$G_3(t, x_0, x_1, x_2) = \frac{1}{2} \frac{\partial^2 g_1}{\partial x^2}(t, x_0)x_1^2 + \frac{\partial g_1}{\partial x}(t, x_0)x_2 + \frac{\partial g_2}{\partial x}(t, x_0)x_1 + g_3(t, x_0), \quad (3.8)$$

$$G_4(t, x_0, x_1, x_2, x_3) = \frac{1}{6} \frac{\partial^3 g_1}{\partial x^3}(t, x_0)x_1^3 + \frac{\partial^2 g_1}{\partial x^2}(t, x_0)x_1x_2 + \frac{\partial g_1}{\partial x}(t, x_0)x_3 \\ + \frac{1}{2} \frac{\partial^2 g_2}{\partial x^2}(t, x_0)x_1^2 + \frac{\partial g_2}{\partial x}(t, x_0)x_2 + \frac{\partial g_3}{\partial x}(t, x_0)x_1 + g_4(t, x_0), \quad (3.9)$$

respectively. We can verify the equality (see Lemma A.2 of Chiba [3] for the proof)

$$\frac{\partial G_i}{\partial x_j} = \frac{\partial G_{i-1}}{\partial x_{j-1}} = \dots = \frac{\partial G_{i-j}}{\partial x_0}, \quad i > j \geq 0, \quad (3.10)$$

and it may help in deriving G_i .

We denote a solution of the unperturbed part $\dot{x}_0 = Fx_0$ by $x_0(t) = X(t)A$, where $X(t) = e^{Ft}$ is the fundamental matrix and $A \in \mathbf{R}^n$ is an initial value. With this $x_0(t)$, the equation of x_1 is written as

$$\dot{x}_1 = Fx_1 + G_1(t, X(t)A), \quad (3.11)$$

a solution to which we denote by

$$x_1 = X(t)X(\tau)^{-1}h + X(t) \int_{\tau}^t X(s)^{-1}G_1(s, X(s)A)ds, \quad (3.12)$$

where $h \in \mathbf{R}^n$ is an initial value at an initial time $\tau \in \mathbf{R}$. Define $R_1(A)$ and $h := h_{\tau}^{(1)}(A)$ by

$$R_1(A) := \lim_{t \rightarrow \infty} \frac{1}{t} \int^t X(s)^{-1}G_1(s, X(s)A)ds, \quad (3.13)$$

$$h_{\tau}^{(1)}(A) := X(\tau) \int^{\tau} (X(s)^{-1}G_1(s, X(s)A) - R_1(A))ds, \quad (3.14)$$

respectively. Since $X(s)^{-1}G_1(s, X(s)A)$ is bounded uniformly in $s \in \mathbf{R}$, one can verify that $R_1(A)$ is well-defined. In this section, we fix integral constants of the indefinite integrals \int^t in Eqs.(3.13), (3.14) arbitrarily. Note that $R_1(A)$ is independent of the integral constant, while $h_{\tau}^{(1)}(A)$ depends on it. In the next section, we choose the integral constant in Eq.(3.14) to be such an value that the RG equation is put in a simple form. With these $R_1(A)$ and $h := h_{\tau}^{(1)}(A)$, the right hand side of Eq.(3.12) is decomposed into two parts;

$$x_1 := x_1(t, \tau, A) = h_{\tau}^{(1)}(A) + X(t)R_1(A)(t - \tau). \quad (3.15)$$

Here, one part $h_{\tau}^{(1)}(A)$ is bounded uniformly in $t \in \mathbf{R}$, as is proved by using almost periodicity of $X(s)^{-1}G_1(s, X(s)A)$ (see Chiba [3]), and the other $X(t)R_1(A)(t - \tau)$ is linearly increasing in t , which is called the *secular term*. We note here that $X(t)$ is bounded in t .

In a similar manner, we solve the equations of x_2, x_3, \dots step by step. The solutions are expressed as

$$x_i := x_i(t, \tau, A) = h_t^{(i)}(A) + \left(X(t)R_i(A) + \sum_{k=1}^{i-1} (Dh_t^{(k)})_A R_{i-k}(A) \right) (t - \tau) + O((t - \tau)^2), \quad (3.16)$$

where $R_i(A)$ and $h_t^{(i)}(A)$ with $i = 2, 3, \dots$ are defined by

$$R_i(A) := \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left(X(s)^{-1} G_i(s, X(s)A, h_s^{(1)}(A), \dots, h_s^{(i-1)}(A)) - X(s)^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_A R_{i-k}(A) \right) ds, \quad (3.17)$$

$$h_t^{(i)}(A) := X(t) \int^t \left(X(s)^{-1} G_i(s, X(s)A, h_s^{(1)}(A), \dots, h_s^{(i-1)}(A)) - X(s)^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_A R_{i-k}(A) - R_i(A) \right) ds, \quad (3.18)$$

respectively, and where $(Dh_t^{(k)})_A$ is the derivative of $h_t^{(k)}(A)$ with respect to $A \in \mathbf{R}^n$. Integral constants of the indefinite integrals in Eqs.(3.17), (3.18) are fixed arbitrarily. We can prove that $h_t^{(k)}(A)$ are bounded uniformly in $t \in \mathbf{R}$. The proof of this fact and the explicit expression of the term $O((t - \tau)^2)$ in Eq.(3.16) are given in Appendix A of [3].

Now we define a *renormalized constant* $A = A(\tau)$ so that the curve $x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ defined as above is independent of τ :

$$\left. \frac{d}{d\tau} \right|_{\tau=t} \left(x_0 + \varepsilon x_1(t, \tau, A(\tau)) + \varepsilon^2 x_2(t, \tau, A(\tau)) + \dots \right) = 0.$$

This equation is called the *RG condition* and it yields an ODE of $A(t)$ as follows:

Definition 3.1. Along with $R_1(A), \dots, R_m(A)$ defined in Eqs.(3.13), (3.17), we define the *m-th order RG equation* for Eq.(3.1) to be

$$\frac{dA}{dt} = \dot{A} = \varepsilon R_1(A) + \varepsilon^2 R_2(A) + \dots + \varepsilon^m R_m(A), \quad A \in \mathbf{R}^n. \quad (3.19)$$

Using $h_t^{(1)}(A), \dots, h_t^{(m)}(A)$ defined in Eqs.(3.14), (3.18), we define the *m-th order RG transformation* $\alpha_t : \mathbf{R}^n \rightarrow \mathbf{R}^n$ to be

$$\alpha_t(A) = X(t)A + \varepsilon h_t^{(1)}(A) + \dots + \varepsilon^m h_t^{(m)}(A). \quad (3.20)$$

Remark 3.2. Since $X(t)$ is nonsingular and $h_t^{(1)}(A), \dots, h_t^{(m)}(A)$ are bounded uniformly in $t \in \mathbf{R}$, for sufficiently small $|\varepsilon|$, there exists an open set $U = U(\varepsilon)$ such that \bar{U} is compact and the restriction of α_t to U is diffeomorphism from U into \mathbf{R}^n .

In general, the infinite order RG equation $\dot{A} = \sum_{k=1}^{\infty} \varepsilon^k R_k(A)$ and the infinite order RG transformation $\alpha_t(A) = X(t)A + \sum_{k=1}^{\infty} \varepsilon^k h_t^{(k)}(A)$ are *formal* power series in ε . In this paper, we consider only the finite order RG equations.

Now we are in a position to construct approximate solutions of Eq.(3.1) by the RG method. Let $A = A(t, t_0, \xi)$ be a solution of the m -th order RG equation (3.19) whose initial time is t_0 and whose initial value is $\xi \in \mathbf{R}^n$. Define a curve $\tilde{x}(t) = \tilde{x}(t, t_0, \xi)$ by

$$\tilde{x}(t) = \alpha_t(A(t, t_0, \xi)) = X(t)A(t, t_0, \xi) + \varepsilon h_t^{(1)}(A(t, t_0, \xi)) + \cdots + \varepsilon^m h_t^{(m)}(A(t, t_0, \xi)). \quad (3.21)$$

Then, the curve $\tilde{x}(t)$ gives an approximate solution of Eq.(3.1).

Fundamental theorems on the RG method are listed below. All proofs are included in Chiba [3].

Theorem 3.3. (Approximation of Vector Fields)

Let φ_t^{RG} be the flow of the m -th order RG equation for Eq.(3.1) and α_t the m -th order RG transformation. Then, there exists a positive constant ε_0 such that the following holds for $\forall |\varepsilon| < \varepsilon_0$:

(i) The map

$$\Phi_{t,t_0} := \alpha_t \circ \varphi_{t-t_0}^{RG} \circ \alpha_{t_0}^{-1} : \alpha_{t_0}(U) \rightarrow \mathbf{R}^n \quad (3.22)$$

defines a local flow on $\alpha_{t_0}(U)$ for each $t_0 \in \mathbf{R}$, where $U = U(\varepsilon)$ is an open set on which α_{t_0} is a diffeomorphism (see Rem.3.2). This Φ_{t,t_0} induces a time-dependent vector field F_ε through

$$F_\varepsilon(t, x) := \left. \frac{d}{da} \right|_{a=t} \Phi_{a,t}(x), \quad x \in \alpha_t(U), \quad (3.23)$$

and its integral curves are given by the approximate solutions $\tilde{x}(t)$ defined by Eq.(3.21).

(ii) There exists a time-dependent vector field $\tilde{F}_\varepsilon(t, x)$ such that

$$F_\varepsilon(t, x) = Fx + \varepsilon g_1(t, x) + \cdots + \varepsilon^m g_m(t, x) + \varepsilon^{m+1} \tilde{F}_\varepsilon(t, x), \quad (3.24)$$

where $\tilde{F}_\varepsilon(t, x)$ and its derivative are bounded uniformly in $t \in \mathbf{R}$ and bounded as $\varepsilon \rightarrow 0$. In particular, the vector field $F_\varepsilon(t, x)$ is close to the original vector field $Fx + \varepsilon g_1(t, x) + \cdots$ within $O(\varepsilon^{m+1})$.

Theorem 3.4. (Error Estimate)

There exist positive constants ε_0, C, T , and a compact subset $V = V(\varepsilon) \subset \mathbf{R}^n$ including the origin such that for $\forall |\varepsilon| < \varepsilon_0$, every solution $x(t)$ of Eq.(3.1) and $\tilde{x}(t)$ defined by Eq.(3.21) with $x(0) = \tilde{x}(0) \in V$ satisfy the inequality

$$\|x(t) - \tilde{x}(t)\| < C\varepsilon^m, \quad \text{for } 0 \leq t \leq T/\varepsilon. \quad (3.25)$$

The following two theorems are concerned with an autonomous equation

$$\dot{x} = Fx + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots, \quad (3.26)$$

where $\varepsilon \in \mathbf{R}$ is a small parameter, F is a diagonalizable $n \times n$ matrix all of whose eigenvalues lie on the imaginary axis, and $g_i(x)$ are C^∞ vector fields on \mathbf{R}^n .

Theorem 3.5. (Existence of Invariant Manifolds)

Let $\varepsilon^k R_k(A)$ be a first non zero term in the RG equation (3.19). If the vector field $\varepsilon^k R_k(A)$ has a normally hyperbolic invariant manifold N , then the original equation (3.1) also has a normally hyperbolic invariant manifold N_ε , which is diffeomorphic to N , for sufficiently small $|\varepsilon|$. In particular, the stability of N_ε coincides with that of N .

Theorem 3.6. (Inheritance of the Symmetries)

- (i) If vector fields Fx and $g_1(x), g_2(x), \dots$ are invariant under the action of a Lie group G , then the m -th order RG equation is also invariant under the action of G .
(ii) The m -th order RG equation commutes with the linear vector field Fx with respect to Lie bracket product. Equivalently, each $R_i(A)$, $i = 1, 2, \dots$, satisfies

$$X(t)R_i(A) = R_i(X(t)A), \quad A \in \mathbf{R}^n. \quad (3.27)$$

In the rest of this section, we apply these theorems to several equations.

Example 3.7. Consider the perturbed harmonic oscillator

$$\ddot{x} + x + \varepsilon x^3 = 0, \quad x \in \mathbf{R}. \quad (3.28)$$

It is convenient to identify \mathbf{R}^2 with \mathbf{C} by introducing a complex variable z through $x = z + \bar{z}$, $\dot{x} = i(z - \bar{z})$. Then, the above equation is rewritten as

$$\begin{cases} \dot{z} = iz + \frac{i\varepsilon}{2}(z + \bar{z})^3, \\ \dot{\bar{z}} = -i\bar{z} - \frac{i\varepsilon}{2}(z + \bar{z})^3. \end{cases} \quad (3.29)$$

In this case, the matrix F and the vector-valued functions G_1, G_2 defined by Eqs.(3.6),(3.7), respectively, are given by

$$F = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad G_1(z_0) = \frac{i}{2} \begin{pmatrix} (z_0 + \bar{z}_0)^3 \\ -(z_0 + \bar{z}_0)^3 \end{pmatrix}, \quad G_2(z_0, z_1) = \frac{3i}{2} \begin{pmatrix} (z_0 + \bar{z}_0)^2(z_1 + \bar{z}_1) \\ -(z_0 + \bar{z}_0)^2(z_1 + \bar{z}_1) \end{pmatrix}. \quad (3.30)$$

To obtain a first order approximate solution, we calculate $R_1(A)$ and $h_t^{(1)}(A)$ with $A \in \mathbf{C}$ as

$$\begin{aligned} R_1(A) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \begin{pmatrix} e^{-is} & 0 \\ 0 & e^{is} \end{pmatrix} G_1(e^{is}A) ds = \frac{3i}{2} \begin{pmatrix} A^2\bar{A} \\ -A\bar{A}^2 \end{pmatrix}, \\ h_t^{(1)}(A) &= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \int^t \left(\begin{pmatrix} e^{-is} & 0 \\ 0 & e^{is} \end{pmatrix} G_1(e^{is}A) - \frac{3i}{2} \begin{pmatrix} A^2\bar{A} \\ -A\bar{A}^2 \end{pmatrix} \right) ds \\ &= \begin{pmatrix} \frac{1}{4}A^3e^{3it} - \frac{3}{4}A\bar{A}^2e^{-it} - \frac{1}{8}A^3e^{-3it} \\ -\frac{1}{8}A^3e^{3it} - \frac{3}{4}A^2\bar{A}e^{it} + \frac{1}{4}A^3e^{-3it} \end{pmatrix}, \end{aligned} \quad (3.31)$$

where we have chosen the integral constant to be zero. Therefore, the first order RG equation is expressed as

$$\dot{A} = \varepsilon \frac{3i}{2} |A|^2 A, \quad A \in \mathbf{C}. \quad (3.33)$$

It is solved by

$$A(t) := A(t, a, \theta) = \frac{1}{2} a \exp i \left(\frac{3\varepsilon}{8} a^2 t + \theta \right), \quad (3.34)$$

where $a, \theta \in \mathbf{R}$ are arbitrary constants. A first order approximate solution in complex variable is written as

$$\begin{aligned}\bar{z}(t) &= e^{it}A(t) + \varepsilon \left(\frac{1}{4}A(t)^3 e^{3it} - \frac{3}{4}A(t)\overline{A(t)}^2 e^{-it} - \frac{1}{8}\overline{A(t)}^3 e^{-3it} \right) \\ &= \frac{1}{2}a \exp i \left(t + \frac{3\varepsilon}{8}a^2t + \theta \right) \\ &\quad + \varepsilon \left(\frac{\varepsilon}{32}a^3 \exp i \left(3t + \frac{9\varepsilon}{8}a^2t + 3\theta \right) - \frac{3\varepsilon}{32}a^3 \exp i \left(-t - \frac{3\varepsilon}{8}a^2t - \theta \right) - \frac{\varepsilon}{64}a^3 \exp i \left(-3t - \frac{9\varepsilon}{8}a^2t - 3\theta \right) \right).\end{aligned}$$

Finally, a first order approximate solution of Eq.(3.28) is given by

$$\begin{aligned}\bar{x}(t) &= \bar{z}(t) + \overline{\bar{z}(t)} \\ &= a \cos \left(t + \frac{3\varepsilon}{8}a^2t + \theta \right) + \frac{\varepsilon}{32}a^3 \cos \left(3t + \frac{9\varepsilon}{8}a^2t + 3\theta \right) - \frac{3\varepsilon}{16}a^3 \cos \left(t + \frac{3\varepsilon}{8}a^2t + \theta \right).\end{aligned}\quad (3.35)$$

Next, to find a second order approximate solution, we calculate (3.17) and (3.18) to obtain, respectively, $R_2(A)$ and $h_t^{(2)}(A)$,

$$R_2(A) = -\frac{51}{16}i \begin{pmatrix} A^3 \overline{A}^2 \\ -A^2 \overline{A}^3 \end{pmatrix}, \quad (3.36)$$

$$h_t^{(2)}(A) = \begin{pmatrix} \frac{3}{64}A^5 e^{5it} - \frac{15}{16}A^4 \overline{A} e^{3it} + \frac{69}{32}A^2 \overline{A}^3 e^{-it} + \frac{21}{64}A \overline{A}^4 e^{-3it} - \frac{1}{32} \overline{A}^5 e^{-5it} \\ -\frac{1}{32}A^5 e^{5it} + \frac{21}{64}A^4 \overline{A} e^{3it} + \frac{69}{32}A^3 \overline{A}^2 e^{-it} - \frac{15}{16}A \overline{A}^4 e^{-3it} + \frac{3}{64} \overline{A}^5 e^{-5it} \end{pmatrix}. \quad (3.37)$$

Therefore, the second order RG equation is expressed as

$$\dot{A} = \varepsilon \frac{3i}{2}|A|^2 A - \varepsilon^2 \frac{51}{16}i|A|^4 A. \quad (3.38)$$

It is solved by

$$A(t) := A(t, a, \theta) = \frac{1}{2}a \exp i \left(\frac{3}{8}\varepsilon a^2 t - \frac{51}{256}\varepsilon^2 a^4 t + \theta \right), \quad (3.39)$$

where $a, \theta \in \mathbf{R}$ are arbitrary constants. With this $A(t)$, a second order approximate solution in complex variables is written as

$$\begin{aligned}\bar{z}(t) &= e^{it}A(t) + \varepsilon \left(\frac{1}{4}A(t)^3 e^{3it} - \frac{3}{4}A(t)\overline{A(t)}^2 e^{-it} - \frac{1}{8}\overline{A(t)}^3 e^{-3it} \right) \\ &\quad + \varepsilon^2 \left(\frac{3}{64}A(t)^5 e^{5it} - \frac{15}{16}A(t)^4 \overline{A(t)} e^{3it} + \frac{69}{32}A(t)^2 \overline{A(t)}^3 e^{-it} + \frac{21}{64}A(t)\overline{A(t)}^4 e^{-3it} - \frac{1}{32}\overline{A(t)}^5 e^{-5it} \right).\end{aligned}$$

Thus a second order approximate solution of Eq.(3.28) is given by

$$\begin{aligned}\bar{x}(t) &= \bar{z}(t) + \overline{\bar{z}(t)} \\ &= a \cos \left(t + \frac{3}{8}\varepsilon a^2 t - \frac{51}{256}\varepsilon^2 a^4 t + \theta \right) \\ &\quad + \varepsilon \left(\frac{a^3}{32} \cos \left(3t + \frac{9}{8}\varepsilon a^2 t - \frac{153}{256}\varepsilon^2 a^4 t + 3\theta \right) - \frac{3a^2}{16} \cos \left(t + \frac{3}{8}\varepsilon a^2 t - \frac{51}{256}\varepsilon^2 a^4 t + \theta \right) \right) \\ &\quad + \varepsilon^2 \left(\frac{a^5}{1024} \cos \left(5t + \frac{15}{8}\varepsilon a^2 t - \frac{255}{256}\varepsilon^2 a^4 t + 5\theta \right) - \frac{39a^5}{1024} \cos \left(3t + \frac{9}{8}\varepsilon a^2 t - \frac{153}{256}\varepsilon^2 a^4 t + 3\theta \right) \right. \\ &\quad \left. + \frac{69a^5}{512} \cos \left(t + \frac{3}{8}\varepsilon a^2 t - \frac{51}{256}\varepsilon^2 a^4 t + \theta \right) \right).\end{aligned}\quad (3.41)$$

Numerical solution of Eq.(3.28) and two approximate solutions Eq.(3.35) and Eq.(3.41) are presented as Fig.1 for comparison. The solid curve denotes an exact solution of Eq.(3.28) for $\varepsilon = 0.1$ with $x(0) = 0.985, \dot{x}(0) = 0$. The dashed and the dotted curves are the first order approximate solution (3.35) and the second order approximate solution (3.41) for $\varepsilon = 0.1, a = 1, \theta = 0$, respectively. In this case, the first order approximate solution $\tilde{x}(t)$ satisfies $\tilde{x}(0) \sim 0.9844, \dot{\tilde{x}}(0) = 0$ and the second order approximate solution $\tilde{x}(t)$ satisfies $\tilde{x}(0) \sim 0.9854, \dot{\tilde{x}}(0) = 0$. When $0 \leq t \leq 20$, three curves almost overlap with one another. However when $80 \leq t \leq 100$, the second order approximate solution is more close to the exact solution than the first order approximate solution.

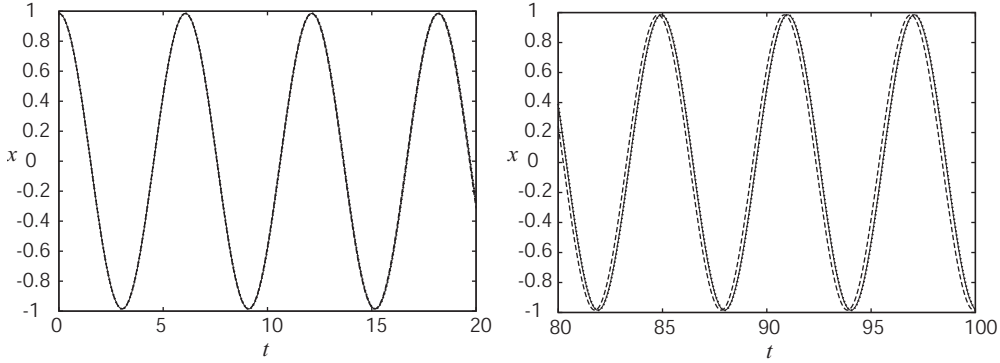


Fig. 1: The solid line denotes an exact solution of Eq.(3.28), the dashed lines denotes the first order approximate solution, and the dotted line denotes the second order approximate solution.

Example 3.8. Consider the system on \mathbf{R}^2

$$\begin{cases} \dot{x} = y - x^3 + \varepsilon x, \\ \dot{y} = -x. \end{cases} \quad (3.42)$$

Changing the coordinates by $(x, y) = (\varepsilon X, \varepsilon Y)$ and substituting them into the above system, we obtain

$$\begin{cases} \dot{X} = Y + \varepsilon X - \varepsilon^2 X^3, \\ \dot{Y} = -X. \end{cases} \quad (3.43)$$

We introduce a complex variable $z \in \mathbf{C}$ by $X = z + \bar{z}, Y = i(z - \bar{z})$. Then, the above system is rewritten as

$$\begin{cases} \dot{z} = iz + \frac{\varepsilon}{2}(z + \bar{z}) - \frac{\varepsilon^2}{2}(z + \bar{z})^3, \\ \dot{\bar{z}} = -i\bar{z} + \frac{\varepsilon}{2}(z + \bar{z}) - \frac{\varepsilon^2}{2}(z + \bar{z})^3. \end{cases} \quad (3.44)$$

For this system, F, g_1, g_2 in Eq.(3.1) are expressed, respectively, as

$$F = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_1(z, \bar{z}) = \frac{1}{2} \begin{pmatrix} z + \bar{z} \\ z + \bar{z} \end{pmatrix}, \quad g_2(z, \bar{z}) = -\frac{1}{2} \begin{pmatrix} (z + \bar{z})^3 \\ (z + \bar{z})^3 \end{pmatrix}. \quad (3.45)$$

The *second* order RG equation for this system is given by

$$\dot{A} = \frac{\varepsilon}{2}A + \varepsilon^2 \left(-\frac{i}{8}A - \frac{3}{2}|A|^2 A \right), \quad A \in \mathbf{C}. \quad (3.46)$$

Introduce the polar coordinates by $A = re^{i\theta}$. Then, the above RG equation is brought into

$$\begin{cases} \dot{r} = \frac{\varepsilon r}{2}(1 - 3\varepsilon r^2), \\ \dot{\theta} = -\frac{\varepsilon^2}{8}. \end{cases} \quad (3.47)$$

It is easy to show that this RG equation has a stable periodic orbit $r = \sqrt{1/3\varepsilon}$ if $\varepsilon > 0$. However, we can not apply Thm.3.5 to conclude that the original equation (3.43) also has a stable periodic orbit because the RG equation (3.46) does not satisfy the condition of Thm.3.5. To handle this problem, we introduce a new variable ε_0 so that $\varepsilon_0(t) \equiv \varepsilon$ may be a solution to Eq.(3.44) extended as

$$\begin{cases} \dot{z} = iz + \frac{\varepsilon}{2}(z + \bar{z} - \varepsilon_0(z + \bar{z})^3), \\ \dot{\bar{z}} = -i\bar{z} + \frac{\varepsilon}{2}(z + \bar{z} - \varepsilon_0(z + \bar{z})^3), \\ \dot{\varepsilon}_0 = 0. \end{cases} \quad (3.48)$$

In this case, F, g_1, g_2 in Eq.(3.1) are put in the form

$$F = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_1(z, \bar{z}, \varepsilon_0) = \frac{1}{2} \begin{pmatrix} z + \bar{z} - \varepsilon_0(z + \bar{z})^3 \\ z + \bar{z} - \varepsilon_0(z + \bar{z})^3 \\ 0 \end{pmatrix}, \quad g_2(z, \bar{z}, \varepsilon_0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.49)$$

The *first* order RG equation for this system is given by

$$\begin{cases} \dot{A} = \frac{\varepsilon}{2}(A - 3\varepsilon_0|A|^2A), \\ \dot{\varepsilon}_0 = 0. \end{cases} \quad (3.50)$$

Putting $A = re^{i\theta}$ provides

$$\begin{cases} \dot{r} = \frac{\varepsilon r}{2}(1 - 3\varepsilon_0 r^2), \\ \dot{\theta} = 0. \end{cases} \quad (3.51)$$

Again it is easy to verify that this RG equation has a stable periodic orbit $r = \sqrt{1/3\varepsilon_0}$ if $\varepsilon_0 > 0$. Thm.3.5 is now applicable, showing that the original system (3.42) also has a stable periodic orbit if $\varepsilon > 0$.

Example 3.9. Consider the system on \mathbf{R}^2

$$\begin{cases} \dot{x} = y + y^2, \\ \dot{y} = -x + \varepsilon^2 y - xy + y^2. \end{cases} \quad (3.52)$$

Changing the coordinates by $(x, y) = (\varepsilon X, \varepsilon Y)$ yields

$$\begin{cases} \dot{X} = Y + \varepsilon Y^2, \\ \dot{Y} = -X + \varepsilon(Y^2 - XY) + \varepsilon^2 Y. \end{cases} \quad (3.53)$$

We introduce a complex variable z by $X = z + \bar{z}$, $Y = i(z - \bar{z})$. Then, the above system is rewritten as

$$\begin{cases} \dot{z} = iz + \frac{\varepsilon}{2}(i(z - \bar{z})^2 - 2z^2 + 2z\bar{z}) + \frac{\varepsilon^2}{2}(z - \bar{z}), \\ \dot{\bar{z}} = -i\bar{z} + \frac{\varepsilon}{2}(-i(z - \bar{z})^2 - 2\bar{z}^2 + 2z\bar{z}) - \frac{\varepsilon^2}{2}(z - \bar{z}). \end{cases} \quad (3.54)$$

For this system, $R_1(A)$ defined by Eq.(3.13) vanishes and the second order RG equation is given by

$$\dot{A} = \frac{1}{2}\varepsilon^2(A - 3|A|^2A - \frac{16i}{3}|A|^2A). \quad (3.55)$$

Putting $A = re^{i\theta}$ results in

$$\begin{cases} \dot{r} = \frac{1}{2}\varepsilon^2r(1 - 3r^2), \\ \dot{\theta} = -\frac{8}{3}\varepsilon^2r^2. \end{cases} \quad (3.56)$$

It is easy to verify that this RG equation has a stable periodic orbit $r = \sqrt{1/3}$ if $\varepsilon > 0$. Since $R_1(A) = 0$, Thm.3.5 implies that the original system (3.52) also has a stable periodic orbit if $\varepsilon > 0$.

Note that all RG equations in Examples 3.7 to 3.9 are invariant under the action of the rotation group on \mathbf{R}^2 , and RG equations split into equations of radius r and of angle θ . This fact results from Thm.3.6.

4 Simplified RG equation

Recall that the definitions of the functions $R_i(A)$ and $h_i^{(i)}(A)$ given in Eqs.(3.13, 14, 17, 18) include the indefinite integrals and we have left the integral constants undetermined in the previous section. In this section, we use the integral constants to simplify the RG equation.

For a given equation (3.1), we have defined the RG equation

$$\dot{A} = \varepsilon R_1(A) + \cdots + \varepsilon^m R_m(A), \quad (4.1)$$

and the RG transformation

$$\alpha_t(A) = X(t)A + \varepsilon h_t^{(1)}(A) + \cdots + \varepsilon^m h_t^{(m)}(A). \quad (4.2)$$

Put $A = X(t)^{-1}x$. Then, the RG equation (4.1) is rewritten as

$$\dot{x} = Fx + \varepsilon X(t)R_1(X(t)^{-1}x) + \cdots + \varepsilon^m X(t)R_m(X(t)^{-1}x). \quad (4.3)$$

Note that if the original equation (3.1) is autonomous, the above equation is reduced to an equation

$$\dot{x} = Fx + \varepsilon R_1(x) + \cdots + \varepsilon^m R_m(x), \quad (4.4)$$

because of Thm.3.6 (ii). We apply the RG method with slight modification to Eq.(4.3). For Eq.(4.3), we define functions $\tilde{R}_i(A)$ and $\tilde{h}_t^{(i)}(A)$, respectively, by

$$\tilde{R}_1(A) := \lim_{t \rightarrow \infty} \frac{1}{t} \int^t X(s)^{-1} G_1(s, X(s)A) ds, \quad (4.5)$$

$$\tilde{h}_t^{(1)}(A) := X(t) \int^t (X(s)^{-1} G_1(s, X(s)A) - \tilde{R}_1(A)) ds + X(t) B_1(A), \quad (4.6)$$

and

$$\begin{aligned} \widetilde{R}_i(A) &:= \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left(X(s)^{-1} G_i(s, X(s)A, \widetilde{h}_s^{(1)}(A), \dots, \widetilde{h}_s^{(i-1)}(A)) \right. \\ &\quad \left. - X(s)^{-1} \sum_{k=1}^{i-1} (D\widetilde{h}_s^{(k)})_A \widetilde{R}_{i-k}(A) \right) ds, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \widetilde{h}_t^{(i)}(A) &:= X(t) \int^t \left(X(s)^{-1} G_i(s, X(s)A, \widetilde{h}_s^{(1)}(A), \dots, \widetilde{h}_s^{(i-1)}(A)) \right. \\ &\quad \left. - X(s)^{-1} \sum_{k=1}^{i-1} (D\widetilde{h}_s^{(k)})_A \widetilde{R}_{i-k}(A) - \widetilde{R}_i(A) \right) ds + X(t) B_i(A), \end{aligned} \quad (4.8)$$

for $i = 2, 3, \dots$, where $B_i(A)$, $i = 1, 2, \dots$, are arbitrary vector fields on \mathbf{R}^n which come from integral constants of the indefinite integrals in Eqs.(4.6), (4.8). The function G_i is defined in a similar manner to that in the previous section. For example, G_1 to G_4 are given by Eq.(3.6) to Eq.(3.9) in which $g_i(t, x)$ is replaced by $X(t)R_i(X(t)^{-1}x)$. With these $\widetilde{R}_i(A)$ and $\widetilde{h}_t^{(i)}(A)$, we define a new RG equation and a new RG transformation for Eq.(4.3) by

$$\dot{A} = \varepsilon \widetilde{R}_1(A) + \dots + \varepsilon^m \widetilde{R}_m(A), \quad (4.9)$$

$$\widetilde{\alpha}_t(A) = X(t)A + \varepsilon \widetilde{h}_t^{(1)}(A) + \dots + \varepsilon^m \widetilde{h}_t^{(m)}(A), \quad (4.10)$$

respectively. It is easy to verify that Thm.3.3 to Thm.3.5 hold for these new RG equation and new RG transformation, because the proof of them are independent of the integral constants in Eqs.(3.14), (3.18). In particular, like Eq.(3.24), the equality

$$\left. \frac{d}{da} \right|_{a=t} \widetilde{\alpha}_a \circ \widetilde{\varphi}_{a-t}^{RG} \circ \widetilde{\alpha}_t^{-1}(x) = Fx + \varepsilon X(t)R_1(X(t)^{-1}x) + \dots + \varepsilon^m X(t)R_m(X(t)^{-1}x) + O(\varepsilon^{m+1}) \quad (4.11)$$

holds, where $\widetilde{\varphi}_t^{RG}$ is the flow of Eq.(4.9). However, in general, Thm.3.6 fails to hold since $B_i(A)$'s depend on $A \in \mathbf{R}^n$.

We now calculate the right hand sides of Eq.(4.5) to Eq.(4.8) to look into relations between $R_i(A)$, $h_t^{(i)}(A)$ and $\widetilde{R}_i(A)$, $\widetilde{h}_t^{(i)}(A)$. Since $G_1(t, x_0) = X(t)R_1(X(t)^{-1}x_0)$, $\widetilde{R}_1(A)$ and $\widetilde{h}_t^{(1)}(A)$ are calculated as

$$\widetilde{R}_1(A) = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t X(s)^{-1} X(s) R_1(X(s)^{-1} X(s) A) ds = R_1(A), \quad (4.12)$$

$$\begin{aligned} \widetilde{h}_t^{(1)}(A) &= X(t) \int^t \left(X(s)^{-1} X(s) R_1(X(s)^{-1} X(s) A) - \widetilde{R}_1(A) \right) ds + X(t) B_1(A) \\ &= X(t) B_1(A), \end{aligned} \quad (4.13)$$

respectively. Since

$$G_2(t, x_0, x_1) = X(t) \frac{\partial R_1}{\partial x_0} (X(t)^{-1} x_0) x_1 + X(t) R_2(X(t)^{-1} x_0), \quad (4.14)$$

$\widetilde{R}_2(A)$ is calculated as

$$\begin{aligned} \widetilde{R}_2(A) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int^t X(s)^{-1} \left(X(s) \frac{\partial R_1}{\partial A} (A) X(t)^{-1} \widetilde{h}_s^{(1)}(A) + X(s) R_2(A) - (D\widetilde{h}_s^{(1)})_A \widetilde{R}_1(A) \right) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left(\frac{\partial R_1}{\partial A} (A) B_1(A) + R_2(A) - (DB_1)_A R_1(A) \right) ds \end{aligned} \quad (4.15)$$

$$= R_2(A) - [B_1, R_1](A), \quad (4.16)$$

where $[B_1, R_1](A)$ is the commutator of vector fields, which is defined by

$$[B_1, R_1](A) := \frac{\partial B_1}{\partial A}(A)R_1(A) - \frac{\partial R_1}{\partial A}(A)B_1(A). \quad (4.17)$$

Similar calculation shows that

$$\widetilde{R}_3 = R_3 - [B_1, R_2] - [B_2, R_1] + \frac{\partial B_1}{\partial A}[B_1, R_1] + \frac{1}{2} \frac{\partial^2 R_1}{\partial A^2} B_1^2, \quad (4.18)$$

$$\begin{aligned} \widetilde{R}_4 = & R_4 - [B_1, R_3] - [B_2, R_2] - [B_3, R_1] \\ & + \frac{1}{6} \frac{\partial^3 R_1}{\partial A^3} B_1^3 + \frac{\partial^2 R_1}{\partial A^2} B_1 B_2 + \frac{1}{2} \frac{\partial^2 R_2}{\partial A^2} B_1^2 + \frac{\partial B_2}{\partial A} [B_1, R_1] \\ & - \frac{\partial B_1}{\partial A} \left(\frac{1}{2} \frac{\partial^2 R_1}{\partial A^2} B_1^2 + \frac{\partial B_1}{\partial A} [B_1, R_1] - [B_1, R_2] - [B_2, R_1] \right), \end{aligned} \quad (4.19)$$

where the argument A is omitted for notational simplicity.

Lemma 4.1. The equalities $\widetilde{h}_i^{(i)}(A) = X(t)B_i(A)$ hold for $i = 1, 2, \dots$.

Proof. We prove the lemma by induction. Assume that $\widetilde{h}_i^{(k)}(A) = X(t)B_k(A)$ for $k = 1, \dots, i-1$. At first, we show that the integrand in Eq.(4.7) is independent of s . By the assumption, the second term of the integrand in Eq.(4.7) is clearly independent of s . Next, note that a function $G_i(s, x_0, \dots, x_{i-1})$ is a linear combination of functions of the form

$$\frac{\partial^j}{\partial x_0^j} \left(X(s)R_l(X(s)^{-1}x_0) \right) x_1^{j_1} x_2^{j_2} \cdots x_{i-1}^{j_{i-1}}, \quad j_1 + j_2 + \cdots + j_{i-1} = j. \quad (4.20)$$

Thus, $G_i(s, X(s)A, X(s)B_1, \dots, X(s)B_{i-1})$ is a linear combination of functions of the form

$$X(s) \frac{\partial^j R_l}{\partial A^j}(A) B_1^{j_1} B_2^{j_2} \cdots B_{i-1}^{j_{i-1}}. \quad (4.21)$$

This proves that the first term of the integrand in Eq.(4.7) is independent of s . Therefore $\widetilde{R}_i(A)$ is equal to the integrand in Eq.(4.7) in which s is replaced by t . This and Eq.(4.8) are put together to prove that

$$\widetilde{h}_i^{(i)}(A) = X(t) \int^t (\widetilde{R}_i(A) - \widetilde{R}_i(A)) ds + X(t)B_i(A) = X(t)B_i(A). \quad (4.22)$$

■

While we have written out $\widetilde{R}_1(A), \dots, \widetilde{R}_4(A)$, we can calculate $\widetilde{R}_1(A), \widetilde{R}_2(A), \dots$, systematically in the following manner. By virtue of Lem.4.1, the RG transformation (4.10) is written as

$$\widetilde{\alpha}_t(A) = X(t)A + \varepsilon X(t)B_1(A) + \cdots + \varepsilon^m X(t)B_m(A). \quad (4.23)$$

Put $x = \widetilde{\alpha}_t(A)$ and substitute it into Eq.(4.11). Then we obtain

$$\frac{d}{da} \Big|_{a=t} \widetilde{\alpha}_a \circ \widetilde{\varphi}_{a-t}^{RG}(A) = F \widetilde{\alpha}_t(A) + \varepsilon X(t)R_1(X(t)^{-1} \widetilde{\alpha}_t(A)) + \cdots + \varepsilon^m X(t)R_m(X(t)^{-1} \widetilde{\alpha}_t(A)) + O(\varepsilon^{m+1}). \quad (4.24)$$

The left hand side of the above is calculated as

$$\begin{aligned} \frac{d}{da} \Big|_{a=t} \widetilde{\alpha}_a \circ \widetilde{\varphi}_{a-t}^{RG}(A) &= \frac{d\widetilde{\alpha}_t}{dt}(A) + (D\widetilde{\alpha}_t)_A \frac{d}{da} \Big|_{a=t} \widetilde{\varphi}_{a-t}^{RG}(A) \\ &= F\widetilde{\alpha}_t(A) + X(t) \left(\text{id} + \varepsilon \frac{\partial B_1}{\partial A} + \cdots + \varepsilon^m \frac{\partial B_m}{\partial A} \right) \left(\varepsilon \widetilde{R}_1(A) + \cdots + \varepsilon^m \widetilde{R}_m(A) \right), \end{aligned}$$

where id is the identity matrix. Hence, Eq.(4.24) is brought into

$$\varepsilon \widetilde{R}_1(A) + \cdots + \varepsilon^m \widetilde{R}_m(A) = \left(\text{id} + \varepsilon \frac{\partial B_1}{\partial A} + \cdots + \varepsilon^m \frac{\partial B_m}{\partial A} \right)^{-1} \sum_{k=1}^m \varepsilon^k R_k(X(t)^{-1} \widetilde{\alpha}_t(A)) + O(\varepsilon^{m+1}). \quad (4.25)$$

To expand the right hand side of the above, we use the following equalities

$$\begin{aligned} \left(\text{id} + \varepsilon \frac{\partial B_1}{\partial A} + \cdots + \varepsilon^m \frac{\partial B_m}{\partial A} \right)^{-1} &= \text{id} + \sum_{k=1}^{\infty} (-1)^k \left(\varepsilon \frac{\partial B_1}{\partial A} + \cdots + \varepsilon^m \frac{\partial B_m}{\partial A} \right)^k, \quad (4.26) \\ R_k(X(t)^{-1} \widetilde{\alpha}_t(A)) &= R_k(A + \varepsilon B_1(A) + \cdots + \varepsilon^m B_m(A)) \\ &= R_k(A) + \sum_{l=1}^{\infty} \frac{1}{l!} \frac{\partial^l R_k}{\partial A^l}(A) (\varepsilon B_1(A) + \cdots + \varepsilon^m B_m(A))^l. \quad (4.27) \end{aligned}$$

Substitution of Eq.(4.26) and Eq.(4.27) into Eq.(4.25) yields $\widetilde{R}_i(A)$ as the coefficients of ε^i in the right hand side of Eq.(4.25). Consequently, we obtain the following lemmas.

Lemma 4.2. Each $\widetilde{R}_k(A)$, $k = 3, 4, \dots$, is of the form

$$\widetilde{R}_k(A) = R_k(A) + P_k(R_1, \dots, R_{k-1}, B_1, \dots, B_{k-2})(A) - [B_{k-1}, R_1](A), \quad (4.28)$$

where P_k is a function of $R_1, \dots, R_{k-1}, B_1, \dots, B_{k-2}$.

Lemma 4.3. Suppose that every $R_k(A)$, $k = 1, 2, \dots$, satisfies $R_k(X(t)A) = X(t)R_k(A)$. If every $B_k(A)$, $k = 1, 2, \dots$, satisfies $B_k(X(t)A) = X(t)B_k(A)$, then $\widetilde{R}_k(A)$, $k = 1, 2, \dots$, also satisfies $\widetilde{R}_k(X(t)A) = X(t)\widetilde{R}_k(A)$.

Now we suppose that we can determine $B_1(A), \dots, B_{k-2}(A)$ appropriately so that $\widetilde{R}_2, \dots, \widetilde{R}_{k-1}$ may take a simple form in some sense. Then, a suitable choice of $B_{k-1}(A)$ may bring $\widetilde{R}_k(A)$ into a simple form through Eq.(4.28).

Let $\mathcal{P}^j(\mathbf{R}^n)$ be the set of homogenous polynomial vector fields of degree j on \mathbf{R}^n . In what follows, to simplify $\widetilde{R}_i(A)$'s systematically, we start with the case where $g_i(t, x)$ in Eq.(3.1) is a homogenous polynomial vector fields of degree $i + 1$ with respect to x . In this case, it is easy to verify that each term of the RG equation (4.1) is also a homogeneous polynomial, $R_i \in \mathcal{P}^{i+1}(\mathbf{R}^n)$ for any i . Note that if $g_1(t, x) \in \mathcal{P}^l(\mathbf{R}^n)$ for some positive integer $l \neq 2$ with respect to x , $R_i(A)$, $i = 2, 3, \dots$, are no longer homogenous polynomial vector fields, although extension to such a case is easy to perform and treated later. If $B_i \in \mathcal{P}^{i+1}(\mathbf{R}^n)$ for $i = 1, \dots, k - 2$, then by using Eq.(4.25) with Eqs.(4.26,27), we can show that $R_k + P_k(R_1, \dots, R_{k-1}, B_1, \dots, B_{k-2})$ in Eq.(4.28) is in $\mathcal{P}^{k+1}(\mathbf{R}^n)$. Since the map ad_{R_1} defined by $\text{ad}_{R_1}(B) = [R_1, B]$ is a linear map from $\mathcal{P}^k(\mathbf{R}^n)$ into $\mathcal{P}^{k+1}(\mathbf{R}^n)$, Eq.(4.28) suggests that we are allowed to choose $B_{k-1} \in \mathcal{P}^k(\mathbf{R}^n)$ so that $\widetilde{R}_k(A)$ may take a value in a complementary subspace to $\text{Im ad}_{R_1}|_{\mathcal{P}^k(\mathbf{R}^n)}$ in $\mathcal{P}^{k+1}(\mathbf{R}^n)$.

Theorem 4.4. Suppose that $g_i(t, x)$ given in (3.1) is a homogenous polynomial vector field of degree $i+1$ in x . Let C_{i+1} be a complementary subspace to $\text{Im ad}_{R_1}|_{\mathcal{P}^i(\mathbf{R}^n)}$ in $\mathcal{P}^{i+1}(\mathbf{R}^n) : \mathcal{P}^{i+1}(\mathbf{R}^n) = \text{Im ad}_{R_1}|_{\mathcal{P}^i(\mathbf{R}^n)} \oplus C_{i+1}$. Then, there exist vector fields $B_i \in \mathcal{P}^{i+1}(\mathbf{R}^n)$, $i = 1, 2, \dots$ such that the new RG equation (4.9) has the properties that $\widetilde{R}_i \in C_{i+1}$ for $i = 2, 3, \dots$. Let $\widetilde{\varphi}_t^{RG}$ be the flow of the new RG equation (4.9). Then the equality

$$\frac{d}{da} \Big|_{a=t} \alpha_a \circ X(a)^{-1} \circ \widetilde{\alpha}_a \circ \widetilde{\varphi}_{a-t}^{RG} \circ \widetilde{\alpha}_t^{-1} \circ X(t) \circ \alpha_t^{-1}(x) = Fx + \varepsilon g_1(t, x) + \dots + \varepsilon^m g_m(t, x) + O(\varepsilon^{m+1}) \quad (4.29)$$

holds, where α_t and $\widetilde{\alpha}_t$ are defined by Eq.(4.2) and (4.23), respectively.

If the new RG equation (4.9) satisfies $\widetilde{R}_i \in C_{i+1}$ for $i = 2, \dots, m$, we call it *the m -th order simplified RG equation*.

If Eq.(3.26) is autonomous, the RG equation (4.1) has the property that $R_k(X(t)A) = X(t)R_k(A)$. Thus it is convenient to define $B_k(A)$'s so that the new RG equation (4.9) may have the same properties $\widetilde{R}_k(X(t)A) = X(t)\widetilde{R}_k(A)$ for $k = 1, 2, \dots$. Let $\mathcal{P}^i(\mathbf{R}^n; F)$ be the subspace of $\mathcal{P}^i(\mathbf{R}^n)$ all of whose elements f satisfy $f(X(t)A) = X(t)f(A)$ (recall that $X(t) := e^{Ft}$). Since $R_1 \in \mathcal{P}^2(\mathbf{R}^n; F)$, the restriction of ad_{R_1} to $\mathcal{P}^i(\mathbf{R}^n; F)$ is a map into $\mathcal{P}^{i+1}(\mathbf{R}^n; F)$. We take an arbitrary complementary subspace $C_{i+1,F}$ to $\text{Im ad}_{R_1}|_{\mathcal{P}^i(\mathbf{R}^n; F)}$ into $\mathcal{P}^{i+1}(\mathbf{R}^n; F)$, and fix it :

$$\mathcal{P}^{i+1}(\mathbf{R}^n; F) = \text{Im ad}_{R_1}|_{\mathcal{P}^i(\mathbf{R}^n; F)} \oplus C_{i+1,F}. \quad (4.30)$$

Theorem 4.5. For a given autonomous equation (3.26), suppose that $g_i(x)$ is a homogenous polynomial vector field of degree $i+1$. Then, there exist vector fields $B_i \in \mathcal{P}^{i+1}(\mathbf{R}^n; F)$, $i = 1, 2, \dots$, such that the new RG equation (4.9) has the properties that $\widetilde{R}_i \in C_{i+1,F}$ for $i = 2, 3, \dots$.

If the new RG equation (4.9) for an autonomous equation has the properties that $\widetilde{R}_i \in C_{i+1,F}$ for $i = 2, \dots, m$, we call it *the m -th order simplified RG equation*.

A few remarks are in order. Note that the simplified RG equation depends on the choice of a complementary subspace C_i . The simplified RG equation is equivalent to the normal form or hypernormal form (simplified normal form) of Eq.(4.1). See Murdock [13] for the normal form theory. If $R_1 = \dots = R_{j-1} = 0$ and $R_j \neq 0$ in the RG equation (4.1), Thms.4.4 and 4.5 hold if ad_{R_1} is replaced by ad_{R_j} . Extension to the case that $R_1 \in \mathcal{P}^l(\mathbf{R}^n)$ for some positive integer l and that R_2, R_3, \dots are inhomogeneous polynomials is easy to perform. If $R_k + P_k(R_1, \dots, R_{k-1}, B_1, \dots, B_{k-2})$ in Eq.(4.28) is an element of $\mathcal{P}^{d_1}(\mathbf{R}^n) \oplus \mathcal{P}^{d_2}(\mathbf{R}^n) \oplus \dots \oplus \mathcal{P}^{d_k}(\mathbf{R}^n)$, we can choose B_{k-1} so that \widetilde{R}_k may take a value in $C_{d_1} \oplus C_{d_2} \oplus \dots \oplus C_{d_k}$ (see Exs.4.6, 4.7). Even if R_1 is a polynomial vector field, we can simplify $\widetilde{R}_i(A)$ systematically by using the grading function (see Kokubu, Oka and Wang [9]) under appropriate assumptions, although we do not give care to this method in this paper.

Example 4.6. Consider the equation on \mathbf{R}^2

$$\dot{x} = Fx + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \dots, \quad x \in \mathbf{R}^2, \quad (4.31)$$

where $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and where $g_i(x)$, $i = 1, 2, \dots$ are homogenous polynomial vector fields whose degree is larger than 1. Like Exs.3.7 to 3.9, we express its RG equation in the sense of

Eq.(4.1), in terms of complex variable $A \in \mathbf{C}$, as

$$\frac{d}{dt} \left(\frac{A}{\bar{A}} \right) = \varepsilon^j R_j(A) + \varepsilon^{j+1} R_{j+1}(A) + \dots, \quad R_1 = \dots = R_{j-1} = 0, R_j \neq 0. \quad (4.32)$$

Since each R_l satisfies $R_l(e^{it}A) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} R_l(A)$, $i = \sqrt{-1}$, R_l for $l \geq j$ take the form

$$R_l(A) = \begin{pmatrix} p_l A |A|^{2k_l} \\ \bar{p}_l \bar{A} |A|^{2k_l} \end{pmatrix}, \quad k_l \geq 0, \quad (4.33)$$

where $i = \sqrt{-1}$ and where $2k_l + 1$ is the degree of $R_l(A)$ and $p_l \in \mathbf{C}$ is a constant. We wish to define a homogenous polynomial vector field $B(A)$ so that $[B, R_j](A)$ may be a homogenous polynomial vector field of degree $2k_{j+1} + 1$ satisfying $[B, R_j](e^{it}A) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} [B, R_j](A)$. It then turns out that $B(A)$ has to be of the form

$$B(A) = \begin{pmatrix} qA |A|^{2(k_{j+1}-k_j)} \\ \bar{q}\bar{A} |A|^{2(k_{j+1}-k_j)} \end{pmatrix}, \quad q \in \mathbf{C}, \quad (4.34)$$

and $[B, R_j](A)$ is given by

$$[B, R_j](A) = \begin{pmatrix} c_j A |A|^{2k_{j+1}} \\ \bar{c}_j \bar{A} |A|^{2k_{j+1}} \end{pmatrix} \quad (4.35)$$

where

$$c_j := -2(2k_j - k_{j+1})\text{Re}(q)\text{Re}(p_j) - i(2(k_j - k_{j+1})\text{Im}(q)\text{Re}(p_j) - k_j\text{Re}(q)\text{Im}(p_j)). \quad (4.36)$$

Then, from Eq.(4.16) with R_1 replaced by R_j , \widetilde{R}_{j+1} has the form

$$\widetilde{R}_{j+1}(A) = \begin{pmatrix} (p_{j+1} - c_j)A |A|^{2k_{j+1}} \\ (\bar{p}_{j+1} - \bar{c}_j)\bar{A} |A|^{2k_{j+1}} \end{pmatrix}, \quad p_{j+1} \in \mathbf{C}. \quad (4.37)$$

Our purpose is to determine a constant $q \in \mathbf{C}$ in $B(A)$ so that \widetilde{R}_{j+1} may be simplified.

Case (i) If $(2k_j - k_{j+1})(k_j - k_{j+1})\text{Re}(p_j) \neq 0$, then we can choose q so that $p_{j+1} - c_j = 0$. In this case, the simplified RG equation (4.9) satisfies $\widetilde{R}_{j+1}(A) = 0$.

Case (ii) If $(2k_j - k_{j+1})\text{Re}(p_j) \neq 0$ and $k_j = k_{j+1}$, then we can choose q so that $\text{Re}(p_{j+1} - c_j) = 0$. In this case, $\widetilde{R}_{j+1}(A)$ is of the form

$$\widetilde{R}_{j+1}(A) = i \bar{p}_{j+1} \begin{pmatrix} A |A|^{2k_{j+1}} \\ -\bar{A} |A|^{2k_{j+1}} \end{pmatrix}, \quad \bar{p}_{j+1} \in \mathbf{R}. \quad (4.38)$$

Case (iii) If $(k_j - k_{j+1})\text{Re}(p_j) \neq 0$ and $2k_j = k_{j+1}$, then we can choose q so that $\text{Im}(p_{j+1} - c_j) = 0$. In this case, $\widetilde{R}_{j+1}(A)$ is of the form

$$\widetilde{R}_{j+1}(A) = \bar{p}_{j+1} \begin{pmatrix} A |A|^{2k_{j+1}} \\ \bar{A} |A|^{2k_{j+1}} \end{pmatrix}, \quad \bar{p}_{j+1} \in \mathbf{R}. \quad (4.39)$$

Case (iv) If $\text{Re}(p_j) = 0$ and $k_j \neq 0$, then we can choose q so that $\text{Im}(p_{j+1} - c_j) = 0$ and $\widetilde{R}_{j+1}(A)$ is of the form (4.39).

Case (v) If $\text{Re}(p_j) = 0$ and $k_j = 0$, or if $k_j = k_{j+1} = 0$, then $c_j = 0$ and $\widetilde{R}_{j+1}(A) = R_{j+1}(A)$.

$\widetilde{R}_{j+2}, \widetilde{R}_{j+3}, \dots$ are calculated in a similar way. Now we restrict the example to a few special cases.

If Eq.(4.31) is a linear equation, the RG equation (4.32) is also linear and the degree of each $R_l(A)$ is one (*i.e.* $k_l = 0$). Then Case (v) in the above applies and this proves that the RG equation for an autonomous linear equation can no longer be reduced in this manner. The simplified RG equation for a nonautonomous linear equation will be treated in the next section.

As a next restricted example, consider an equation on \mathbf{R}^2

$$\dot{x} = Fx + P_2(x) + P_3(x) + \dots, \quad x \in \mathbf{R}^2, \quad (4.40)$$

where $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and where $P_i(x)$ is a homogenous polynomial vector field of degree i . Changing the coordinate by $x = \varepsilon X$ brings Eq.(4.40) into

$$\dot{X} = FX + \varepsilon P_2(X) + \varepsilon^2 P_3(X) + \dots, \quad X \in \mathbf{R}^2. \quad (4.41)$$

The RG equation for this equation takes the form

$$\frac{d}{dt} \left(\frac{A}{A} \right) = \varepsilon^2 R_2(A) + \varepsilon^4 R_4(A) + \varepsilon^6 R_6(A) + \dots, \quad R_{2i-1} = 0 \text{ for } i = 1, 2, \dots, \quad (4.42)$$

where $R_{2i}(A)$ is a monomial vector field of the form

$$R_{2i}(A) = \begin{pmatrix} p_{2i+1} A |A|^{2i} \\ \bar{p}_{2i+1} \bar{A} |A|^{2i} \end{pmatrix}, \quad (4.43)$$

whose degree is $2i + 1$. In this case, on account of $k_i \neq k_j$ ($i \neq j$) and $k_i > 0$, Cases (i),(iii) and (iv) are applicable. Suppose that $R_2 = R_4 = \dots = R_{2j-2} = 0$ and $R_{2j} \neq 0$. Then, the simplified RG equation takes the following form:

(I) If $\text{Re}(p_{2j+1}) \neq 0$, Case (i) and (iii) applies and the simplified RG equation is of the form

$$\dot{A} = \varepsilon^{2j} \widetilde{p}_{2j+1} A |A|^{2j} + \varepsilon^{4j} \widetilde{p}_{4j+1} A |A|^{4j}, \quad A \in \mathbf{C}, \quad (4.44)$$

where $\text{Re}(\widetilde{p}_{2j+1}) \neq 0$ and $\text{Im}(\widetilde{p}_{4j+1}) = 0$.

(II) If $\text{Re}(p_{2j+1}) = 0$ and $\text{Im}(p_{2j+1}) \neq 0$, Case (iv) applies and the simplified RG equation is of the form

$$\dot{A} = \varepsilon^{2j} \widetilde{p}_{2j+1} A |A|^{2j} + \varepsilon^{2j+2} \widetilde{p}_{2j+3} A |A|^{2j+2} + \varepsilon^{2j+4} \widetilde{p}_{2j+5} A |A|^{2j+4} + \dots, \quad (4.45)$$

where $\text{Re}(\widetilde{p}_{2j+1}) = 0$, $\text{Im}(\widetilde{p}_{2j+1}) \neq 0$ and $\text{Im}(\widetilde{p}_i) = 0$ for $i = 2j + 3, 2j + 5, \dots$.

Put $A = re^{i\theta}$. Then Eqs.(4.44) and (4.45) are brought into

$$\begin{cases} \dot{r} = \varepsilon^{2j} \alpha_{2j+1} r^{2j+1} + \varepsilon^{4j} \alpha_{4j+1} r^{4j+1}, \\ \dot{\theta} = \varepsilon^{2j} \beta_{2j+1} r^{2j}, \end{cases} \quad (4.46)$$

$$\begin{cases} \dot{r} = \varepsilon^{2j+2} \alpha_{2j+3} r^{2j+3} + \varepsilon^{2j+4} \alpha_{2j+5} r^{2j+5} + \dots, \\ \dot{\theta} = \varepsilon^{2j} \beta_{2j+1} r^{2j}, \end{cases} \quad (4.47)$$

respectively, where $\alpha_i = \text{Re}(\bar{p}_i)$, $\beta_i = \text{Im}(\bar{p}_i)$.

Note that Eq.(4.47) can be further simplified by applying the hypernormal form theory (see Murdock [13]). Indeed, the simplified RG equation is not unique, because we can express $\widetilde{R}_2(A)$ given by (4.16) as

$$\widetilde{R}_2(A) = R_2(A) - [B_1, R_1](A) - [B'_1, R_1](A),$$

where $B'_1 \in \text{Ker ad}_{R_1}$. Though $B'_1(A)$ does not affect $\widetilde{R}_2(A)$, it may change $\widetilde{R}_3(A), \widetilde{R}_4(A), \dots$.

Example 4.7. Consider the equation

$$\begin{cases} \dot{x} = y + \varepsilon x - \varepsilon^2 x^3, \\ \dot{y} = -x, \end{cases} \quad (4.48)$$

where $\varepsilon > 0$ is a small parameter. The second order RG equation for this equation is given by Eq.(3.46) in the complex variable or by Eq.(3.47) in the polar coordinate. Since Eq.(3.47) has a stable periodic orbit $r = \sqrt{1/3\varepsilon}$ and an unstable fixed point $r = 0$, the original equation (4.48) also has a stable periodic orbit and an unstable fixed point $x = 0$, as is shown in Example 3.8. We now calculate the simplified RG equation. Case (i) applies and the term $-3|A|^2 A/2$ in Eq.(3.46) vanishes. However, the term $-iA/8$ does not vanish on account of Case (v). Therefore the second order simplified RG equation is expressed as

$$\dot{A} = \frac{\varepsilon}{2}A - \frac{i\varepsilon^2}{8}A, \quad A \in \mathbf{C}, \quad (4.49)$$

or

$$\begin{cases} \dot{r} = \frac{1}{2}\varepsilon r, \\ \dot{\theta} = -\frac{1}{8}\varepsilon^2, \end{cases} \quad (4.50)$$

in the polar coordinate. This equation has an unstable fixed point $r = 0$, but does *not* have a periodic orbit. To find out the reason why the periodic orbit mentioned above disappear, we derive the RG transformation to explore a region on which the RG transformation is a diffeomorphism (see Remark 3.2).

The second order RG transformation associated with the RG equation (3.46) is given by

$$\alpha_t(A) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} A \\ \bar{A} \end{pmatrix} + \frac{i\varepsilon}{4} \begin{pmatrix} \bar{A}e^{-it} \\ -Ae^{it} \end{pmatrix} + \frac{\varepsilon^2}{8} \begin{pmatrix} -\bar{A}e^{-it} + 2iA^3 e^{3it} - 6i\bar{A}|A|^2 e^{-it} - i\bar{A}^3 e^{-3it} \\ -Ae^{it} + iA^3 e^{3it} + 6iA|A|^2 e^{it} - 2i\bar{A}^3 e^{-3it} \end{pmatrix}. \quad (4.51)$$

Near the periodic orbit $r = \sqrt{1/3\varepsilon}$ of Eq.(3.46), the first, second and third terms of the right hand side of the above are of order $O(\sqrt{1/\varepsilon})$, $O(\sqrt{\varepsilon})$, and $O(\sqrt{\varepsilon})$, respectively. Therefore, if $\varepsilon > 0$ is sufficiently small, $\alpha_t(A)$ is well approximated by its first term, and this proves that $\alpha_t(A)$ is a diffeomorphism for each $t \in \mathbf{R}$, if $|A| \sim O(\sqrt{1/\varepsilon})$.

On the other hand, the RG transformation associated with the simplified RG equation (4.49), which brings the simplified RG equation (4.49) into the original equation (4.48), is given by (see Eq.(4.29))

$$\alpha_t \circ X(t)^{-1} \circ \widetilde{\alpha}_t(A) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} A \\ \bar{A} \end{pmatrix} + \frac{\varepsilon}{4} \begin{pmatrix} i\bar{A}e^{-it} - 6A|A|^2 e^{it} \\ -iAe^{it} - 6\bar{A}|A|^2 e^{-it} \end{pmatrix} + O(\varepsilon^2), \quad (4.52)$$

where $X(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$, and where α_t and $\tilde{\alpha}_t$ are given by Eq.(4.51) and by

$$\tilde{\alpha}_t(A) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} A \\ \bar{A} \end{pmatrix} - \frac{3\varepsilon}{2} \begin{pmatrix} A|A|^2 e^{it} \\ \bar{A}|A|^2 e^{-it} \end{pmatrix} + O(\varepsilon^2), \quad (4.53)$$

respectively. In this case, if $|A| \sim O(\sqrt{1/\varepsilon})$, the first and second terms of the right hand side of Eq.(4.52) are of same order $O(\sqrt{1/\varepsilon})$. As a result, $\alpha_t \circ X(t) \circ \tilde{\alpha}_t(A)$ is not a diffeomorphism near a region $|A| \sim O(\sqrt{1/\varepsilon})$ no matter how small $\varepsilon > 0$ is.

Now the reason we have looked for is clear. The simplified RG equation (4.49) can not imply the existence of the periodic orbit, since the periodic orbit lies out of the region on which the RG transformation associated with the simplified RG equation (4.49) is a diffeomorphism.

In general, the more the RG equation is simplified, the more the RG transformation, which brings the RG equation into the original equation, becomes complex and a region on which the RG transformation is a diffeomorphism may become small.

5 Simplified RG equation for time-dependent linear equations

In this section, the simplified RG equation is applied to time-dependent linear equations. In particular, it is shown that hyperbolic stability of a trivial solution of a time-dependent linear equation is determined by the simplified RG equation along with metanormal form theory proposed by Murdock [13].

Consider a linear equation on \mathbf{R}^n

$$\dot{x} = Fx + \varepsilon G_1(t)x + \varepsilon^2 G_2(t)x + \cdots, \quad x \in \mathbf{R}^n, \quad (5.1)$$

where $\varepsilon > 0$ is a small parameter, F is a diagonalizable $n \times n$ matrix all of whose eigenvalues lie on the imaginary axis, and where $G_1(t), G_2(t), \cdots$ are $n \times n$ matrices which are of C^1 class and periodic in $t \in \mathbf{R}$. For this equation, functions $R_i(A)$ and $h_t^{(i)}(A)$ defined by Eqs.(3.13, 14, 17, 18) are linear with respect to A . In view of this, we define matrices R_i and $h_t^{(i)}$ by

$$R_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t X(s)^{-1} G_1(s) X(s) ds, \quad (5.2)$$

$$h_t^{(1)} = X(t) \int^t (X(s)^{-1} G_1(s) X(s) - R_1) ds, \quad (5.3)$$

and

$$R_i = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left(X(s)^{-1} \sum_{k=1}^{i-1} G_{i-k}(s) h_s^{(k)} + X(s)^{-1} G_i(s) X(s) - X(s)^{-1} \sum_{k=1}^{i-1} h_s^{(k)} R_{i-k} \right) ds, \quad (5.4)$$

$$h_t^{(i)} = X(t) \int^t \left(X(s)^{-1} \sum_{k=1}^{i-1} G_{i-k}(s) h_s^{(k)} + X(s)^{-1} G_i(s) X(s) - X(s)^{-1} \sum_{k=1}^{i-1} h_s^{(k)} R_{i-k} - R_i \right) ds \quad (5.5)$$

for $i = 2, 3, \cdots$, where $X(t) = e^{Ft}$ and the integral constants of the indefinite integrals in the above equations are fixed arbitrary as in Sec.3. With these matrices, the m -th order RG equation and the m -th order RG transformation for Eq.(5.1) are given by

$$\dot{v} = \varepsilon R_1 v + \varepsilon^2 R_2 v + \cdots + \varepsilon^m R_m v, \quad v \in \mathbf{R}^n, \quad (5.6)$$

$$\alpha_t(v) = X(t)v + \varepsilon h_t^{(1)} v + \cdots + \varepsilon^m h_t^{(m)} v, \quad (5.7)$$

respectively.

Next, we specialize the simplified RG equation for Eq.(5.6). Because of linearity, Eq.(4.5) to Eq.(4.8) are reduced to

$$\widetilde{R}_1 = R_1, \quad (5.8)$$

$$\widetilde{R}_i = R_i + \sum_{k=1}^{i-1} R_{i-k} B_k - \sum_{k=1}^{i-1} B_k \widetilde{R}_{i-k}, \quad i = 2, 3, \dots, \quad (5.9)$$

$$\widetilde{h}_t^{(i)} = X(t) B_i, \quad i = 1, 2, \dots, \quad (5.10)$$

where B_i 's are arbitrary constant matrices. For example, matrices $\widetilde{R}_2, \widetilde{R}_3$ and \widetilde{R}_4 are put in the form

$$\widetilde{R}_2 = R_2 - [B_1, R_1], \quad (5.11)$$

$$\widetilde{R}_3 = R_3 - [B_1, R_2] - [B_2, R_1] + B_1[B_1, R_1], \quad (5.12)$$

$$\begin{aligned} \widetilde{R}_4 = & R_4 - [B_1, R_3] - [B_2, R_2] - [B_3, R_1] \\ & - B_1^2[B_1, R_1] + B_2[B_1, R_1] + B_1[B_1, R_2] + B_1[B_2, R_1], \end{aligned} \quad (5.13)$$

respectively, where the bracket denotes the usual one for matrices. Let $M(n, \mathbf{C})$ be the set of $n \times n$ complex matrices and define an operator $\text{ad}_{R_1} : M(n, \mathbf{C}) \rightarrow M(n, \mathbf{C})$ by

$$\text{ad}_{R_1}(B) = [R_1, B] := R_1 B - B R_1. \quad (5.14)$$

It is easy to verify that

$$M(n, \mathbf{C}) = \text{Im ad}_{R_1} \oplus \text{Ker ad}_{R_1^*}, \quad (5.15)$$

where R_1^* denotes the conjugate transpose matrix of R_1 . Since $\widetilde{R}_i, i = 2, 3, \dots$ are rewritten as

$$\widetilde{R}_i = R_i + \sum_{k=1}^{i-2} R_{i-k} B_k - \sum_{k=1}^{i-2} B_k \widetilde{R}_{i-k} + \text{ad}_{R_1}(B_{i-1}), \quad (5.16)$$

we can choose matrices B_1, \dots, B_{m-1} such that

$$\widetilde{R}_i \in \text{Ker ad}_{R_1^*}, \quad \text{for } i = 2, 3, \dots, m. \quad (5.17)$$

Then, the equation

$$\dot{v} = \varepsilon \widetilde{R}_1 v + \varepsilon^2 \widetilde{R}_2 v + \dots + \varepsilon^m \widetilde{R}_m v \quad (5.18)$$

is called *the m-th order simplified RG equation* for Eq.(5.1). If we define a matrix $\widetilde{\alpha}_t$ by

$$\widetilde{\alpha}_t(v) = X(t)v + \varepsilon X(t) B_1 v + \dots + \varepsilon^m X(t) B_m v, \quad (5.19)$$

then the equality

$$\left. \frac{d}{da} \right|_{a=t} (\alpha_a X(a)^{-1} \widetilde{\alpha}_a) \circ \widetilde{\varphi}_{a-t}^{RG} \circ (\alpha_t X(t)^{-1} \widetilde{\alpha}_t)^{-1}(x) = Fx + \varepsilon G_1(t)x + \dots + \varepsilon^m G_m(t)x + \varepsilon^{m+1} S(\varepsilon, t)x \quad (5.20)$$

holds, where $\widetilde{\varphi}_t^{RG} = e^{(\varepsilon \widetilde{R}_1 + \dots + \varepsilon^m \widetilde{R}_m)t}$ is the flow of Eq.(5.18), and where S is a matrix-valued function which is bounded in $t \in \mathbf{R}$ and bounded as $\varepsilon \rightarrow 0$.

Our purpose in this section is to study the stability of the trivial solution $x(t) \equiv 0$ of Eq.(5.1). Changing coordinates by $x = \alpha_t X(t)^{-1} \tilde{\alpha}_t y$ brings Eq.(5.1) into the equation

$$\dot{y} = \varepsilon \tilde{R}_1 y + \cdots + \varepsilon^m \tilde{R}_m y + \varepsilon^{m+1} \tilde{S}(\varepsilon, t) y, \quad (5.21)$$

where \tilde{S} is a matrix-valued function which is bounded in $t \in \mathbf{R}$ and bounded as $\varepsilon \rightarrow 0$. Since $\alpha_t X(t)^{-1} \tilde{\alpha}_t$ is almost periodic in $t \in \mathbf{R}$ and bounded in $t \in \mathbf{R}$, the stability of the trivial solution $x(t) \equiv 0$ of Eq.(5.1) coincides with that of $y(t) \equiv 0$ of Eq.(5.21) (the fact that α_t is almost periodic is shown in Chiba [3]).

Now a question arises : Can we use the truncated equation (5.18) to decide as to whether the trivial solution $y(t) \equiv 0$ to Eq.(5.21) is stable or not? In general, this is impossible. An illustrative example is shown in Murdock [13]. Consider the equation on \mathbf{R}^2

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y + \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y + \varepsilon^2 \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} y. \quad (5.22)$$

Since eigenvalues of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ are 3ε and $-\varepsilon$, $y = 0$ is a saddle point. However, if we truncate the second order term $\varepsilon^2 \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are ε (double root), so that $y = 0$ is an unstable fixed point if $\varepsilon > 0$. This example shows that if we truncate the higher order term of ε , the stability of $y = 0$ of Eq.(5.21) may change. To handle this problem, we need two propositions about stability of the trivial solutions of linear equations. By using metanormal form theory, we can put Eq.(5.21) in the form to which the propositions are applicable.

Proposition 5.1. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of \tilde{R}_1 . There exists a positive constant ε_0 such that the following holds for $0 < \forall \varepsilon < \varepsilon_0$: there exist positive constants D_1, D_2, t_0 , a positive valued function $\phi(\varepsilon)$ with $\phi(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and a solution $y(t) = y^{(k)}(t)$ of Eq.(5.21) such that the inequality

$$D_2 e^{\varepsilon \operatorname{Re}(\lambda_k) t - \varepsilon \phi(\varepsilon) t} \leq \|y^{(k)}(t)\| \leq D_1 e^{\varepsilon \operatorname{Re}(\lambda_k) t + \varepsilon \phi(\varepsilon) t} \quad (5.23)$$

holds for $t \geq t_0$ and $k = 1, \dots, n$.

Proposition 5.2. Suppose that $\tilde{R}_1, \dots, \tilde{R}_m$ are diagonal matrices. Let $\lambda_1(\varepsilon), \dots, \lambda_n(\varepsilon)$ be eigenvalues of $\varepsilon \tilde{R}_1 + \cdots + \varepsilon^m \tilde{R}_m$. Then, there exists a positive constant ε_0 such that the following holds for $0 < \forall \varepsilon < \varepsilon_0$: there exist positive constants D_1, D_2, D_3, D_4, t_0 and a solution $y(t) = y^{(k)}(t)$ of Eq.(5.21) such that the inequality

$$D_2 e^{\operatorname{Re}(\lambda_k(\varepsilon)) t - \varepsilon^{m+1} D_4 t} \leq \|y^{(k)}(t)\| \leq D_1 e^{\operatorname{Re}(\lambda_k(\varepsilon)) t + \varepsilon^{m+1} D_3 t} \quad (5.24)$$

holds for $t \geq t_0$ and $k = 1, \dots, n$.

Prop.5.1 is shown in Sec.7 of [3]. Prop.5.2 is proved in a similar way as that for Prop.5.1, and we omit it here.

In particular, if $\tilde{R}_1 = R_1$ is hyperbolic, namely, none of eigenvalues of R_1 lies on the imaginary axis, then the stability of the trivial solution $y = 0$ of Eq.(5.21) coincides with that of the truncated

equation $\dot{v} = \varepsilon R_1 v$. In addition, if all $\widetilde{R}_1, \dots, \widetilde{R}_m$ are diagonal and if $\varepsilon \widetilde{R}_1 + \dots + \varepsilon^m \widetilde{R}_m$ is hyperbolic, then the stability of the trivial solution $y = 0$ of Eq.(5.21) coincides with that of the truncated equation $\dot{v} = (\varepsilon \widetilde{R}_1 + \dots + \varepsilon^m \widetilde{R}_m)v$. If the simplified RG equation does not satisfies the assumption of Prop.5.1 or Prop.5.2, we use the metanormal form theory to try to transform the simplified equation into a desired form.

Now we show the procedure for deciding as to the stability of the trivial solution of Eq.(5.1), along with equations on \mathbf{R}^2 , although the procedure in general cases are available in the same way (see Rem.5.4).

Step (I) For a given equation (5.1), we calculate the RG equation up to some finite order m . After changing coordinates so that R_1 may take the Jordan form, we calculate the simplified RG equation up to order m .

Step (II) Prop.5.1 shows that if $\widetilde{R}_1 = R_1$ is hyperbolic, the stability of the trivial solution $x = 0$ of Eq.(5.1) coincides with that of the first order RG equation $\dot{v} = \varepsilon R_1 v$. If R_1 is not hyperbolic, go to Step (III).

Step (III) Step (III) is divided into three cases according to the type of R_1 .

Case (i) Suppose that R_1 is a diagonal matrix all of whose eigenvalues are distinct. Since

$$\text{Ker ad}_{R_1^*} = \text{the set of diagonal matrices}, \quad (5.25)$$

the m -th order simplified RG equation is of the form

$$\dot{v} = \varepsilon \begin{pmatrix} \lambda_1^{(1)} & 0 \\ 0 & \lambda_2^{(1)} \end{pmatrix} v + \varepsilon^2 \begin{pmatrix} \lambda_1^{(2)} & 0 \\ 0 & \lambda_2^{(2)} \end{pmatrix} v + \dots + \varepsilon^m \begin{pmatrix} \lambda_1^{(m)} & 0 \\ 0 & \lambda_2^{(m)} \end{pmatrix} v. \quad (5.26)$$

Now Prop.5.2 applies to show that if eigenvalues $\varepsilon \lambda_1^{(1)} + \dots + \varepsilon^m \lambda_1^{(m)}$ and $\varepsilon \lambda_2^{(1)} + \dots + \varepsilon^m \lambda_2^{(m)}$ do *not* lie on the imaginary axis, the stability of the trivial solution of Eq.(5.1) coincides with that of Eq.(5.26). If some of the eigenvalues are on the imaginary axis, we need higher order simplified RG equation.

Case (ii) Suppose that R_1 is a diagonal matrix all of whose eigenvalues are equal. In this case, we have

$$\text{Ker ad}_{R_1^*} = M(2, \mathbf{C}), \quad (5.27)$$

so that \widetilde{R}_i are not diagonal matrices, and we can not apply Prop.5.2. However, since all eigenvalues of R_1 are on the imaginary axis, we can apply the RG method to the simplified RG equation (5.18). We calculate the simplified RG equation for Eq.(5.18) and put its first term in the Jordan form. With this new simplified RG equation, we go back to Step (II).

Case (iii) Suppose that R_1 is of the form $R_1 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. In this case, since

$$\text{Ker ad}_{R_1^*} = \left\{ \begin{pmatrix} p & 0 \\ q & p \end{pmatrix} \mid p, q \in \mathbf{C} \right\}, \quad (5.28)$$

the simplified RG equation (5.18) takes the form

$$\dot{v} = \varepsilon \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} v + \varepsilon^2 \begin{pmatrix} p_2 & 0 \\ q_2 & p_2 \end{pmatrix} v + \varepsilon^3 \begin{pmatrix} p_3 & 0 \\ q_3 & p_3 \end{pmatrix} v + \dots + \varepsilon^m \begin{pmatrix} p_m & 0 \\ q_m & p_m \end{pmatrix} v. \quad (5.29)$$

If $q_2 = q_3 = \dots = q_m = 0$, since $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ and $\begin{pmatrix} p_i & 0 \\ 0 & p_i \end{pmatrix}$ commute with each other, changing the variable v into u by $v = \varepsilon \left(\exp \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t \right) u$ brings the above equation into an equation

$$\dot{u} = \varepsilon^2 \begin{pmatrix} p_2 & 0 \\ 0 & p_2 \end{pmatrix} u + \varepsilon^3 \begin{pmatrix} p_3 & 0 \\ 0 & p_3 \end{pmatrix} u + \dots + \varepsilon^m \begin{pmatrix} p_m & 0 \\ 0 & p_m \end{pmatrix} u. \quad (5.30)$$

Since λ is on the imaginary axis, the stability of $v = 0$ coincides with that of $u = 0$. Now that Prop.5.2 is applicable to Eq.(5.30). It follows that if the eigenvalue $\varepsilon^2 p_2 + \dots + \varepsilon^m p_m$ does not lie on the imaginary axis, the stability of the trivial solution of Eq.(5.1) coincides with that of Eq.(5.30). If it is on the imaginary axis, we need more higher order simplified RG equation.

If $q_2 = \dots = q_{j-1} = 0$ and $q_j \neq 0$, $j \leq m$, Eq.(5.29) is written as

$$\dot{v} = \varepsilon \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} v + \varepsilon^2 \begin{pmatrix} p_2 & 0 \\ 0 & p_2 \end{pmatrix} v + \dots + \varepsilon^{j-1} \begin{pmatrix} p_{j-1} & 0 \\ 0 & p_{j-1} \end{pmatrix} v + \varepsilon^j \begin{pmatrix} p_j & 0 \\ q_j & p_j \end{pmatrix} v + \dots + \varepsilon^m \begin{pmatrix} p_m & 0 \\ q_m & p_m \end{pmatrix} v. \quad (5.31)$$

Changing coordinates by $v = (x, y) = (X, \varepsilon^{(j-1)/2} Y)$ yields

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} X \\ Y \end{pmatrix} &= \varepsilon \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \varepsilon^2 \begin{pmatrix} p_2 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \dots + \varepsilon^m \begin{pmatrix} p_m & 0 \\ 0 & p_m \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\ &+ \varepsilon^{(j+1)/2} \begin{pmatrix} 0 & 1 \\ q_j & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \varepsilon^{(j+3)/2} \begin{pmatrix} 0 & 0 \\ q_{j+1} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \dots + \varepsilon^{(2m-j+1)/2} \begin{pmatrix} 0 & 0 \\ q_m & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \end{aligned} \quad (5.32)$$

Further, putting $\eta = \varepsilon^{1/2}$ and changing the time scale by $t = s/\eta^2$ result in

$$\begin{aligned} \frac{d}{ds} \begin{pmatrix} X \\ Y \end{pmatrix} &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \eta^2 \begin{pmatrix} p_2 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \dots + \eta^{2m-2} \begin{pmatrix} p_m & 0 \\ 0 & p_m \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\ &+ \eta^{j-1} \begin{pmatrix} 0 & 1 \\ q_j & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \eta^{j+1} \begin{pmatrix} 0 & 0 \\ q_{j+1} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \dots + \eta^{2m-j-1} \begin{pmatrix} 0 & 0 \\ q_m & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \end{aligned} \quad (5.33)$$

Since λ is on the imaginary axis, we can apply the RG method to this equation. We calculate the simplified RG equation for this equation, and put the first term in the Jordan form. The resultant equation is called the *metanormal form* for Eq.(5.29). With this equation, we go back to Step (II).

If $x = 0$ is a hyperbolic fixed point of Eq.(5.1), the above procedure is sufficient for deciding its stability. However, if Eq.(5.1) is not analytic but C^∞ with respect to ε , the strength of the stability may get exponentially small (i.e. constants α and β below are of order $O(e^{-1/\varepsilon})$). In this case, we can not determine the stability by using the perturbation method within finite steps. To avoid such a difficulty, we assume analyticity in the next theorem.

Theorem 5.3. Suppose that Eq.(5.1) is analytic in small ε and there exist an open set U including the origin and positive constants C_1, C_2, α, β such that every solution $x(t)$ of Eq.(5.1), whose initial value is in U , satisfies either

$$\|x(t)\| \leq C_1 \|x(0)\| e^{-\alpha t}, \quad \text{if } t \geq 0, \quad (5.34)$$

or

$$\|x(t)\| \leq C_2 \|x(0)\| e^{\beta t}, \quad \text{if } t \leq 0. \quad (5.35)$$

Then, there exists an integer M such that the stability of the solution $x(t) \equiv 0$ of Eq.(5.1) coincides with that of the solution $v(t) \equiv 0$ of the m -th order simplified RG equation (5.18) if $m \geq M$. In particular, we can decide as to the stability of $x = 0$ within finite steps by using the above procedure.

Proof. If m is sufficiently large, the simplified RG equation (5.18) is sufficiently close to Eq.(5.21) which is equivalent to Eq.(5.1). Then, our theorem is an immediate consequence from the persistency of hyperbolic invariant manifolds (see Thm.2.1). ■

Remark 5.4. It is easy to extend the above procedure to n -dimensional case. For example, if

$R_1 \in M(3, \mathbf{C})$ is of a form $R_1 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $\text{Ker ad}_{R_1^*}$ is given by

$$\text{Ker ad}_{R_1^*} = \left\{ \begin{pmatrix} p & 0 & 0 \\ q & p & 0 \\ r & q & p \end{pmatrix} \mid p, q, r \in \mathbf{C} \right\}. \quad (5.36)$$

Furthermore, if R_1 is of the $n \times n$ Jordan block which is not diagonal, then $\text{Ker ad}_{R_1^*}$ is given by the set of lower triangular matrices such that the entries on the diagonal and those on each lower line parallel to the diagonal are equal in each line. If R_1 has many Jordan block, we can apply the above procedure to each Jordan block.

Example 5.5. Consider the Mathieu equation

$$\ddot{x} = -ax - 2\varepsilon \cos t x, \quad x \in \mathbf{R}, \quad (5.37)$$

where a and $\varepsilon > 0$ are parameters. If $a = 1/4$, then $x = 0$ is an unstable fixed point, which is verified by using the first order RG equation and Prop.5.1, see Chiba [3]. To examine the parameter dependency of the instability more precisely, we put

$$a = \frac{1}{4} + \varepsilon b_1 + \varepsilon^2 b_2 + \dots, \quad (5.38)$$

and introduce a new variable by $\dot{x} = y/2$. Then Eq.(5.37) is rewritten as

$$\begin{cases} \dot{x} = \frac{1}{2}y, \\ \dot{y} = -\frac{1}{2}x - 4\varepsilon \cos t x - 2\varepsilon b_1 x - 2\varepsilon^2 b_2 x - \dots \end{cases} \quad (5.39)$$

Introducing a complex variable through $x = z + \bar{z}$, $y = i(z - \bar{z})$, $z \in \mathbf{C}$, we express (5.39) as

$$\begin{cases} \dot{z} = \frac{1}{2}iz + i\varepsilon(e^{it} + e^{-it})(z + \bar{z}) + i\varepsilon b_1(z + \bar{z}) + i\varepsilon^2 b_2(z + \bar{z}) + \dots, \\ \dot{\bar{z}} = -\frac{1}{2}i\bar{z} - i\varepsilon(e^{it} + e^{-it})(z + \bar{z}) - i\varepsilon b_1(z + \bar{z}) - i\varepsilon^2 b_2(z + \bar{z}) + \dots \end{cases} \quad (5.40)$$

The second order RG equation for this equation is given by

$$\dot{v} = \varepsilon \begin{pmatrix} 0 & 1 - b_1 \\ 1 + b_1 & 0 \end{pmatrix} v + \varepsilon^2 \begin{pmatrix} 0 & 1/2 - 2b_1 + b_1^2 - b_2 \\ -1/2 - 2b_2 - b_1^2 + b_2 & 0 \end{pmatrix} v. \quad (5.41)$$

Since eigenvalues of the matrix $R_1 = \begin{pmatrix} 0 & 1 - b_1 \\ 1 + b_1 & 0 \end{pmatrix}$ are $\lambda = \pm \sqrt{1 - b_1^2}$, if $|b_1| < 1$, Step (II) shows that $x = 0$ of Eq.(5.37) is unstable because of the positive eigenvalue $\lambda = \sqrt{1 - b_1^2}$. On the other hand, if $|b_1| \geq 1$, two eigenvalues are on the imaginary axis and we go to Step (III).

In what follows, we fix $b_1 = 1$. By putting $v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u$, Eq.(5.41) is rewritten as

$$\dot{u} = \varepsilon \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} u + \varepsilon^2 \begin{pmatrix} 0 & b_2 - 7/2 \\ -b_2 - 1/2 & 0 \end{pmatrix} u. \quad (5.42)$$

Because of Eq.(5.28), the simplified RG equation for this equation is given by

$$\dot{u} = \varepsilon \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} u + \varepsilon^2 \begin{pmatrix} 0 & 0 \\ -b_2 - 1/2 & 0 \end{pmatrix} u. \quad (5.43)$$

Changing coordinates by $u = (u_1, u_2) = (w_1, \varepsilon^{1/2} w_2)$, we obtain

$$\frac{d}{dt} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \varepsilon^{3/2} \begin{pmatrix} 0 & 2 \\ -b_2 - 1/2 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \quad (5.44)$$

This is of the metanormal form for Eq.(5.41) with $b_1 = 1$ (Step (III)-Case (iii)). Since eigenvalues of the matrix $\begin{pmatrix} 0 & 2 \\ -b_2 - 1/2 & 0 \end{pmatrix}$ is $\lambda = \pm \sqrt{-2(b_2 + 1/2)}$, if $b_2 < -1/2$, $x = 0$ of Eq.(5.37) is unstable because of a positive eigenvalue. If $b_2 \geq -1/2$, two eigenvalues are on the imaginary axis, and we need more higher order term.

In what follows, we fix $b_1 = 1, b_2 = -1/2$. Then, the third order RG equation for Eq.(5.40) is given by

$$\dot{v} = \varepsilon \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} v + \varepsilon^2 \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix} v + \varepsilon^3 \begin{pmatrix} 0 & -1/4 - b_3 \\ 39/4 + b_3 & 0 \end{pmatrix} v. \quad (5.45)$$

Putting $v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u$ provides

$$\dot{u} = \varepsilon \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} u + \varepsilon^2 \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} u + \varepsilon^3 \begin{pmatrix} 0 & 39/4 + b_3 \\ -1/4 - b_3 & 0 \end{pmatrix} u. \quad (5.46)$$

Because of Eq.(5.28), the simplified RG equation of this equation is given by

$$\dot{u} = \varepsilon \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} u + \varepsilon^3 \begin{pmatrix} 0 & 0 \\ -1/4 - b_3 & 0 \end{pmatrix} u. \quad (5.47)$$

To put this equation in the metanormal form, we introduce w_i by $u = (u_1, u_2) = (w_1, \varepsilon w_2)$. Then, the above equation is brought into

$$\frac{d}{dt} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \varepsilon^2 \begin{pmatrix} 0 & 2 \\ -b_3 - 1/4 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \quad (5.48)$$

Since eigenvalues of the matrix $\begin{pmatrix} 0 & 2 \\ -b_3 - 1/4 & 0 \end{pmatrix}$ are given by $\lambda = \pm \sqrt{-2(b_3 + 1/4)}$, $x = 0$ of Eq.(5.37) is unstable if $b_3 < -1/4$. However, if $b_3 \geq -1/4$, we need more higher order term to investigate the stability.

Now we have obtained one of the boundary curve of the Arnold tongue in the (ε, a) plane,

$$a = \frac{1}{4} + \varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{4}\varepsilon^3 + O(\varepsilon^4). \quad (5.49)$$

In an area on one side of this curve, $x = 0$ of Eq.(5.37) is unstable, and in an area on the other side $x = 0$ of Eq.(5.37) is not unstable.

Our strategy may be effective also for nonlinear equations because variational equations are linear equations.

Example. 5.6. Consider the system

$$\dot{\theta}_i = \omega_i + \varepsilon \sum_{j=1}^3 b_{ij} \sin(\theta_j - \theta_i), \quad \theta_i \in [0, 2\pi), \quad i = 1, 2, 3, \quad (5.50)$$

where ω_i, ε and b_{ij} are parameters. If $b_{ij} = 1$ for $1 \leq i, j \leq 3$, this system is well known as the Kuramoto model of coupled oscillators [Kuramoto, 16]. The Kuramoto model is one of the most studied model of synchronization in a population of oscillators although a few open problems remain [Strogatz, 17]. In what follows, we suppose that

$$\omega_1 = \omega_3, \quad \omega_2 = 0, \quad b_{21} = -b_{23}, \quad b_{32} = b_{12}. \quad (5.51)$$

In this case, it is easy to verify that there exists a synchronous solution $\theta_2(t) = \text{const.}$, $\theta_1(t) = \theta_3(t)$. Our purpose is to investigate the stability of the solution. In addition to the assumptions above, we suppose that $b_{31} = -b_{13} + \varepsilon\delta$, where δ is a constant. Then, Eq.(5.50) is rewritten as

$$\begin{cases} \dot{\theta}_1 = \omega_1 + \varepsilon(b_{12} \sin(\theta_2 - \theta_1) + b_{13} \sin(\theta_3 - \theta_1)), \\ \dot{\theta}_2 = \varepsilon(b_{21} \sin(\theta_1 - \theta_2) - b_{21} \sin(\theta_3 - \theta_2)), \\ \dot{\theta}_3 = \omega_1 + \varepsilon(-b_{13} \sin(\theta_1 - \theta_3) + b_{12} \sin(\theta_2 - \theta_3)) + \varepsilon^2\delta \sin(\theta_1 - \theta_3). \end{cases} \quad (5.52)$$

Put $\cos \theta_k = z_k + \bar{z}_k$, $\sin \theta_k = i(z_k - \bar{z}_k)$, $z_k \in \mathbf{C}$ to express the above equation in the Cartesian coordinate. The RG equation for the resultant equation is as follows:

$$\begin{aligned} \dot{z}_i &= \varepsilon R_1(z) + \varepsilon^2 R_2(z) + O(\varepsilon^3), \quad z = (z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3)^t, \quad (5.53) \\ R_1(z) &= 2b_{13} \left(z_1 \bar{z}_1 z_3 - z_1^2 \bar{z}_3, z_1 \bar{z}_1 \bar{z}_3 - \bar{z}_1^2 z_3, 0, 0, \bar{z}_1 z_3^2 - z_1 z_3 \bar{z}_3, z_1 \bar{z}_3^2 - \bar{z}_1 z_3 \bar{z}_3 \right)^t, \\ R_2(z) &= \frac{4}{\omega_1} \left(iz_1 z_2 \bar{z}_2 b_{12} (-z_3 \bar{z}_1 b_{21} - z_1 \bar{z}_3 b_{21} + 2z_1 \bar{z}_1 b_{12} + 2z_1 \bar{z}_1 b_{21}), \right. \\ &\quad -iz_2 \bar{z}_1 \bar{z}_2 b_{12} (-z_3 \bar{z}_1 b_{21} - z_1 \bar{z}_3 b_{21} + 2z_1 \bar{z}_1 b_{12} + 2z_1 \bar{z}_1 b_{21}), \\ &\quad -2iz_2^2 \bar{z}_2 b_{21} (-z_3 \bar{z}_3 b_{12} + z_3 \bar{z}_3 b_{21} - z_3 \bar{z}_1 b_{21} - z_1 \bar{z}_3 b_{21} + z_1 \bar{z}_1 b_{12} + z_1 \bar{z}_1 b_{21}), \\ &\quad 2iz_2 \bar{z}_2^2 b_{21} (-z_3 \bar{z}_3 b_{12} + z_3 \bar{z}_3 b_{21} - z_3 \bar{z}_1 b_{21} - z_1 \bar{z}_3 b_{21} + z_1 \bar{z}_1 b_{12} + z_1 \bar{z}_1 b_{21}), \\ &\quad (z_1 z_3 \bar{z}_3 - z_3^2 \bar{z}_1) \omega_1 \delta / 2 + iz_2 \bar{z}_2 z_3 b_{12} (z_1 \bar{z}_3 b_{21} + \bar{z}_1 z_3 b_{21} + 2z_3 \bar{z}_3 b_{12} - 2z_3 \bar{z}_3 b_{21}), \\ &\quad \left. + (z_3 \bar{z}_3 \bar{z}_1 - z_1 \bar{z}_3^2) \omega_1 \delta / 2 + iz_2 \bar{z}_2 \bar{z}_3 b_{12} (z_1 \bar{z}_3 b_{21} + \bar{z}_1 z_3 b_{21} + 2z_3 \bar{z}_3 b_{12} - 2z_3 \bar{z}_3 b_{21}) \right)^t. \end{aligned}$$

Putting $z_k = e^{i\phi_k}$, $k = 1, 2, 3$, we express the RG equation in the polar coordinate as

$$\begin{cases} \dot{\phi}_1 = -4\epsilon b_{13} \sin(\phi_1 - \phi_3) + \frac{8\epsilon^2}{\omega_1} (b_{12}^2 + b_{12}b_{21} - b_{12}b_{21} \cos(\phi_1 - \phi_3)) + O(\epsilon^3), \\ \dot{\phi}_2 = -\frac{32\epsilon^2}{\omega_1} b_{21}^2 \sin^2 \frac{1}{2}(\phi_1 - \phi_3) + O(\epsilon^3), \\ \dot{\phi}_3 = -4\epsilon b_{13} \sin(\phi_1 - \phi_3) + \frac{8\epsilon^2}{\omega_1} (b_{12}^2 - b_{12}b_{21} + b_{12}b_{21} \cos(\phi_1 - \phi_3)) + \frac{1}{2}\omega_1 \delta \sin(\phi_1 - \phi_3) + O(\epsilon^3). \end{cases} \quad (5.54)$$

The variational equation along the orbit $\phi_2 = \text{const.}$, $\phi_1 = \phi_3$ for the above equation is given by

$$\frac{d}{dt} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 4\epsilon b_{13} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + 4\epsilon^2 \delta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + O(\epsilon^3). \quad (5.55)$$

If the trivial solution $(v_1, v_2, v_3) = (0, 0, 0)$ of (5.55) is hyperbolically stable, the invariant set $\{\phi_2 = \text{const.}, \phi_1 = \phi_3\}$ for the RG equation (5.54) is stable invariant manifold and we can conclude that the solution $\theta_2 = \text{const.}$, $\theta_1 = \theta_3$ to the original equation (5.52) is stable. However, the

eigenvalues of the matrix $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ are all zero and we can not apply Props.5.1 and 5.2. To

handle this problem, we simplify the term $R_2(z)$ by using our method of simplified RG equation.

Let $B_1(z)$ be an undetermined vector field and $\widetilde{R}_2(z) = R_2(z) - [B_1, R_1](z)$ be a simplified second order term of the RG equation, as is shown in Eq.(4.16). To simplify $\widetilde{R}_2(z)$, it seems that $B_1(z)$ has to be a polynomial of degree 3 because $R_1(z)$ and $R_2(z)$ are polynomial of degree 3 and 5, respectively. However, it is sufficient to simplify the variational equation (5.55), and we may define $B_1(z)$ to be a linear vector field of the form

$$B_1(z) = (0, 0, 0, 0, cz_3, c\bar{z}_3)^t, \quad (5.56)$$

where $c \in \mathbf{R}$ is an undetermined constant. With this $B_1(z)$, we bring the new RG equation $\dot{z} = \epsilon R_1(z) + \epsilon^2 (R_2(z) - [B_1, R_1](z)) + O(\epsilon^3)$ into the equation in the polar coordinate by putting $z_k = e^{i\phi_k}$, and we calculate the variational equation along the orbit $\phi_2 = \text{const.}$, $\phi_1 = \phi_3$. The resultant equation is given by

$$\frac{d}{dt} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 4\epsilon b_{13} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + 4\epsilon^2 \begin{pmatrix} -\alpha & 0 & \alpha \\ 0 & 0 & 0 \\ \beta & 0 & -\beta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + O(\epsilon^3), \quad (5.57)$$

where $\alpha = b_{13}c$ and $\beta = \delta - b_{13}c$. Putting $(v_1, v_2, v_3)^t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} (u_1, u_2, u_3)^t$, we express this equation as

$$\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 4\epsilon b_{13} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + 4\epsilon^2 \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha - \beta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + O(\epsilon^3). \quad (5.58)$$

Now we choose $c = \delta/b_{13}$ so that $\beta = 0$. Further, we put $u_3 = \varepsilon U_3$ to obtain

$$\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \\ U_3 \end{pmatrix} = 4\varepsilon^2 \begin{pmatrix} 0 & 0 & -b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & -\delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ U_3 \end{pmatrix} + O(\varepsilon^3). \quad (5.59)$$

The eigenvalues of the matrix in the right hand side of the above are given by $0, 0, -\delta$. In the original coordinate (v_1, v_2, v_3) , the eigenvectors associated with the zero eigenvalues are given by $(0, 1, 0)^t$ and $(1, 0, 1)^t$, which are sitting along the invariant set $\{\phi_2 = \text{const.}, \phi_1 = \phi_3\}$ of the RG equation. On the other hand, the eigenvector associated with the eigenvalue $-\delta$ is transverse to the invariant set. Therefore, we conclude that the invariant set $\{\theta_2 = \text{const.}, \theta_1 = \theta_3\}$ of the original equation Eq.(5.52) is stable if $|\varepsilon|$ is sufficiently small and if $\delta > 0$.

Fig.2 presents the numerical solution to Eq.(5.52) for parameters $\omega_1 = 1$, $b_{12} = b_{13} = b_{21} = 1$, $\delta = 10$, $\varepsilon = 0.1$ and with an initial values $\theta_1(0) = 0$, $\theta_2(0) = 0$, $\theta_3(0) = 1$.

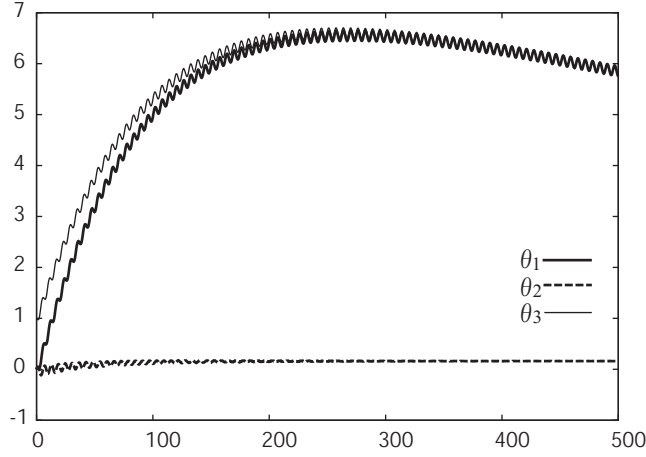


Fig. 2: A numerical result for Eq.(5.52) with $\omega_1 = 1$, $b_{12} = b_{13} = -b_{12} = 1$, $\delta = 10$, $\varepsilon = 0.1$. It is shown that $|\theta_1 - \theta_3|$ tends to zero as $t \rightarrow \infty$.

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