C^1 Approximation of Vector Fields based on the Renormalization Group Method

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Abstract

The renormalization group (RG) method for differential equations is one of the perturbation methods for obtaining solutions which approximate exact solutions for a long time interval. This article shows that, for a differential equation associated with a given vector field on a manifold, a family of approximate solutions obtained by the RG method defines a vector field which is close to the original vector field in the C^1 topology under appropriate assumptions. Furthermore, some topological properties of the original vector field, such as the existence of a normally hyperbolic invariant manifold and its stability are shown to be inherited from those of the RG equation. This fact is applied to the bifurcation theory.

1 Introduction

The renormalization group (RG) method for differential equations is one of the perturbation methods for obtaining solutions which approximate exact solutions for a long time interval. In their papers [1,2], Chen, Goldenfeld, Oono have established the RG method for ordinary differential equations of the form

$$\dot{x} = \frac{dx}{dt} = f(t, x) + \varepsilon g(t, x), \quad x \in \mathbf{R}^n, \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter. For this equation, the method for deriving approximate solutions of the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \tag{1.2}$$

is called the *naive expansion* or the *regular perturbation method*, where $x_i(t)$'s are governed by inhomogeneous linear ODEs obtained by putting Eq.(1.2) into Eq.(1.1) and equating the coefficients of ε^i of the both sides of Eq.(1.1). It is well known that approximate solutions constructed by the naive expansion are valid only in a time interval of $O(1)$ in general, since secular terms diverge as $t \to \infty$. Many techniques for obtaining approximate solutions which are valid in a long time interval have been developed until now, which are collectively called singular perturbation methods.

The RG method proposed by Chen *et al.* is one of the singular perturbation methods looking like the variation-of-constant method, in which the secular terms included in $x_1(t), x_2(t), \cdots$ of Eq.(1.2) are renormalized into the integral constant of $x_0(t)$. The ODE to be satisfied by the renormalized integral constant is called the *RG equation*. Chen *et al.* showed that the RG method unifies the conventional singular perturbation methods such as the multi-scale method, the boundary layer technique, WKB analysis, and the reductive perturbation method, by giving

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explicit examples. Though the multi-scale method requires occasionally fractional power laws or logarithmic functions of ε in the expansion of $x(t)$, the RG method needs only a power-series expansion of $x(t)$ in ε, and it starts with the naive expansion of $x(t)$ to reach the same result as the multi-scale method does.

Kunihiro [3],[4] interpreted the RG method as a theory of envelopes for approximate solutions constructed by the naive expansion. His insight revealed why the RG method works well. Nozaki, Oono [5] and Goto, Masutomi, Nozaki [6] proposed a proto-RG equation or translational Lie group method to renormalize secular terms up to arbitrary order and to obtain higher order approximate solutions. Ei, Fujii, Kunihiro [7] apply the RG method to obtain approximate center manifolds and slow manifolds. Ziane [8] and DeVille *et al.* [9] proved that an orbit constructed on the RG method approximates an exact solution for a long time interval. Further DeVille *et al.* [9] showed that if the unperturbed part of a given ODE is linear and diagonalizable, the RG equation for the ODE is equivalent to the normal form of the vector field.

Despite the active interest in the RG method, little attention has been paid to date to the question as to whether a family of approximate solutions to exact solutions of the original ODE (vector field), which is obtained by varying initial values, forms a well-defined vector field or not. Put another way, a question is to be asked as to whether approximate solutions intersect with one other or not. Further, the RG method has been applied to differential equations only on the Euclidean space, but not extended to a method applicable to differential equations on manifolds, yet.

In the present paper, it is shown that for a given vector field of the form $f(t, x) + \varepsilon g(t, x)$ on an arbitrary manifold, approximate solutions obtained by the RG method define a vector field which is close to the original vector field in the C^1 topology on appropriate assumptions of boundedness for the flow of $f(t, x)$ and for other functions. This implies that the approximate vector field works well in investigating properties of the original vector field that are persistent under C^1 perturbation. In particular, if the approximate vector field has a normally hyperbolic invariant manifold, then the original vector field is expected also to have an invariant manifold because the Fenichel theory assures that normally hyperbolic invariant manifolds are persistent under C^1 perturbation. In fact, it is shown that the existence of an invariant manifold and its stability are inherited from those of the RG equation since the flow of the RG equation is proved to be conjugate to that of the approximate vector field. In view of this, it is desirable that the RG equation is easier to solve than the original equation. In fact, it will be proved that the RG equation has larger symmetry than the original equation. This method will be applied in the bifurcation theory to show that a periodic orbit is emerged far away from a fixed point, which is an example of the global bifurcation other than the ordinary Hopf bifurcation.

In particular, the RG method is applied to a time-dependent linear equation of the form

$$\dot{x} = F(t)x + \varepsilon G(t)x, \quad x \in \mathbf{R}^n, \tag{1.3}$$

where $F(t)$ and $G(t)$ are $n \times n$ matrix functions. On appropriate assumptions, the stability of the trivial solution $x = 0$ of Eq.(1.3) is shown to coincide with that of the RG equation for Eq.(1.3), which is time-*independent* linear equation. By using this result, synchronous solution of coupled oscillators is shown to be stable.

This paper is organized as follows: Sec.2 presents basic facts and definitions in dynamical systems. Sec.3 contains a simple example of the RG method. In Sec.4, a main theorem on approximate vector fields is proved. Sec.5 gives a few properties of the RG equation in term of symmetries. In Sec.6, an invariant manifold of a given equation is shown to be inherited from its RG equation. In Sec.7, the RG method is applied to time-dependent linear equations (1.3). In Appendix A, we discuss the higher order RG equation to prove Thm.6.1.

2 Notations

Let f be a time independent C^r vector field on a C^r manifold M and $\varphi : \mathbf{R} \times M \to M$ its flow. We denote by $\varphi_t(x_0) \equiv x(t)$, $t \in \mathbf{R}$, a solution to the ODE $\dot{x} = f(x)$ through $x_0 \in M$, which satisfies $\varphi_t \circ \varphi_s = \varphi_{t+s}$, $\varphi_0 = id_M$, where id_M denotes the identity map of M. For a fixed $t \in \mathbf{R}$, $\varphi_t : M \to M$ defines a diffeomorphism of M. We assume φ_t is defined for all $t \in \mathbf{R}$.

For a time-dependent vector field, let $x(t, \tau, \xi)$ denote a solution to an ODE $\dot{x}(t) = f(t, x)$ through ξ at $t = \tau$, which defines a flow $\varphi : \mathbf{R} \times \mathbf{R} \times M \to M$ by $\varphi_{t,\tau}(\xi) = x(t, \tau, \xi)$. For fixed $t, \tau \in \mathbf{R}$, $\varphi_{t,\tau} : M \to M$ is a diffeomorphism of M satisfying

$$\varphi_{t,t'} \circ \varphi_{t',\tau} = \varphi_{t,\tau}, \quad \varphi_{t,t} = id_M. \tag{2.1}$$

Conversely, a family of diffeomorphisms $\varphi_{t,\tau}$ of M, which are C^1 with respect to t and τ, satisfying the above equality for any $t, \tau \in \mathbf{R}$ defines a time-dependent vector field on M through

$$f(t, x) = \frac{d}{d\tau}\Big|_{\tau=t} \varphi_{\tau,t}(x). \tag{2.2}$$

3 A brief review of the renormalization group method

Before describing a general theory of the RG method in the next section, we review the RG method for obtaining approximate solutions of an ODE with a simple example.

Let us consider an ODE

$$\ddot{x} + x + \varepsilon x^3 = 0, \quad x \in \mathbf{R}, |\varepsilon| \ll 1. \tag{3.1}$$

Assume that the ODE admits a solution of the form $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$. Then the substitution provides

$$\ddot{x}_0 + \varepsilon \ddot{x}_1 + x_0 + \varepsilon x_1 + \varepsilon(x_0 + \varepsilon x_1)^3 + O(\varepsilon^2) = 0.$$

Expanding this into a power series in ε and equating the coefficients of $\varepsilon^0, \varepsilon^1$ to zero, respectively, we get

$$\ddot{x}_0 + x_0 = 0, \tag{3.2}$$
$$\ddot{x}_1 + x_1 = -x_0^3. \tag{3.3}$$

We denote a general solution of the former whose initial time is $t = 0$ by

$$x_0(t, 0, A) = Ae^{it} + \overline{A}e^{-it}, \quad A \in \mathbf{C}. \tag{3.4}$$

Then (3.3) and (3.4) are put together to give

$$\ddot{x}_1 + x_1 = -(A^3 e^{3it} + 3|A|^2 A e^{it} + 3|A|^2 \overline{A} e^{-it} + \overline{A}^3 e^{-3it}).$$

A special solution of this equation, whose initial time is $t = \tau$, is written as

$$x_1(t, \tau; A) = \frac{A^3}{8} e^{3it} + \frac{3i}{2}|A|^2 A(t - \tau)e^{it} + \text{c.c.}, \tag{3.5}$$

where c.c. is the complex conjugate of the first two terms of the right hand side. Note that a secular term arises, which diverges to infinity as $t \to \infty$. The reason for taking the initial time $t = \tau$ is that we want to construct a family of curves parameterized by τ since approximate solutions obtained by the RG method are given as envelopes of the family (see Kunihiro [3,4]).

Now let us define $\hat{x}$ as

$$\hat{x}(t, \tau; A) = x_0(t, 0, A) + \varepsilon x_1(t, \tau; A).$$

Then $\hat{x}$ is an approximate solution to Eq.(3.1) on short time intervals. Indeed, $\hat{x}$ satisfies the equation

$$\ddot{\hat{x}} + \hat{x} + \varepsilon\hat{x}^3 = 3\varepsilon^2(Ae^{it} + \overline{A}e^{-it})^2(\frac{A^3}{8}e^{3it} + \frac{3i}{2}|A|^2A(t-\tau)e^{it} + \text{c.c.}) + O(\varepsilon^3), \tag{3.6}$$

which implies that if A is bounded and t is sufficiently close to τ, then $\hat{x}$ approximates to an exact solution of (3.1) well. This procedure for obtaining a local approximate solution is called *naive expansion*.

The RG method employs two additional steps to obtain solutions approximating to exact solutions on a long time intervals. At first, we regard the constant A as a differentiable function of τ and differentiate $\hat{x}$ with respect to τ at t,

$$\begin{aligned}\frac{d\hat{x}}{d\tau}\Big|_{\tau=t}(t, \tau, A(\tau)) &= \frac{\partial x_0}{\partial A}\frac{dA}{d\tau}\Big|_{\tau=t} + \varepsilon\frac{\partial x_1}{\partial \tau}\Big|_{\tau=t} + \varepsilon\frac{\partial x_1}{\partial A}\frac{dA}{d\tau}\Big|_{\tau=t} \\ &= A'e^{it} + \overline{A}'e^{-it} + \varepsilon\left(-\frac{3i}{2}|A|^2Ae^{it} + \frac{3A^2}{8}A'e^{3it} + \text{c.c.}\right).\end{aligned}$$

We impose the condition on $A(t)$ that $d\overline{x}/d\tau|_{\tau=t} = 0$, which is called the *RG condition*. Then we obtain the following ODE for $A(t)$

$$\frac{dA}{dt} = \varepsilon\frac{3i}{2}|A|^2A + O(\varepsilon^2).$$

Truncating the higher order term $O(\varepsilon^2)$, we obtain the *RG equation*

$$\frac{dA}{dt} = \varepsilon\frac{3i}{2}|A|^2A, \tag{3.7}$$

which is solved by

$$A(t) := A(t, a, \theta) = \frac{1}{2}a\exp i\left(\frac{3\varepsilon}{8}a^2t + \theta\right), \tag{3.8}$$

where a, θ are arbitrary constants. With this $A(t)$, we define $X(t, a, \theta)$ by

$$X(t, a, \theta) := \hat{x}(t, t, A(t, a, \theta)). \tag{3.9}$$

Then this $X(t)$ gives a solution which approximates an exact solution of (3.1) for a long time interval. The condition $d\overline{x}/d\tau|_{\tau=t} = 0$ means that the curve $X(t, a, \theta) = \hat{x}(t, t; A(t, a, \theta))$ is an envelope for the family of curves $\{\hat{x}(t, \tau; A(\tau, a, \theta))\}_{\tau\in\mathbf{R}}$ (see Kunihiro [3],[4]). Our general definition of the RG equation is shown in the next section.

4 Main theorem

In this section, under appropriate assumptions, we prove that a family of orbits constructed by the RG method defines a vector field which approximates the original vector field in the C^1 topology. Though we show Thm.4.4 for vector fields on Euclidean space, it can be easily extended to vector fields on an arbitrary manifold. See Remark 4.5.

Let $f(t, x)$ and $g(t, x)$ be C^4 and C^3 time-dependent vector fields on $\mathbf{R}^n$, respectively, and consider an ODE

$$\dot{x}(t) = f(t, x) + \varepsilon g(t, x) \tag{4.1}$$

and its unperturbed system

$$\dot{x}_0(t) = f(t, x_0). \tag{4.2}$$

We denote a general solution to the latter by

$$x_0(t) := x_0(t, 0, A) = \varphi^0_{t,0}(A), \tag{4.3}$$

whose initial value is $x_0(0) = A \in \mathbf{R}^n$ at $t = 0$, and where φ^0 is its flow. With this x_0, we further consider an ODE

$$\dot{x}_1(t) = \frac{\partial f}{\partial x}(t, x_0)x_1 + g(t, x_0). \tag{4.4}$$

A general solution to this equation is written as

$$x_1 = (D\varphi^0_{t,0})_A \circ (D\varphi^0_{\tau,0})^{-1}_A h(\tau, A) + (D\varphi^0_{t,0})_A \int_\tau^t (D\varphi_{s,0})^{-1}_A g(s, \varphi^0_{s,0}(A))ds, \tag{4.5}$$

where τ is an initial time, $h(\tau, A)$ is an initial value, and $(D\varphi^0_{t,0})_A$ is the derivative of $\varphi^0_{t,0}$ at A. In what follows, we denote by $\mathbf{R}_{\geq T}$ the set of the real numbers which are larger than or equal to $T \in \mathbf{R}$: $\mathbf{R}_{\geq T} = \{t \in \mathbf{R} \,|\, t \geq T\}$. Set $\mathbf{R}_{\geq T} = \mathbf{R}$ if $T = -\infty$.

Definition 4.1. A function $p(t)$ is said to be KBM on $\mathbf{R}_{\geq T}$ if the number

$$\lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^t p(s)ds \tag{4.6}$$

converges for all $t_0 \geq T$.

The notation of KBM vector fields was introduced in [14] and used in DeVille *et al.*[9] to define the RG equation. Note that periodic functions and almost periodic functions are KBM on $\mathbf{R}$ (see Fink [13]).

Next definition is proposed by DeVille *et al.* [9].

Definition 4.2. Suppose that $(D\varphi^0_{t,0})^{-1}_A g(t, \varphi^0_{t,0}(A))$ is KBM on $\mathbf{R}_{\geq T}$ for each $A \in \mathbf{R}^n$. Then a C^3 function $R : \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$R(A) = \lim_{t\to\infty} \frac{1}{t - T} \int_T^t (D\varphi^0_{s,0})^{-1}_A g(s, \varphi^0_{s,0}(A))ds \tag{4.7}$$

is called the *resonance* or *secular* part for the solution x_1 defined by Eq.(4.5).

By using Eq.(4.7), Eq.(4.5) is rewritten as

$$x_1 = (D\varphi^0_{t,0})_A \circ (D\varphi^0_{\tau,0})^{-1}_A h(\tau, A) + (D\varphi^0_{t,0})_A \int_\tau^t \Big((D\varphi^0_{s,0})^{-1}_A g(s, \varphi^0_{s,0}(A)) - R(A)\Big)\, ds + (D\varphi^0_{t,0})_A R(A)(t-\tau).$$

Define the initial value $h(\tau, A)$ to be

$$h(\tau, A) := (D\varphi^0_{\tau,0})_A \int^\tau \Big((D\varphi^0_{s,0})^{-1}_A g(s, \varphi^0_{s,0}(A)) - R(A)\Big)\, ds, \tag{4.8}$$

where $\int^\tau$ is the indefinite integral, whose integral constant is fixed arbitrarily. Then, x_1 is expressed as

$$\begin{aligned} x_1 := x_1(t,\tau; A) &= (D\varphi^0_{t,0})_A \int^t \Big((D\varphi^0_{s,0})^{-1}_A g(s, \varphi^0_{s,0}(A)) - R(A)\Big)\, ds + (D\varphi^0_{t,0})_A R(A)(t-\tau) \\ &= h(t, A) + (D\varphi^0_{t,0})_A R(A)(t-\tau). \end{aligned} \tag{4.9}$$

In perturbation theory, the second term of the right hand side is called the *secular term*. The reason for defining the initial value $h(\tau, A)$ as (4.8) is that we want to divide x_1 into two terms : the one is the secular term which diverges as $t \to \infty$, and the other is the bounded term $h(t, A)$ (see also the norm condition (N) below). With this $x_1(t, \tau; A)$, we associate a curve defined by

$$\hat{x}(t) := \hat{x}(t, \tau; A) = x_0(t, 0, A) + \varepsilon x_1(t, \tau; A), \tag{4.10}$$

which provides a locally approximate solution of (4.1). Now we define the RG equation.

Definition 4.3. Suppose that $(D\varphi^0_{t,0})^{-1}_A g(t, \varphi^0_{t,0}(A))$ is KBM on $\mathbf{R}_{\geq T}$ for each $A \in \mathbf{R}^n$. Then, the equation defined by

$$\frac{dA}{dt} = \varepsilon R(A), \quad A \in \mathbf{R}^n \tag{4.11}$$

is called the *RG equation* for $f + \varepsilon g$, and the vector field $\varepsilon R(A)$ on $\mathbf{R}^n$ is called the *RG vector field* for $f + \varepsilon g$. We denote by φ^{RG}_t the flow generated by the RG vector field.

In the literature, the RG equation is defined so that its solution $A := A(t)$ may satisfy $d\hat{x}/d\tau|_{\tau=t}(t, \tau; A(\tau)) = 0$. According to our definition of the RG vector field, $d\hat{x}/d\tau|_{\tau=t}$ is calculated as

$$\frac{d\hat{x}}{d\tau}\Big|_{\tau=t}(t, \tau; A(\tau)) = \varepsilon^2 \frac{\partial x_1}{\partial A}(t, t; A(t)) R(A(t)). \tag{4.12}$$

Truncated the higher order term $O(\varepsilon^2)$, Eq.(4.12) implies that solutions to (4.11) satisfy $d\hat{x}/d\tau|_{\tau=t}(t, \tau; A(\tau)) = 0$.

To state our main theorem, we assume the following norm conditions (N) for the functions $f(t, x)$, $g(t, x)$, $x_0(t, 0, A)$ and $h(t, A) = x_1(t, t; A)$ on $\mathbf{R}_{\geq T} \times \mathbf{R}^n$. These conditions will be used to prove that the vector field F_ε defined in Eq.(4.16) is sufficiently close to the original vector field $f + \varepsilon g$ in the C^1 topology (see Eqs.(4.18, 19)).

Norm Conditions (N) Let $K \subset \mathbf{R}^n$ be an arbitrary compact subset. We assume that there exists T such that $(D\varphi^0_{t,0})^{-1}_A g(t, \varphi^0_{t,0}(A))$ is KBM on $\mathbf{R}_{\geq T}$ for each $A \in K$ and the following functions are bounded uniformly on $\mathbf{R}_{\geq T} \times K$.

(N1) $h(t, A)$

(N2) $\partial^2 f/\partial x^2$, $\partial f/\partial x$, $\partial g/\partial x$, $x_0(-t, 0, A)$, $(\partial x_0/\partial A)^{-1}$, $\partial^2 x_0/\partial A^2$, $\partial h/\partial A$, $\partial h^2/\partial A^2$
(N3) f, $\partial^2 f/\partial x\partial t$, $\partial^3 f/\partial x^3$, $\partial^3 f/\partial x^2\partial t$, g, $\partial^2 g/\partial x^2$, $\partial^2 g/\partial x\partial t$, $\partial^3 x_0/\partial A^3$, $\partial^3 h/\partial A^3$

In Sec.6 and Appendix A, we consider a system of the form

$$\dot{x} = Fx + \varepsilon g(t, x), \quad x \in \mathbf{R}^n, \tag{4.13}$$

where F is a diagonalizable $n \times n$ constant matrix all of whose eigenvalues lie on the imaginary axis. In this case, the following is a sufficient condition for this system to satisfy the norm conditions (N1) to (N3).

(i) $g(t, x)$ is polynomial in x and periodic in t.
(ii) $g(t, x)$ is polynomial in x and almost periodic in t the set of whose Fourier exponents has no accumulation points.

See Appendix A for the proof. The case where F has eigenvalues on the left half plane will be treated in a forthcoming paper. In Example 4.6 , we show another example satisfying norm conditions (N) whose unperturbed part is nonlinear.

In what follows, we fix an open subset $U \subset \mathbf{R}^n$ such that $\overline{U}$ is compact. Define $\alpha_t : U \to \mathbf{R}^n$ to be

$$\alpha_t(A) = x_0(t, 0, A) + \varepsilon h(t, A), \tag{4.14}$$

for all $t \in \mathbf{R}_{\geq T}$. The set U is defined so that α_t is diffeomorphism on U (see the proof of Thm.4.4 (i) below). Note that the smaller $|\varepsilon|$ is, the largest set U we can take.

Our main theorem is stated as follows.

Theorem 4.4. Let $f, g, x_0(t, 0, A), x_1(t, \tau; A)$ be vector fields and solutions to differential equations defined in (4.1) to (4.4) and (4.9), respectively. Let $\varepsilon R(A)$ be the RG vector field for $f + \varepsilon g$ and denote its integral curves, whose initial time is t_0 and initial value is $\xi \in U$, by $A(t) := A(t, t_0, \xi) = \varphi^{RG}_{t-t_0}(\xi)$. Then, there exist $\varepsilon_0 > 0$ such that the following holds for all $|\varepsilon| < \varepsilon_0$:

(i) Suppose that the norm condition (N1) is satisfied. Then,

$$\Phi_{t,t_0} := \alpha_t \circ \varphi^{RG}_{t-t_0} \circ \alpha^{-1}_{t_0} : \alpha_{t_0}(U) \to \mathbf{R}^n \tag{4.15}$$

defines a flow on $U_\varepsilon := \{(t, x) \,|\, t \in \mathbf{R}_{\geq T}, x \in \alpha_t(U)\}$ associated with a time-dependent vector field

$$F_\varepsilon(t, x) := \frac{d}{da}\Big|_{a=t} \Phi_{a,t}(x). \tag{4.16}$$

The integral curves of F_ε are put in the form

$$X(t, t_0; \xi) := \hat{x}(t, t; A(t, t_0, \xi)), \tag{4.17}$$

where $\hat{x}$ is defined by (4.10).
(ii) Suppose that the norm conditions (N1), (N2) are satisfied. Then, there exists a non-negative constant L_1 such that the vector field F_ε defined by (4.16) satisfies an inequality

$$\sup_{U_\varepsilon} \|f + \varepsilon g - F_\varepsilon\| < \varepsilon^2 L_1. \tag{4.18}$$

(iii) Suppose that the norm conditions (N1) to (N3) are satisfied. Then, there exists a non-negative constant L_2 such that the vector field F_ε satisfies an inequality

$$\sup_{U_\varepsilon} \|D_{t,x}f + \varepsilon D_{t,x}g - D_{t,x}F_\varepsilon\| < \varepsilon^2 L_2, \tag{4.19}$$

where $D_{t,x}f = (\partial f/\partial t, \partial f/\partial x)$ and $\|D_{t,x}f\| = \|\partial f/\partial x\| + \|\partial f/\partial t\|$. In particular, F_ε is sufficiently close to $f + \varepsilon g$ in the C^1 topology if $|\varepsilon|$ is sufficiently small.

Proof of (i). Since $h(t,x)$ is bounded on $\mathbf{R}_{\geq T} \times U$ by the norm condition (N1), $\varepsilon h(t,x)$ can be sufficiently close to a null function as a C^3 function of x for sufficiently small ε. Since the flow $\varphi^0_{t,0}$ is a C^4 diffeomorphism and since the set of diffeomorphisms is open in the space of C^1 maps in the C^1 topology, it follows that for a sufficiently small ε, the map α_t given by (4.14) is a diffeomorphism from U into $\mathbf{R}^n$ for each $t \in \mathbf{R}_{\geq T}$. Therefore the map $\Phi_{t,t_0} : \alpha_{t_0}(U) \to \mathbf{R}^n$ defined by (4.15) is a diffeomorphism from $\alpha_{t_0}(U)$ into $\mathbf{R}^n$ as well, and satisfies $\Phi_{t,t'} \circ \Phi_{t',t_0} = \Phi_{t,t_0}$, $\Phi_{t,t} = id_{\alpha_t(U)}$. This shows that Φ_{t,t_0} is a flow associated with a vector field F_ε defined by (4.16). Then, it turns out that

$$\Phi_{t,t_0}(\alpha_{t_0}(\xi)) = \alpha_t \circ \varphi^{RG}_{t-t_0}(\xi) = \alpha_t(A(t,t_0,\xi)) = \hat{x}(t,t;A(t,t_0,\xi)) = X(t,t_0;\xi),$$

which implies that $X(t,t_0;\xi)$ gives an integral curve of F_ε, namely,

$$\frac{dX}{dt}(t,t_0;\xi) = F_\varepsilon(t, X(t,t_0;\xi)). \tag{4.20}$$

This ends the proof. ∎

Proof of (ii),(iii). Denote $h(t,A)$ as $h_t(A)$. The vector field $F_\varepsilon(t,x)$ is calculated as

$$\begin{aligned}
F_\varepsilon(t,x) &= \frac{d}{da}\Big|_{a=t}\Big((\varphi^0_{a,0} + \varepsilon h_a) \circ \varphi^{RG}_{a-t} \circ \alpha_t^{-1}(x)\Big) \\
&= \frac{d}{da}\Big|_{a=t}(\varphi^0_{a,0} + \varepsilon h_a) \circ \alpha_t^{-1}(x) + (D\varphi^0_{t,0} + \varepsilon Dh_t)_{\alpha_t^{-1}(x)} \frac{d}{da}\Big|_{a=t}\varphi^{RG}_{a-t} \circ \alpha_t^{-1}(x) \\
&= f(t, x_0(t,0,\alpha_t^{-1}(x))) + \varepsilon \frac{\partial f}{\partial x}(t, x_0(t,0,\alpha_t^{-1}(x)))x_1(t,t;\alpha_t^{-1}(x)) + \varepsilon g(t, x_0(t,0,\alpha_t^{-1}(x))) \\
&\quad + \varepsilon \frac{d}{da}\Big|_{a=t} x_1(t,a,\alpha_t^{-1}(x)) + \varepsilon (D\varphi^0_{t,0} + \varepsilon Dh_t)_{\alpha_t^{-1}(x)} R(\alpha_t^{-1}(x)) \\
&= f(t, x_0(t,0,\alpha_t^{-1}(x))) + \varepsilon g(t, x_0(t,0,\alpha_t^{-1}(x))) \\
&\quad + \varepsilon \frac{\partial f}{\partial x}(t, x_0(t,0,\alpha_t^{-1}(x)))h_t(\alpha_t^{-1}(x)) + \varepsilon^2 (Dh_t)_{\alpha_t^{-1}(x)} R(\alpha_t^{-1}(x)).
\end{aligned}$$

On account of $\alpha_t(x) = x_0(t,0,x) + \varepsilon h_t(x)$, the above equation is expanded as

$$\begin{aligned}
F_\varepsilon(t,x) &= f(t,x) + \varepsilon \frac{df}{d\varepsilon}\Big|_{\varepsilon=0}(t, x_0(t,0,\alpha_t^{-1}(x))) + \frac{\varepsilon^2}{2}\frac{d^2 f}{d\varepsilon^2}\Big|_{\varepsilon=\theta_1\varepsilon}(t, x_0(t,0,\alpha_t^{-1}(x))) + \varepsilon g(t,x) \\
&\quad + \varepsilon^2 \frac{dg}{d\varepsilon}\Big|_{\varepsilon=\theta_2\varepsilon}(t, x_0(t,0,\alpha_t^{-1}(x))) + \varepsilon \frac{\partial f}{\partial x}(t,x)h_t((\varphi^0_{t,0})^{-1}(x)) + \varepsilon^2 \frac{\partial f}{\partial x}(t,x)\frac{dh_t}{d\varepsilon}\Big|_{\varepsilon=\theta_3\varepsilon}(\alpha_t^{-1}(x)) \\
&\quad + \varepsilon^2 \frac{d}{d\varepsilon}\Big|_{\varepsilon=\theta_4\varepsilon}\left(\frac{\partial f}{\partial x}(t, x_0(t,0,\alpha_t^{-1}(x)))\right)h_t(\alpha_t^{-1}(x)) + \varepsilon^2 (Dh_t)_{\alpha_t^{-1}(x)} R(\alpha_t^{-1}(x)),
\end{aligned}$$

where $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$ are constants in the Taylor's formula. The second term of the right hand side of the above is calculated as

$$\frac{df}{d\varepsilon}\Big|_{\varepsilon=0}(t, x_0(t,0,\alpha_t^{-1}(x))) = \frac{\partial f}{\partial x}(t,x)\frac{\partial x_0}{\partial A}(t,0,(\varphi^0_{t,0})^{-1}(x))\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\alpha_t^{-1}(x) = -\frac{\partial f}{\partial x}(t,x)h_t((\varphi^0_{t,0})^{-1}(x)).$$

Therefore we obtain

$$\begin{aligned}F_\varepsilon(t,x)-f(t,x)-\varepsilon g(t,x) = {}& \frac{\varepsilon^2}{2}\frac{d^2 f}{d\varepsilon^2}\Big|_{\varepsilon=\theta_1\varepsilon}(t,x_0(t,0,\alpha_t^{-1}(x))) + \varepsilon^2\frac{dg}{d\varepsilon}\Big|_{\varepsilon=\theta_2\varepsilon}(t,x_0(t,0,\alpha_t^{-1}(x)))\\ &+\varepsilon^2\frac{\partial f}{\partial x}(t,x)\frac{dh_t}{d\varepsilon}\Big|_{\varepsilon=\theta_3\varepsilon}(\alpha_t^{-1}(x)) + \varepsilon^2\frac{d}{d\varepsilon}\Big|_{\varepsilon=\theta_4\varepsilon}\left(\frac{\partial f}{\partial x}(t,x_0(t,0,\alpha_t^{-1}(x)))\right)h_t(\alpha_t^{-1}(x))\\ &+\varepsilon^2(Dh_t)_{\alpha_t^{-1}(x)}R(\alpha_t^{-1}(x)). \end{aligned}\tag{4.21}$$

We have to estimate the norm of the right hand side of the above equation. At first, $df/d\varepsilon$ is given by

$$\begin{aligned}&\frac{df}{d\varepsilon}(t,x_0(t,0,\alpha_t^{-1}(x)))\\ &= -\frac{\partial f}{\partial x}(t,x_0(t,0,\alpha_t^{-1}(x)))\frac{\partial x_0}{\partial A}(t,0,\alpha_t^{-1}(x))\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)^{-1} h_t(\alpha_t^{-1}(x)).\end{aligned}\tag{4.22}$$

Note that equations

$$\alpha_t^{-1}(x) = (\varphi_{t,0}^0+\varepsilon h_t)^{-1}(x) = (id+\varepsilon(\varphi_{t,0}^0)^{-1}\circ h_t)^{-1}\circ(\varphi_{t,0}^0)^{-1}(x),\tag{4.23}$$

$$x_0(t,0,\alpha_t^{-1}(x)) = \varphi_{t,0}^0\circ\alpha_t^{-1}(x) = (id-\varepsilon h_t\circ\alpha_t^{-1})(x),\tag{4.24}$$

$$\begin{aligned}&\frac{\partial x_0}{\partial A}(t,0,\alpha_t^{-1}(x))\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)^{-1}\\ &= id-\varepsilon\left(\frac{\partial h_t}{\partial A}\right)_{\alpha_t^{-1}(x)}\sum_{k=0}^{\infty}\left(-\varepsilon\left(\frac{\partial x_0}{\partial A}\right)^{-1}_{\alpha_t^{-1}(x)}\circ\left(\frac{\partial h_t}{\partial A}\right)_{\alpha_t^{-1}(x)}\right)^k\circ\left(\frac{\partial x_0}{\partial A}\right)^{-1}_{\alpha_t^{-1}(x)}\end{aligned}\tag{4.25}$$

hold and the left hand side of the above three equations are bounded by the norm conditions (N1),(N2). Therefore the right hand side of Eq.(4.22) is bounded uniformly in $\mathbf{R}_{\geq T}$. To show the boundedness of the first term of right hand side of Eq.(4.21), it is sufficient to show that the derivative of each factor of the right hand side of Eq.(4.22) is bounded. They are calculated as

$$\begin{aligned}&\frac{d}{d\varepsilon}\frac{\partial f}{\partial x}(t,x_0(t,0,\alpha_t^{-1}(x)))\\ &= -\frac{\partial^2 f}{\partial x^2}(t,x_0(t,0,\alpha_t^{-1}(x)))\frac{\partial x_0}{\partial A}(t,0,\alpha_t^{-1}(x))\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)^{-1} h_t(\alpha_t^{-1}(x)),\end{aligned}\tag{4.26}$$

$$\begin{aligned}&\frac{d}{d\varepsilon}\frac{\partial x_0}{\partial A}(t,0,\alpha_t^{-1}(x))\\ &= -\frac{\partial^2 x_0}{\partial A^2}(t,0,\alpha_t^{-1}(x))\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)^{-1} h_t(\alpha_t^{-1}(x))\end{aligned}\tag{4.27}$$

$$\begin{aligned}&\frac{d}{d\varepsilon}\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)^{-1}\\ &= -\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)^{-1}\frac{d}{d\varepsilon}\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)^{-1},\end{aligned}\tag{4.28}$$

$$\begin{aligned}&\frac{d}{d\varepsilon}\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)\\ &= \left(\frac{\partial h_t}{\partial A}\right)_{\alpha_t^{-1}(x)} - \left(\frac{\partial^2}{\partial A^2}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)^{-1} h_t(\alpha_t^{-1}(x)),\end{aligned}\tag{4.29}$$

$$\begin{aligned}&\frac{d}{d\varepsilon}h_t(\alpha_t^{-1}(x))\\ &= -\left(\frac{\partial h_t}{\partial A}\right)_{\alpha_t^{-1}(x)}\left(\frac{\partial}{\partial A}(\varphi_{t,0}^0+\varepsilon h_t)_{\alpha_t^{-1}(x)}\right)^{-1} h_t(\alpha_t^{-1}(x)).\end{aligned}\tag{4.30}$$

By the norm conditions and Eq.(4.23, 24, 25), these are bounded uniformly in $\mathbf{R}_{\geq T}$. Therefore the first term of the right hand side of Eq.(4.21) is bounded.

The boundedness of the second term of the right hand side of Eq.(4.21) is verified from Eq.(4.22) by using g instead of f, and the boundedness of other terms of the right hand side of Eq.(4.21) are verified from Eq.(4.26, 30) and the norm conditions (N1),(N2). This proves Thm 4.4 (ii). Thm 4.4 (iii) is verified by differentiating both sides of Eq.(4.21) with respect to x, t and estimating the norm as above. This calculation is elementary and omitted here. ■

Remark 4.5. Though we have treated the vector field F_ε on an open set of $\mathbf{R}^n$, the vector field F_ε may be defined in the case of an arbitrary manifold M. Let $\{U_i\}_{i\in\Lambda}$ be an open covering of M such that each $\overline{U_i}$ is compact. We identify U_i with an open subset on $\mathbf{R}^n$. Suppose that $U_i \cap U_j \neq \emptyset$ and let $\psi_{ij} : U_i \cap U_j \to U_i \cap U_j$ be a coordinate transformation function from U_i to U_j. Let $\varepsilon R^i(A)$ and $\varepsilon R^j(A)$ be the RG vector fields constructed on U_i and U_j, respectively, and let $\varphi_t^{RG(i)}, \varphi_t^{RG(j)}$ be respective flows. By Eq.(4.7), it is easy to verify that $R^i(A) = (D\psi_{ij})^{-1}R^j(\psi_{ij}(A))$ and $\varphi_t^{RG(i)} = \psi_{ij}^{-1} \circ \varphi_t^{RG(j)} \circ \psi_{ij}$. Let $F_\varepsilon^i, F_\varepsilon^j$ be approximate vector fields constructed on U_i, U_j defined by (4.16), respectively. Then F_ε^i is transformed by the coordinate transformation as follows:

$$\begin{aligned} D\psi_{ij}F_\varepsilon^i(t,x) &= D\psi_{ij}\frac{d}{da}\Big|_{a=t}\Phi_{a,t}(x) \\ &= \frac{d}{da}\Big|_{a=t}\psi_{ij}\circ\alpha_t\circ\varphi_{t-t_0}^{RG(i)}\circ\alpha_{t_0}^{-1}(x) \\ &= \frac{d}{da}\Big|_{a=t}\psi_{ij}\circ(x_0+\varepsilon h)\circ\psi_{ij}^{-1}\circ(\psi_{ij}\circ\varphi_{t-t_0}^{RG(i)}\circ\psi_{ij}^{-1})\circ(\psi_{ij}\circ(x_0+\varepsilon h)\circ\psi_{ij}^{-1})^{-1}(\psi_{ij}(x)), \end{aligned}$$

where $\psi_{ij}\circ x_0(t,0,\psi_{ij}^{-1}(x))$ and $\psi_{ij}\circ h(t,\psi_{ij}^{-1}(x)) = \psi_{ij}\circ x_1(t,t,\psi_{ij}^{-1}(x))$ are coordinate representations on U_j of $x_0(t,0,x)$ and of $x_1(t,t,x)$, respectively, which are represented in the coordinates on U_i. This means that

$$D\psi_{ij}F_\varepsilon^i(t,x) = F_\varepsilon^j(t,\psi_{ij}(x)), \quad x \in U_i. \tag{4.31}$$

Let $\{\rho_i\}_{i\in\Lambda}$ be a partition of unity subordinate to the cover $\{U_i\}_{i\in\Lambda}$ and define $F_\varepsilon(t,x) := \sum_{i\in\Lambda}\rho_i(x)F_\varepsilon^i(t,x)$, then F_ε is a well-defined vector field on M which approximates to $f+\varepsilon g$.

Remark 4.6. Now that we have the approximate vector field $F_\varepsilon(t,x) = f(t,x)+\varepsilon g(t,x)+O(\varepsilon^2)$, the Gronwall inequality immediately proves the error estimate for approximate solutions :

Let $x(t,t_0)$ be a solution of Eq.(4.1) satisfying the norm conditions (N) whose initial time is t_0. Let $X(t,t_0;\xi)$ be a curve defined by Eq.(4.17). Suppose that $x(t_0,t_0) = X(t_0,t_0;\xi) \in \alpha_t(U)$. Then, there exist positive constants ε_0, T, C such that the inequality

$$||x(t,t_0) - X(t,t_0;\xi)|| < C\varepsilon, \quad 0 < t < T/\varepsilon \tag{4.32}$$

holds for $0 < \varepsilon < \varepsilon_0$.

This fact was essentially proved in Ziane [8] and DeVille *et al.* [9]. Note that DeVille *et al.* also treated the case that the norm conditions (N) are not satisfied, for example, $g(t,x) = x/\sqrt{t}$. The above fact is also followed by putting $m=1$ and replacing e^{Ft} by $(D\varphi_{t,0}^0)_A$ in the proof of Thm.A.8, in which the error estimate for a higher order case by using the higher order RG equation is proved.

In the next example, the RG method is applied to a vector field whose unperturbed part is nonlinear. Application to vector fields with linear unperturbed parts will be treated in Sec.6.

Example 4.7. Consider a system on $\{(x,y)\,|\,x>0, y\in\mathbf{R}\}\subset\mathbf{R}^2$

$$\begin{cases} \dot{x} = xy + \varepsilon xy^2, \\ \dot{y} = -\log x + \varepsilon y, \end{cases} \tag{4.33}$$

where $\varepsilon \in \mathbf{R}$ is a small constant. Note that unperturbed part is nonlinear. In order to obtain approximate solutions to (4.33), we apply the RG method. The unperturbed system of (x_0, y_0) is written as $\dot{x}_0 = x_0 y_0$, $\dot{y}_0 = -\log x_0$. Its general solution, whose initial value is $(x_0(0), y_0(0)) = (A, B)$, is given by

$$x_0(t) = e^{B\sin t + (\log A)\cos t},\ \ y_0(t) = B\cos t - (\log A)\sin t. \tag{4.34}$$

The RG equation defined by Eq.(4.11) is calculated as

$$\frac{d}{dt}\begin{pmatrix} A \\ B \end{pmatrix} = \frac{\varepsilon}{2}\begin{pmatrix} A\log A \\ B \end{pmatrix}, \tag{4.35}$$

which is solved as

$$A(t) = \exp\left(pe^{\varepsilon t/2}\right),\ \ B(t) = qe^{\varepsilon t/2}, \tag{4.36}$$

where $p, q \in \mathbf{R}$ are arbitrary constants. On the other hand, $h(t, A, B)$ defined by Eq.(4.8) is given by $h(t, A, B) = (D\varphi^0_{t,0})_{(A,B)} M(t)$, where

$$(D\varphi^0_{t,0})_{(A,B)} = \begin{pmatrix} \cos t\cdot e^{B\sin t + (\log A)\cos t}/A & \sin t\cdot e^{B\sin t+(\log A)\cos t} \\ -\sin t/A & \cos t \end{pmatrix}, \tag{4.37}$$

$$M(t) = \begin{pmatrix} \dfrac{A(\log A)^2 - AB^2}{3}\sin^3 t + \dfrac{2AB\log A}{3}\cos^3 t - \dfrac{AB}{2}\sin^2 t + AB^2\sin t - \dfrac{A\log A}{4}\sin^2 t \\ \dfrac{(\log A)^2 - B^2}{3}\cos^3 t - \dfrac{2B\log A}{3}\sin^3 t - \dfrac{\log A}{2}\sin^2 t - (\log A)^2\cos t + \dfrac{B}{4}\sin 2t \end{pmatrix}. \tag{4.38}$$

It is easy to verify that the norm conditions (N) are satisfied. According to (4.17) with the present $A(t), B(t)$, an approximate solution to (4.33) is given by

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} e^{B(t)\sin t - (\log A(t))\cos t} \\ B(t)\cos t - (\log A(t))\cos t \end{pmatrix} + \varepsilon h(t, A(t), B(t)). \tag{4.39}$$

Note that the RG vector field $\dfrac{\varepsilon}{2}(x\log x, y)$ commutes with the vector field $(xy, -\log x)$, which is the unperturbed part of Eq.(4.33) with respect to the Lie bracket product. This fact is proved generally in the next section.

5 RG vector fields with symmetry

In this section, we consider an autonomous equation on a manifold M

$$\dot{x} = f(x) + \varepsilon g(x),\quad x\in M. \tag{5.1}$$

For this equation, we suppose that $(D\varphi^0_s)^{-1}_A g(\varphi^0_s(A))$ is KBM on $\mathbf{R}_{\geq T}$ and the RG equation for $f + \varepsilon g$

$$\frac{dA}{dt} = \varepsilon R(A) = \varepsilon \lim_{t\to\infty}\frac{1}{t-T}\int_T^t (D\varphi^0_s)^{-1}_A g(\varphi^0_s(A))ds \tag{5.2}$$

is defined, where φ^0 is a flow of $f(x)$ satisfying $\varphi^0_{t+t'} = \varphi^0_t \circ \varphi^0_{t'}$.

Assume that a Lie group G acts on the manifold M. If a vector field f on M satisfies

$$(Da)_x f(x) = f(ax),\quad \forall a\in G,\ \forall x\in M, \tag{5.3}$$

then f is called *invariant* under the action of G, where $(Da)_x$ is the derivative at x of the map determined by $a : M \to M$ at x.

Proposition 5.1. If vector fields f and g are invariant under the action of a Lie group G, then so is the RG vector field for $f + \varepsilon g$.

Proof. For all $a \in G$, $R(aA)$ is calculated as

$$\begin{aligned}R(aA) &= \lim_{t\to\infty} \frac{1}{t-T} \int_T^t (D\varphi^0_s)^{-1}_{aA} g(\varphi^0_s(aA))ds \\ &= \lim_{t\to\infty} \frac{1}{t-T} \int_T^t (Da)_A (D\varphi^0_s)^{-1}_A (Da)^{-1}_A g(a\varphi^0_s(A))ds \\ &= (Da)_A \lim_{t\to\infty} \frac{1}{t-T} \int_T^t (D\varphi^0_s)^{-1}_A (Da)^{-1}_A (Da)_A g(\varphi^0_s(A))ds = (Da)_A R(A).\end{aligned}$$

This proves the proposition. ■

The next proposition was proved by Ziane [8] for the case that $f(t, x)$ is a linear vector field.

Proposition 5.2. The RG vector field $\varepsilon R(A)$ for $f + \varepsilon g$ commutes with f with respect to the Lie bracket product. Equivalently, $R(A)$ satisfies

$$(D\varphi^0_t)_A R(A) = R(\varphi^0_t(A)), \tag{5.4}$$

for all $t \in \mathbf{R}$ and all $A \in M$.

Proof. For all $s' \in \mathbf{R}$ and for all $A \in M$, $R(\varphi^0_{s'}(A))$ is calculated as

$$\begin{aligned}R(\varphi^0_{s'}(A)) &= \lim_{t\to\infty} \frac{1}{t-T} \int_T^t (D\varphi^0_s)^{-1}_{\varphi^0_{s'}(A)} g(\varphi^0_s \circ \varphi^0_{s'}(A))ds \\ &= \lim_{t\to\infty} \frac{1}{t-T} \int_T^t (D\varphi^0_{s'})_A \circ (D\varphi^0_s)^{-1}_A \circ (D\varphi^0_{s'})^{-1}_A g(\varphi^0_{s+s'}(A))ds \\ &= (D\varphi^0_{s'})_A \lim_{t\to\infty} \frac{1}{t-T} \int_T^t (D\varphi^0_{s+s'})^{-1}_A g(\varphi^0_{s+s'}(A))ds.\end{aligned}$$

Putting $s + s' = s''$ provides

$$\begin{aligned}R(\varphi^0_{s'}(A)) &= (D\varphi^0_{s'})_A \lim_{t\to\infty} \frac{1}{t-T} \int_{T+s'}^{t+s'} (D\varphi^0_{s''})^{-1}_A g(\varphi^0_{s''}(A))ds'' \\ &= (D\varphi^0_{s'})_A R(A) + (D\varphi^0_{s'})_A \lim_{t\to\infty} \frac{1}{t-T} \int_t^{t+s'} (D\varphi^0_{s''})^{-1}_A g(\varphi^0_{s''}(A))ds'' \\ &\quad -(D\varphi^0_{s'})_A \lim_{t\to\infty} \frac{1}{t-T} \int_T^{T+s'} (D\varphi^0_{s''})^{-1}_A g(\varphi^0_{s''}(A))ds'' \\ &= (D\varphi^0_{s'})_A R(A).\end{aligned}$$

This proves the proposition. ■

Propositions 5.1 and 5.2 show that if vector fields f and g are invariant under the action of a Lie group G, then the RG vector field $\varepsilon R(A)$ is invariant under the action of G and the one-parameter group $\{\varphi^0_t\}_{t\in\mathbf{R}}$. In this sense, the RG vector field has a simpler structure than the original vector field $f + \varepsilon g$.

6 Invariant Manifolds

In this section, we consider an equation of the form

$$\dot{x} = Fx + \varepsilon g(x), \quad x \in \mathbf{R}^n, \tag{6.1}$$

where F is a diagonalizable $n \times n$ constant matrix all of whose eigenvalues lie on the imaginary axis, and where g is a polynomial vector field on $\mathbf{R}^n$. Note that in this situation, the norm conditions (N) are satisfied.

Theorem 6.1. If the RG vector field $\varepsilon R(x)$ for Eq.(6.1) has a boundaryless compact normally hyperbolic invariant manifold N, then Eq.(6.1) also has a normally hyperbolic invariant manifold N_ε for sufficiently small $\varepsilon > 0$. This invariant manifold N_ε is diffeomorphic to N and its stability coincides with that of N.

We will prove this theorem in Appendix A, while we give a brief sketch of the proof below.

Suppose that the RG vector field has a normally hyperbolic invariant manifold N. Then, the approximate vector field $F_\varepsilon(t, x)$ defined by Eq.(4.16) has a normally hyperbolic invariant manifold $\tilde{N}$ which is diffeomorphic to $\mathbf{R} \times N$ in the (t, x) space since the flow of the approximate vector field is related to the flow of the RG vector field through Eq.(4.15). Now we need the Fenichel's theorem :

Theorem. (Fenichel, [10])

Let M be a C^r manifold ($r \geq 1$), and $\mathcal{X}^r(M)$ the set of C^r vector fields on M with the C^1 topology. Let f be a C^r vector field on M and suppose that $N \subset M$ is a boundaryless compact connected normally hyperbolic f-invariant manifold. Then, the following holds:

(i) There is a neighborhood $\mathcal{U} \subset \mathcal{X}^r(M)$ of f such that there exists an normally hyperbolic g-invariant C^r manifold $N_g \subset M$ for $\forall g \in \mathcal{U}$.
(ii) N_g is diffeomorphic to N and the diffeomorphism $h : N_g \to N$ is close to the identity $id : N \to N$ in the C^1 topology.

See [10],[11],[12] for the proof of the theorem and the definition of normal hyperbolicity. Since the approximate vector field $F_\varepsilon(t, x)$ is C^1 close to the original vector field $Fx + \varepsilon g(x)$, we expect that Fenichel's theorem concludes that the original vector field $Fx + \varepsilon g(x)$ has an invariant manifold which is diffeomorphic to $\mathbf{R} \times N$ in the (t, x) space. Since Eq.(6.1) is an autonomous equation, $Fx + \varepsilon g(x)$ has an invariant manifold which is diffeomorphic to N in the x space.

The above argument need to be modified because the approximate vector field is time-dependent vector field even if the original vector is independent of t, while Fenichel's theorem holds for time-independent vector fields. In Appendix A, we define the higher order RG equation to refine the error estimate of the approximate vector field to prove Thm.6.1.

Note that for the case of compact normally hyperbolic invariant manifolds with boundary, Fenichel's theorem is modified as follows : If a vector field f has a compact connected normally hyperbolic invariant manifold N with boundary, then a vector field g, which is C^1 close to f, has a *locally* invariant manifold N_g which is diffeomorphic to N. In this case, an orbit of the flow of g through a point on N_g may go out from N_g through its boundary.

According to this theorem, Thm.6.1 has to be modified so that N_ε is locally invariant if N has boundary.

Example 6.2. Consider the system on $\mathbf{R}^2$

$$\begin{cases} \dot{x} = y - x^3 + \varepsilon x, \\ \dot{y} = -x. \end{cases} \tag{6.2}$$

The unperturbed system $\dot{x} = y - x^3$, $\dot{y} = -x$ has the origin as a fixed point which is *not* hyperbolic. By using Thm.6.1, we show the occurrence of the Hopf bifurcation at $\varepsilon = 0$ and a stable periodic orbit appears for $\varepsilon > 0$.

Changing the coordinate by $(x, y) = (\varepsilon X, \varepsilon Y)$, we obtain

$$\begin{cases} \dot{X} = Y + \varepsilon(X - \varepsilon X^3), \\ \dot{Y} = -X. \end{cases} \tag{6.3}$$

We want to regard the term $\varepsilon^2 X^3$ as a *first* order term with respect to ε since at this time, we define only the *first* order RG equation while the higher order RG equation will be defined in Appendix A. To do so, define the function $\varepsilon_0(t)$ by $\varepsilon_0(t) \equiv \varepsilon$ and rewrite Eq.(6.3) as

$$\begin{cases} \dot{X} = Y + \varepsilon(X - \varepsilon_0 X^3), \\ \dot{Y} = -X, \\ \dot{\varepsilon}_0 = 0. \end{cases} \tag{6.4}$$

Then this system takes the form (6.1). The RG method is applicable to (6.4). Substitute $X = X_0 + \varepsilon X_1$, $Y = Y_0 + \varepsilon Y_1$ into (6.4) and equate the coefficients of $\varepsilon^0, \varepsilon^1$ to zero, respectively. Then we get

$$\begin{cases} \dot{X}_0 = Y_0, \\ \dot{Y}_0 = -X_0, \end{cases} \quad \begin{cases} \dot{X}_1 = Y_1 + X_0 - \varepsilon_0 X_0^3, \\ \dot{Y}_1 = -X_1. \end{cases} \tag{6.5}$$

We denote a solution to the former by

$$X_0(t) = Ae^{it} + \overline{A}e^{-it}, \quad A \in \mathbf{C}. \tag{6.6}$$

With this $X_0(t)$, a special solution to the latter defined by (4.9), whose initial time is $t = \tau$, is written as

$$X_1(t) = \frac{1}{2}(A - 3\varepsilon_0 A|A|^2)(t - \tau)e^{it} + \frac{3i}{8}A^3 e^{3it} + \text{c.c.}, \tag{6.7}$$

where c.c. is the complex conjugate of the first two terms of the right hand side. Therefore, the RG equation for (6.3) is given by

$$\frac{dA}{dt} = \frac{1}{2}\varepsilon(A - 3\varepsilon_0 A|A|^2). \tag{6.8}$$

Substituting $A = re^{i\theta}$ into the above equation provides

$$\begin{cases} \dot{r} = \dfrac{\varepsilon}{2}(r - 3\varepsilon_0 r^3), \\ \dot{\theta} = 0. \end{cases} \tag{6.9}$$

Fixed points of this system are $r = 0$ and $r = \sqrt{1/3\varepsilon_0} := r_0$, when $\varepsilon_0 > 0$. Further, we obtain

$$\frac{d}{dr}\Big|_{r=r_0} \frac{\varepsilon}{2}(r - 3\varepsilon_0 r^3) = \frac{\varepsilon}{2}(1 - 9\varepsilon_0 \cdot \frac{1}{3\varepsilon_0}) = -\varepsilon < 0.$$

This means that the RG equation (6.9) has a circle $\{r = r_0\}$ as a stable normally hyperbolic invariant manifold (the set of fixed points) if $\varepsilon > 0$. By Thm 6.1, the system (6.2) also has a stable periodic orbit if $\varepsilon > 0$ is sufficiently small. This proves that the Hopf bifurcation occurs for (6.2). Note that the radius of the invariant circle for the RG equation is of order $O(1/\sqrt{\varepsilon})$. In the original coordinate (x, y), the radius of the periodic orbit for the system (6.2) is of order $O(\sqrt{\varepsilon})$. Indeed, the periodic solution is approximately given by $x(t) = 2\sqrt{\varepsilon/3}\cos t$ in the (x, y) coordinate.

We can show that the second order RG equation defined in Def.A.5 for Eq.(6.3) is given as $\dot{r} = \varepsilon(r - 3\varepsilon r^3)/2$, $\dot{\theta} = -\varepsilon^2/8$. Thus we can obtain the same result as above without introducing ε_0 by using the second order RG equation, although it provides a modification to the motion in the θ direction.

We have just seen in Example 6.2 that the RG method can be used on problems in which there is an ordinary Hopf bifurcation. In the next example, we show that the RG method can also be used for systems in which a limit cycle is created far away from a fixed point, namely with $O(1)$ radius.

Example 6.3. Consider the system on $\mathbf{R}^2$

$$\begin{cases} \dot{x} = y + \varepsilon(x - x^3), \\ \dot{y} = -x. \end{cases} \tag{6.10}$$

Substituting $x = x_0 + \varepsilon x_1$, $y = y_0 + \varepsilon y_1$ into (6.10) and equating the coefficients of $\varepsilon^0, \varepsilon^1$ to zero, respectively, we get

$$\begin{cases} \dot{x}_0 = y_0, \\ \dot{y}_0 = -x_0, \end{cases} \quad \begin{cases} \dot{x}_1 = y_1 + x_0 - x_0^3, \\ \dot{y}_1 = -x_1. \end{cases} \tag{6.11}$$

We denote a solution to the former by

$$x_0(t) = Ae^{it} + \overline{A}e^{-it}, \quad A \in \mathbf{C}. \tag{6.12}$$

With this $x_0(t)$, a special solution to the latter defined by (4.9), whose initial time is $t = \tau$, is written as

$$x_1(t) = \frac{1}{2}(A - 3A|A|^2)(t - \tau)e^{it} + \frac{3i}{8}A^3e^{3it} + \text{c.c.}, \tag{6.13}$$

where c.c. is the complex conjugate of the first two terms of the right hand side. Therefore, the RG equation for (6.10) is given by

$$\frac{dA}{dt} = \frac{1}{2}\varepsilon(A - 3A|A|^2). \tag{6.14}$$

Substituting $A = re^{i\theta}$ into the above equation provides

$$\begin{cases} \dot{r} = \dfrac{\varepsilon}{2}(r - 3r^3), \\ \dot{\theta} = 0. \end{cases} \tag{6.15}$$

Fixed points of this system are $r = 0$ and $r = \sqrt{1/3} := r_0$, when $\varepsilon > 0$. It is easy to verify that $r = r_0$ is the stable fixed point. Therefore the system (6.10) has a stable periodic orbit if $\varepsilon > 0$ is sufficiently small. Note that since the radius of the invariant circle for the RG equation is of $O(1)$, the radius of the periodic orbit of the system (6.10) is also of $O(1)$. This can be verified numerically. For each ε, points $y_0 > 0$ at which the periodic orbit for the system (6.10) crosses the y axis are calculated numerically to provide the Fig.1 below. The radius y_0 is almost independent of ε when $\varepsilon > 0$ is sufficiently small.

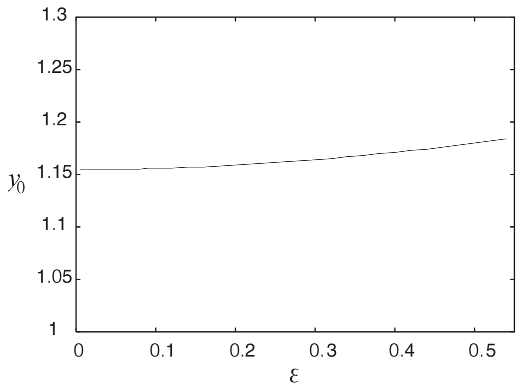

Fig. 1: The radius y_0 of the periodic orbit of the system (6.10) for each ε.

7 Linear Equations

We apply the RG method to a time-dependent linear equation

$$\dot{x} = F(t)x + \varepsilon G(t)x, \quad x \in \mathbf{R}^n, \tag{7.1}$$

where $F(t)$ and $G(t)$ are $n \times n$ matrix functions which are of C^1 class with respect to t. A solution to the equation $\dot{x}_0 = F(t)x_0$ is denoted by $x_0(t, 0, v) = X(t)v$, where $X(t)$ is the fundamental matrix and $v \in \mathbf{R}^n$ is an initial value. We assume that $X(t)^{-1}G(t)X(t)$ is KBM on $t \geq 0$, and we define a constant matrix

$$R := \lim_{t\to\infty} \frac{1}{t} \int_0^t X^{-1}(s)G(s)X(s)ds. \tag{7.2}$$

We call it a *secular matrix* for Eq.(7.1). Then, a special solution to an equation $\dot{x}_1 = F(t)x_1 + G(t)x_0(t, 0, v)$ defined by (4.9) is given by

$$x_1(t, \tau; v) = X(t)\widetilde{G}(t)v + X(t)(t - \tau)Rv, \quad \widetilde{G}(t) = \int^t (X(s)^{-1}G(s)X(s) - R)ds, \tag{7.3}$$

and the RG equation for (7.1) is given by a linear equation

$$\dot{v} = \varepsilon Rv, \quad v \in \mathbf{R}^n. \tag{7.4}$$

If $X(t)$ and $\widetilde{G}(t)$ is bounded in $t \geq 0$, then Thm 4.4 (i) holds and the flow Φ_{t,t_0} defined by (4.15) is put in the form

$$\Phi_{t,t_0} = X(t)(I + \varepsilon\widetilde{G}(t))e^{\varepsilon R(t-t_0)}(I + \varepsilon\widetilde{G}(t_0))^{-1}X(t_0)^{-1}, \tag{7.5}$$

where I is the $n \times n$ identity matrix. Accordingly, the approximate vector field F_ε defined by (4.16) is expressed as

$$F_\varepsilon(t, x) = F(t)x + \varepsilon G(t)X(t)(I + \varepsilon\widetilde{G}(t))^{-1}X(t)^{-1}x + \varepsilon^2 X(t)\widetilde{G}(t)R(I + \varepsilon\widetilde{G}(t))^{-1}X(t)^{-1}x. \tag{7.6}$$

The following proposition means that the stability of $X(t)^{-1}x(t)$ is inherited from that of the RG equation if $\varepsilon > 0$ is sufficiently small. In fact, the proposition shows that if real parts of all eigenvalues of R are negative, then $\|X(t)^{-1}x(t)\| \to 0$ as $t \to \infty$ for arbitrary solution $x(t)$ of (7.1), and that if there exists an eigenvalue of R whose real part is positive, then there exists a solution $x(t)$ of (7.1) such that $\|X(t)^{-1}x(t)\| \to \infty$ as $t \to \infty$.

Proposition 7.1. Suppose that $X(t)$ and $\widetilde{G}(t)$ defined in (7.3) is bounded in $t \geq 0$. Let R be a secular matrix for (7.1) and $\lambda_1, \cdots, \lambda_n$ its eigenvalues. Then, for each integer k with $1 \leq k \leq n$, there exist positive constants D_1, D_2, t_0, a positive valued function $\phi(\varepsilon)$ with $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$, and a solution $x(t)$ of (7.1) such that the inequality

$$D_2 e^{\varepsilon \mathrm{Re}(\lambda_k)t - 2\varepsilon\phi(\varepsilon)t} \leq \|X(t)^{-1}x(t)\| \leq D_1 e^{\varepsilon \mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t} \tag{7.7}$$

holds for $t \geq t_0$.

Proof. Since $\widetilde{G}(t) = \int_0^t (X(s)^{-1}G(s)X(s) - R)ds$ is bounded, $(I + \varepsilon\widetilde{G}(t))^{-1}$ is expanded into the Neumann series as $(I + \varepsilon\widetilde{G}(t))^{-1} = \sum_{n=0}^{\infty}(-\varepsilon)^n \widetilde{G}(t)^n$. With this expansion inserted into (7.6), $F_\varepsilon(t,x)$ is rewritten as

$$F_\varepsilon(t,x) = F(t)x + \varepsilon G(t)x + \varepsilon^2 H(t,\varepsilon)x, \tag{7.8}$$

$$H(t,\varepsilon) := \sum_{n=0}^{\infty}(-\varepsilon)^n \left(X(t)\widetilde{G}(t)R\widetilde{G}(t)^n X(t)^{-1} - G(t)X(t)\widetilde{G}(t)^{n+1}X(t)^{-1} \right). \tag{7.9}$$

Let us rewrite the equation (7.1) as

$$\dot{x} = F_\varepsilon(t,x) - \varepsilon^2 H(t,\varepsilon)x. \tag{7.10}$$

Introducing a new function $y(t)$ by $x(t) = X(t)y(t)$, we verify that y satisfies the differential equation

$$\dot{y} = \widetilde{F}_\varepsilon(t)y - \varepsilon^2 \widetilde{H}(t,\varepsilon)y, \tag{7.11}$$

where

$$\widetilde{F}_\varepsilon(t) := \varepsilon X(t)^{-1}G(t)X(t) + \varepsilon^2 X(t)^{-1}H(t,\varepsilon)X(t), \tag{7.12}$$

$$\widetilde{H}(t,\varepsilon) := \sum_{n=0}^{\infty}(-\varepsilon)^n \left(\widetilde{G}(t)R\widetilde{G}(t)^n - X(t)^{-1}G(t)X(t)\widetilde{G}(t)^{n+1} \right), \tag{7.13}$$

and further that the flow of the linear vector field $\widetilde{F}_\varepsilon(t)y$ is given by

$$\widetilde{\Phi}_{t,t_0} = (I + \varepsilon\widetilde{G}(t))e^{\varepsilon R(t-t_0)}(I + \varepsilon\widetilde{G}(t_0))^{-1}. \tag{7.14}$$

To prove the proposition, we can suppose that the secular matrix R is put in the Jordan form. In fact, if we change the variable x in (7.1) by $x \mapsto Px$, where P is an arbitrary nonsingular constant matrix, then $F(t), G(t)$ and $X(t)^{-1}G(t)X(t)$ are brought into $P^{-1}F(t)P, P^{-1}G(t)P$, and $P^{-1}X(t)^{-1}G(t)X(t)P$, respectively. This means that R turns into $P^{-1}RP$. In what follows, we assume that R is of the Jordan form:

$$R = \begin{pmatrix} \lambda_1 & p_1 & & & \\ & \lambda_2 & p_2 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{n-1} & p_{n-1} \\ & & & & \lambda_n \end{pmatrix}, \tag{7.15}$$

where λ_i $(i = 1, \cdots, n)$ are the eigenvalues of R such that $\mathrm{Re}(\lambda_1) \leq \cdots \leq \mathrm{Re}(\lambda_n)$ and where p_i $(i = 1, \cdots, n-1)$ are either 0 or 1.

Now let us fix an integer $k < n$ such that $\mathrm{Re}(\lambda_{k+1}) - \mathrm{Re}(\lambda_k) > 0$. The case that $n = k$ and the case that there are no such a $k < n$ are treated later. Define matrices $Q_1(t), Q_2(t)$ to be upper triangle matrices

$$Q_1(t) = \left(\begin{array}{ccc|ccc} e^{\varepsilon\lambda_1 t} & & * & & & \\ & \ddots & & & 0 & \\ & & e^{\varepsilon\lambda_k t} & & & \\ \hline & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{array}\right), \quad Q_2(t) = \left(\begin{array}{ccc|ccc} 0 & & & & & \\ & \ddots & & & 0 & \\ & & 0 & & & \\ \hline & & & e^{\varepsilon\lambda_{k+1} t} & & * \\ & & & & \ddots & \\ & & & & & e^{\varepsilon\lambda_n t} \end{array}\right),$$

such that $Q_1(t) + Q_2(t) = e^{\varepsilon R t}$. Then, a solution $y(t)$ to (7.11) satisfies an integral equation

$$y(t) = \widetilde{\Phi}_{t,0}\boldsymbol{e}_k - \varepsilon^2 \int_0^t (I + \varepsilon\widetilde{G}(t))Q_1(t-s)(I + \varepsilon\widetilde{G}(s))^{-1} \circ \widetilde{H}(s,\varepsilon)y(s)ds \\ +\varepsilon^2 \int_t^\infty (I + \varepsilon\widetilde{G}(t))Q_2(t-s)(I + \varepsilon\widetilde{G}(s))^{-1} \circ \widetilde{H}(s,\varepsilon)y(s)ds, \tag{7.16}$$

where $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ are the canonical basis of $\mathbf{R}^n$. The first term of the right hand side of the above is written as $\widetilde{\Phi}_{t,0}\boldsymbol{e}_k = (I + \varepsilon\widetilde{G}(t))(q_1(t)e^{\varepsilon\lambda_k t}, \cdots, q_{k-1}(t)e^{\varepsilon\lambda_k t}, e^{\varepsilon\lambda_k t}, 0, \cdots, 0)^t$, where $q_i(t)$ $(i = 1, \cdots, k-1)$ are monomials of t whose degrees are at most $k-1$. The fact that $\widetilde{G}(t) = \int_0^t (X(s)^{-1}G(s)X(s) - R)ds$ is bounded uniformly in t implies that $(I + \varepsilon\widetilde{G}(t))^{\pm 1}$ and $X(t)^{-1}G(t)X(t)$ are also bounded uniformly in t, and thereby so is $\widetilde{H}(t,\varepsilon)$. Consequently, there exist positive constants C_0, C_1 such that

$$||\widetilde{H}(t,\varepsilon)|| \leq C_0, \quad ||(I + \varepsilon\widetilde{G}(t))^{\pm 1}|| \leq C_1. \tag{7.17}$$

Further, there exist positive constants C_2, C_3 and a positive valued function $\phi(\varepsilon)$ satisfying $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$||Q_1(t)|| \leq \frac{C_2}{\phi(\varepsilon)^n} e^{\varepsilon \mathrm{Re}(\lambda_k)t + \varepsilon\phi(\varepsilon)t}, \quad \text{for } t \geq 0, \\ ||\widetilde{\Phi}_{t,0}\boldsymbol{e}_k|| \leq \frac{C_1 C_2}{\phi(\varepsilon)^n} e^{\varepsilon \mathrm{Re}(\lambda_k)t + \varepsilon\phi(\varepsilon)t}, \quad \text{for } t \geq 0, \tag{7.18} \\ ||Q_2(t)|| \leq \frac{C_3}{\phi(\varepsilon)^n} e^{\varepsilon \mathrm{Re}(\lambda_{k+1})t - \varepsilon\phi(\varepsilon)t}, \quad \text{for } t \leq 0.$$

Indeed, if $\varepsilon t \geq 1$, there exists a constant C such that $||Q_1(t)|| \leq C\varepsilon^n t^n e^{\varepsilon \mathrm{Re}(\lambda_k)t}$. Suppose that there exists a function $q(\varepsilon)$ such that

$$||Q_1(t)|| \leq C\varepsilon^n t^n e^{\varepsilon \mathrm{Re}(\lambda_k)t} \leq Cq(\varepsilon)e^{\varepsilon \mathrm{Re}(\lambda_k)t + \varepsilon\phi(\varepsilon)t}.$$

This inequality is equivalent to the inequality $\varepsilon t \leq q(\varepsilon)^{1/n} e^{\varepsilon\phi(\varepsilon)t/n}$, and it is easy to verify that this inequality holds when $q(\varepsilon) = (n/(\phi(\varepsilon)e))^n$. Putting $C_2 = C(n/e)^n$, we obtain $||Q_1(t)|| \leq \dfrac{C_2}{\phi(\varepsilon)^n} e^{\varepsilon \mathrm{Re}(\lambda_k)t + \varepsilon\phi(\varepsilon)t}$, for $\varepsilon t \geq 1$. This inequality also holds when $0 \leq \varepsilon t < 1$ because $||Q_1(t)|| \leq Ce^{\varepsilon \mathrm{Re}(\lambda_k)t}$ holds if $0 \leq \varepsilon t < 1$. The inequalities for $||\widetilde{\Phi}_{t,0}\boldsymbol{e}_k||$ and $||Q_2(t)||$ above are verified in a similar way.

We define a sequence of functions $\{y_m(t)\}_{m\geq 0}$ by

$$y_0(t) = \widetilde{\Phi}_{t,0}\boldsymbol{e}_k, \\ y_{m+1}(t) = y_0(t) - \varepsilon^2 \int_0^t (I + \varepsilon\widetilde{G}(t))Q_1(t-s)(I + \varepsilon\widetilde{G}(s))^{-1} \circ \widetilde{H}(s,\varepsilon)y_m(s)ds \\ +\varepsilon^2 \int_t^\infty (I + \varepsilon\widetilde{G}(t))Q_2(t-s)(I + \varepsilon\widetilde{G}(s))^{-1} \circ \widetilde{H}(s,\varepsilon)y_m(s)ds.$$

We need two lemmas to prove the proposition.

Lemma 7.2. Let $\phi(\varepsilon) = \varepsilon^{1/(2n+2)}$ and fix $\varepsilon > 0$ small so that $\mathrm{Re}(\lambda_{k+1}) - \mathrm{Re}(\lambda_k) - 3\phi(\varepsilon) > 0$. Then there exists a constant $0 < p < 1$ such that

$$||y_m(t) - y_{m-1}(t)|| \leq p^m e^{\varepsilon \mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t}, \quad m = 1, 2, \cdots, \tag{7.19}$$

for $t \geq 0$.

Proof. We prove (7.19) by induction. For $m = 1$, the quantity $\|y_1(t) - y_0(t)\|$ is estimated as follows :

$$
\begin{aligned}
\|y_1 - y_0\| &\leq \varepsilon^2 \int_0^t \|I + \varepsilon\widetilde{G}(t)\| \cdot \|Q_1(t-s)\| \cdot \|(I + \varepsilon\widetilde{G}(s))^{-1}\| \cdot \|\widetilde{H}(s,\varepsilon)\| \cdot \|y_0(s)\| ds \\
&\quad + \varepsilon^2 \int_t^\infty \|I + \varepsilon\widetilde{G}(t)\| \cdot \|Q_2(t-s)\| \cdot \|(I + \varepsilon\widetilde{G}(s))^{-1}\| \cdot \|\widetilde{H}(s,\varepsilon)\| \cdot \|y_0(s)\| ds \\
&\leq \frac{\varepsilon^2 C_0 C_1^3 C_2^2}{\phi(\varepsilon)^{2n}} \int_0^t e^{\varepsilon \mathrm{Re}(\lambda_k)t + \varepsilon\phi(\varepsilon)t} e^{\varepsilon\phi(\varepsilon)s} ds \\
&\quad + \frac{\varepsilon^2 C_0 C_1^3 C_2 C_3}{\phi(\varepsilon)^{2n}} \int_t^\infty e^{\varepsilon \mathrm{Re}(\lambda_{k+1})t - \varepsilon\phi(\varepsilon)t} e^{-\varepsilon \mathrm{Re}(\lambda_{k+1})s + \varepsilon \mathrm{Re}(\lambda_k)s + 3\varepsilon\phi(\varepsilon)s} ds \\
&\leq \frac{\varepsilon C_0 C_1^3 C_2}{\phi(\varepsilon)^{2n+1}} \left(C_2 + \frac{C_3 \phi(\varepsilon)}{\mathrm{Re}(\lambda_{k+1}) - \mathrm{Re}(\lambda_k) - 3\phi(\varepsilon)} \right) e^{\varepsilon \mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t} \\
&= \varepsilon^{1/(2n+2)} C_0 C_1^3 C_2 \left(C_2 + \frac{C_3 \phi(\varepsilon)}{\mathrm{Re}(\lambda_{k+1}) - \mathrm{Re}(\lambda_k) - 3\phi(\varepsilon)} \right) e^{\varepsilon \mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t}.
\end{aligned}
$$

Define

$$
p = \varepsilon^{1/(2n+2)} C_0 C_1^3 C_2 \left(C_2 + \frac{C_3 \phi(\varepsilon)}{\mathrm{Re}(\lambda_{k+1}) - \mathrm{Re}(\lambda_k) - 3\phi(\varepsilon)} \right), \tag{7.20}
$$

then Eq.(7.19) holds for $m = 1$. Further, if ε is sufficiently small, the inequality $0 < p < 1$ holds. With this p, if we suppose Eq.(7.19) holds, then we can verify that $\|y_{m+1} - y_m\| \leq p^{m+1} e^{\varepsilon \mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t}$ by the same calculation as above. ■

This lemma implies that the sequence $\{y_m(t)\}_{m \geq 0}$ converges to a solution of (7.11).

Lemma 7.3. Under the same conditions as Lem.7.2, there exist positive constants D_1 and t_0 such that

$$
\|y_m(t)\| \leq D_1 e^{\varepsilon \mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t}, \quad m = 0, 1, \cdots \tag{7.21}
$$

for $t \geq t_0$.

Proof. We prove the lemma by induction. When $m = 0$, the above inequality is clear if $D_1 \geq C_1 C_2 / \phi(\varepsilon)^n$. Suppose that the above inequality holds for m, then

$$
\begin{aligned}
\|y_{m+1}\| &\leq \|y_0\| + \varepsilon^2 \int_0^t \|I + \varepsilon\widetilde{G}(t)\| \cdot \|Q_1(t-s)\| \cdot \|(I + \varepsilon\widetilde{G}(s))^{-1}\| \cdot \|\widetilde{H}(s,\varepsilon)\| \cdot \|y_m(s)\| ds \\
&\quad + \varepsilon^2 \int_t^\infty \|I + \varepsilon\widetilde{G}(t)\| \cdot \|Q_2(t-s)\| \cdot \|(I + \varepsilon\widetilde{G}(s))^{-1}\| \cdot \|\widetilde{H}(s,\varepsilon)\| \cdot \|y_m(s)\| ds \\
&= D_1 e^{\varepsilon \mathrm{Re}(\lambda_k)t + \varepsilon\phi(\varepsilon)t} + \frac{\varepsilon^2 C_0 C_1^2 C_2 D_1}{\phi(\varepsilon)^n} \int_0^t e^{\varepsilon \mathrm{Re}(\lambda_k)t + \varepsilon\phi(\varepsilon)t} e^{\varepsilon\phi(\varepsilon)s} ds \\
&\quad + \frac{\varepsilon^2 C_0 C_1^2 C_3 D_1}{\phi(\varepsilon)^n} \int_t^\infty e^{\varepsilon \mathrm{Re}(\lambda_{k+1})t - \varepsilon\phi(\varepsilon)t} e^{-\varepsilon \mathrm{Re}(\lambda_{k+1})s + \varepsilon \mathrm{Re}(\lambda_k)s + 3\varepsilon\phi(\varepsilon)s} \\
&\leq D_1 e^{\varepsilon \mathrm{Re}(\lambda_k)t + \varepsilon\phi(\varepsilon)t} + \frac{\varepsilon C_0 C_1^2 C_2 D_1}{\phi(\varepsilon)^{n+1}} e^{\varepsilon \mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t} + \frac{\varepsilon C_0 C_1^2 C_3 D_1}{\phi(\varepsilon)^n (\mathrm{Re}(\lambda_{k+1}) - \mathrm{Re}(\lambda_k) - 3\phi(\varepsilon))} e^{\varepsilon \mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t} \\
&\leq D_1 e^{\varepsilon \mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t} \left(e^{-\varepsilon\phi(\varepsilon)t} + \frac{\varepsilon^{n/(2n+2)}}{C_1 C_2} p \right),
\end{aligned}
$$

where p is defined by Eq.(7.20). Since $0 < p < 1$, we can take sufficiently large t_0 and sufficiently small ε such that

$$
0 < e^{-\varepsilon\phi(\varepsilon)t} + \frac{\varepsilon^{n/(2n+2)}}{C_1 C_2} p < 1
$$

for $t \geq t_0$. This proves the lemma. ∎

We return to the proof of Prop.7.1. By taking the limit $m \to \infty$ in (7.21), we obtain a solution $y(t) = y^{(k)}(t)$ of Eq.(7.11) satisfying the right part of the inequality of (7.7) when $k \neq n$. If there exist eigenvalues $\lambda_{k'}$ of R satisfying $\mathrm{Re}(\lambda_{k'}) = \mathrm{Re}(\lambda_k)$, we repeat the above discussion with $\boldsymbol{e}_{k'}$ instead of $\boldsymbol{e}_k$ included in Eq.(7.16). Then we obtain a solution $y^{(k')}(t)$ of Eq.(7.11), which is linearly independent of $y^{(k)}(t)$, satisfying the right part of the inequality of (7.7). To prove the same inequality for $k = n$, instead of (7.16), we use the integral equation

$$y(t) = \widetilde{\Phi}_{t,0}\boldsymbol{e}_n - \varepsilon^2 \int_0^t (I + \varepsilon\widetilde{G}(t))e^{\varepsilon R(t-s)}(I + \varepsilon\widetilde{G}(s))^{-1} \circ \widetilde{H}(s,\varepsilon)y(s)ds. \tag{7.22}$$

The same procedure as above applied to this equation to yield the right part inequality of (7.7) for $k = n$.

We proceed to prove the left part inequality of (7.7). Let $y^{(k)}(t)$ be a solution of (7.11) which satisfies the right part inequality of (7.7), and denote the fundamental matrix to (7.11) by Y, whose column vectors are $y^{(1)}(t), \cdots, y^{(n)}(t)$. Define a matrix Z by $Z = Y^{-t}$, where Y^{-t} is the abbreviation of $(Y^{-1})^t$, this notation will be used in the sequel. Each column vector $z^{(1)}(t), \cdots, z^{(n)}(t)$ of Z satisfies an adjoint equation of (7.11)

$$\dot{z} = -(\widetilde{F}_\varepsilon(t) - \varepsilon^2\widetilde{H}(t,\varepsilon))^t z. \tag{7.23}$$

Since the flow of the linear vector field $-(\widetilde{F}_\varepsilon)^t z$ is given by $\widetilde{\Phi}^{-t}_{t,t_0} = (I + \varepsilon\widetilde{G}(t))^{-t}e^{-\varepsilon R^t(t-t_0)}(I + \varepsilon\widetilde{G}(t_0))^t$, we can prove that there exists a solution $z(t) = u^{(k)}(t)$ of (7.23) such that

$$||u^{(k)}(t)|| \leq D'_1 e^{-\varepsilon\mathrm{Re}(\lambda_k)t + 2\varepsilon\phi(\varepsilon)t}, \quad k = 1, \cdots, n, \quad t \geq t_0, \tag{7.24}$$

by the same procedure as that for the proof of the inequality (7.7), where D'_1 is some positive constant. Let U be the fundamental matrix for (7.23) whose column vectors are $u^{(1)}(t), \cdots, u^{(n)}(t)$. By the uniqueness of solutions of (7.23), there exists a constant matrix K such that $U = ZK$. Let k_{ij} be the (i, j) component of K. Since $Y^t U = K$, the inequality $|k_{ii}| = |(y^{(i)}(t), u^{(i)}(t))| \leq ||y^{(i)}(t)|| \cdot ||u^{(i)}(t)||$ holds, which then proves the left part inequality of (7.7) if $k_{ii} \neq 0$, where $(\ ,\ )$ denotes the standard inner product of the $\mathbf{R}^n$. If $k_{ii} = 0$, we define $\widetilde{u}^{(k)}(t)$ by $\widetilde{u}^{(k)}(t) = u^{(k)}(t) + \sum_{i=k+1}^n \alpha_i u^{(i)}(t)$, with $\alpha_i \in \mathbf{R}$. And define a matrix $\widetilde{U}$ whose column vectors are $\widetilde{u}^{(1)}(t), \cdots, \widetilde{u}^{(n)}(t)$. Each $\widetilde{u}^{(k)}(t)$ satisfies the inequality (7.24) for some constant D'_1. Then there exists a constant matrix $\widetilde{K}$ such that $Y^t\widetilde{U} = \widetilde{K}$ and we can assume that its diagonal component $k_{ii} \neq 0$ by defining $\alpha_i \in \mathbf{R}$ appropriately. This ends the proof of Prop.7.1. ∎

Corollary 7.4. Consider an equation $\dot{x} = Fx + \varepsilon G(t)x$ with $x \in \mathbf{R}^n$, where F is an $n \times n$ constant matrix and $G(t)$ is an $n \times n$ matrix which is of C^1 class with respect to t. Suppose that all eigenvalues of F lie on the imaginary axis, and suppose that $G(t)$ and $\widetilde{G}(t)$ defined by Eq.(7.3) are bounded in $t \in \mathbf{R}$. If $\varepsilon > 0$ is sufficiently small, then the stability of a trivial solution $x(t) \equiv 0$ coincides with that of a trivial solution of the RG equation $\dot{v} = \varepsilon Rv$, where R is a secular matrix for $Fx + \varepsilon G(t)x$.

In the above corollary, the boundedness of $G(t)$ and $\widetilde{G}(t)$ are satisfied if $G(t)$ is periodic or almost periodic function in t whose Fourier exponents do not have accumulation points in $\mathbf{R}$.

Example 7.5. Let us consider the Mathieu equation:

$$\ddot{y} = -(a + 2\varepsilon\cos t)y, \tag{7.25}$$

where a and ε are positive parameters. It is well known that there exists an area in (a, ε) plane such that the origin is an unstable fixed point for (7.25) if (a, ε) is in this area. We calculate the area approximately by the RG method.

Let $a = a_0 + \varepsilon a_1$ and $y = y_0 + \varepsilon y_1$. Substituting them into (7.25), and comparing the coefficients of ε^0 and ε^1 in both sides of (7.25) provides

$$\ddot{y}_0 = -b^2 y_0, \tag{7.26}$$

$$\ddot{y}_1 = -b^2 y_1 - a_1 y_0 - 2\cos t \cdot y_0, \tag{7.27}$$

where $a_0 = b^2$. A general solution to the former is given by

$$y_0(t) = Ae^{ibt} + \overline{A}e^{-ibt}, \ \ A \in \mathbf{C}. \tag{7.28}$$

With this y_0, Eq.(7.27) is rewritten as

$$\ddot{y}_1 = -b^2 y_1 - \left(a_1 A e^{ibt} + Ae^{i(1+b)t} + \overline{A}e^{i(1-b)t} + \text{c.c.}\right). \tag{7.29}$$

If $b = 1/2$ (*i.e.* $a_0 = 1/4$), the secular term appears for all a_1. In fact, the equation

$$\ddot{y}_1 = -\frac{1}{4} y_1 - \left(a_1 A e^{it/2} + Ae^{3it/2} + \overline{A}e^{it/2} + \text{c.c.}\right) \tag{7.30}$$

admits a special solution given by

$$y_1(t, \tau; A) = i(a_1 A + \overline{A})(t - \tau)e^{it/2} + \frac{A}{2}e^{3it/2} + \text{c.c.}, \tag{7.31}$$

where the initial time has been chosen to be $t = \tau$. Then, the RG equation for (7.25) is given by

$$\dot{A} = i\varepsilon(a_1 A + \overline{A}). \tag{7.32}$$

Putting $A = B + iC, \ B, C \in \mathbf{R}$, we break up (7.32) into

$$\left\{ \begin{array}{l} \dot{B} = \varepsilon(1 - a_1)C \\ \dot{C} = \varepsilon(1 + a_1)B. \end{array} \right. \tag{7.33}$$

A general solution to this equation is given by

$$B(t) = \left\{ \begin{array}{ll} pe^{\varepsilon\sqrt{1-a_1^2}t} + qe^{-\varepsilon\sqrt{1-a_1^2}t} & (|a_1| \leq 1) \\ pe^{i\varepsilon\sqrt{a_1^2-1}t} + qe^{-i\varepsilon\sqrt{a_1^2-1}t} & (|a_1| > 1), \end{array} \right. \tag{7.34}$$

where $p, q \in \mathbf{R}$ are arbitrary constants. This shows that the origin is an unstable fixed point for the RG equation (7.33) if $|a_1| < 1$. This proves the instability of the fixed point of the Mathieu equation (7.25) if $a = 1/4 + \varepsilon a_1 + O(\varepsilon^2), \ |a_1| < 1$.

Example 7.6. Consider the coupled Mathieu equations

$$\left\{ \begin{array}{l} \ddot{x} = -(a + 2\varepsilon\cos t)x - \varepsilon p(x - y) - \varepsilon q(\dot{x} - \dot{y}) \\ \ddot{y} = -(a + 2\varepsilon\cos t)y - \varepsilon p(y - x) - \varepsilon q(\dot{y} - \dot{x}), \end{array} \right. \tag{7.35}$$

where $\varepsilon > 0$ and $a, p, q \in \mathbf{R}$ are constants. Put $u = x + y$, then the equation for $u(t)$ is the Mathieu equation (7.25). In Example 7.5, we proved that if $a = 1/4$, the trivial solution $u = 0$ of the Mathieu equation (7.25) is unstable. In what follows, we assume that $a = 1/4$. Put $z = x - y$. Then z satisfies the equation

$$\ddot{z} = -\frac{1}{4}z + \varepsilon(-2q\dot{z} - 2pz - 2\cos t \cdot z). \tag{7.36}$$

Put further $z = z_0 + \varepsilon z_1$, where z_0 is subjected to the unperturbed equation $\ddot{z}_0 = -\frac{1}{4}z_0$, and has a general solution of the form $z_0(t) = Ae^{it/2} + \overline{A}e^{-it/2}$. With this $z_0(t)$, the equation for z_1 proves to be given by

$$\ddot{z}_1 = -\frac{1}{4}z_1 - iqAe^{it/2} - 2pAe^{it/2} - Ae^{3it/2} - \overline{A}e^{it/2} + \text{c.c.}, \tag{7.37}$$

where c.c. denote the complex conjugate of the last four terms of the right hand side. A special solution of this equation, whose initial time is $t = \tau$, is given by

$$z_1(t) = i(iqA + 2pA + \overline{A})(t - \tau)e^{it/2} + \frac{A}{2}e^{3it/2} + \text{c.c.} \tag{7.38}$$

Therefore the RG equation for Eq.(7.36) is put in the form

$$\dot{A} = i\varepsilon(iqA + 2pA + \overline{A}), \quad A \in \mathbf{C}. \tag{7.39}$$

Put $A = \alpha + i\beta, \alpha, \beta \in \mathbf{R}$. Then the above equation is rewritten as

$$\frac{d}{dt}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \varepsilon \begin{pmatrix} -q & -2p+1 \\ 2p+1 & -q \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{7.40}$$

Eigenvalues of the matrix in the right hand side of the above equation are $\lambda_{\pm} = -q \pm \sqrt{1 - 4p^2}$. Therefore the stability of the trivial solution $(\alpha, \beta) = (0, 0)$ of the RG equation is as given in Figure 2.

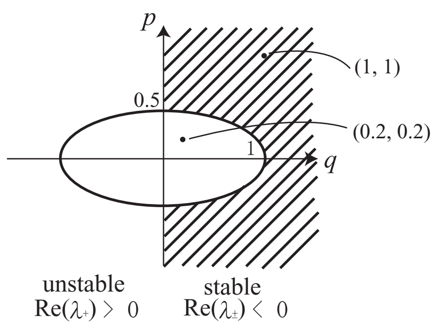

Fig. 2: The trivial solution $(\alpha, \beta) = (0, 0)$ is stable on the shaded area.

Corollary 7.4. shows that the stability of the trivial solution $z(t) = 0$ of Eq.(7.36) coincides with that of the stability of $(\alpha, \beta) = (0, 0)$. This proves that if $\text{Re}(\lambda_{\pm}) < 0$, then $|x(t) - y(t)| \to 0$ as $t \to \infty$ although each $|x(t)|, |y(t)|$ diverges as $t \to \infty$.

A numerical solution to Eq.(7.35) for $\varepsilon = 0.01,\ x(0) = 0.5, y(0) = 0.1$ is presented in Fig.3.

Acknowledgment

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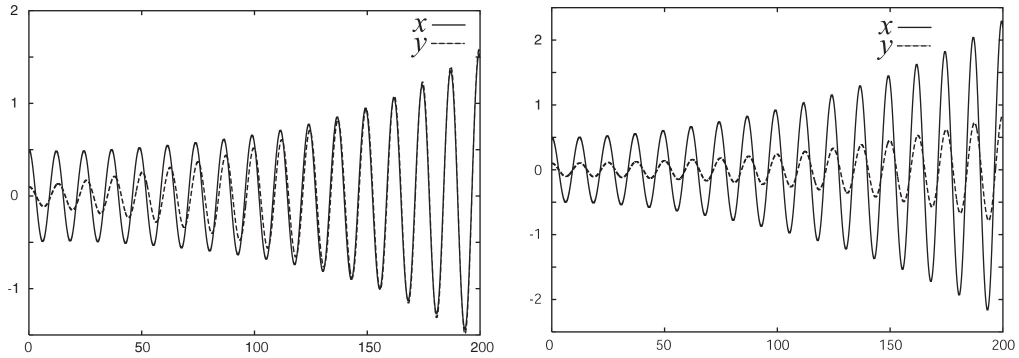
A Higher order RG equation

In this appendix, we define the higher order RG equation for constructing an approximate vector field which is $O(\varepsilon^{m+1})$ close to a given original vector field. The result is used in proving Theorem 6.1.



(a) $p = q = 1$ (b) $p = \; q = 0.2$

Fig. 3: Numerical results for Eq.(7.35). The synchronous solution $x(t) = y(t)$ is (a) stable if $p = q = 1$, (b) unstable if $p = q = 0.2$.

Let F be a diagonalizable $n{\times}n$ matrix all of whose eigenvalues lie on the imaginary axis and $g_1(t,x), \cdots, g_m(t,x)$ C^∞ vector fields on $\mathbf{R}\times\mathbf{R}^n$ which are polynomial in x and periodic in t. Consider an ODE

$$\dot{x} = Fx + \varepsilon g_1(t,x) + \varepsilon^2 g_2(t,x) + \cdots + \varepsilon^m g_m(t,x), \quad x \in \mathbf{R}^n, \tag{A.1}$$

where $\varepsilon \in \mathbf{R}$ is a small parameter. Put $x = x_0 + \varepsilon x_1 + \cdots + \varepsilon^m x_m$. Then the above equation is rewritten as

$$\dot{x}_0 + \varepsilon\dot{x}_1 + \cdots + \varepsilon^m \dot{x}_m = F(x_0 + \varepsilon x_1 + \cdots + \varepsilon^m x_m) + \sum_{i=1}^{m} \varepsilon^i g_i(t, x_0 + \varepsilon x_1 + \cdots + \varepsilon^m x_m). \tag{A.2}$$

Expanding the right hand side of the above equation with respect to ε and equating the coefficients of each ε^i of the both sides of the above, we obtain ODEs for $x_0, x_1, \cdots, x_m$

$$\dot{x}_0 = Fx_0, \tag{A.3}$$

$$\dot{x}_1 = Fx_1 + G_1(t, x_0), \tag{A.4}$$

$$\vdots$$

$$\dot{x}_i = Fx_i + G_i(t, x_0, x_1, \cdots, x_{i-1}), \tag{A.5}$$

$$\vdots$$

$$\dot{x}_m = Fx_m + G_m(t, x_0, x_1, \cdots, x_{m-1}), \tag{A.6}$$

where G_i is some smooth function of $t, x_0, x_1, \cdots, x_{i-1}$ which is periodic in t. For example, G_1, G_2 and G_3 are given by

$$G_1(t, x_0) = g_1(t, x_0), \tag{A.7}$$

$$G_2(t, x_0, x_1) = \frac{\partial g_1}{\partial x}(t, x_0)x_1 + g_2(t, x_0), \tag{A.8}$$

$$G_3(t, x_0, x_1, x_2) = \frac{1}{2}\frac{\partial^2 g_1}{\partial x^2}(t, x_0)x_1^2 + \frac{\partial g_1}{\partial x}(t, x_0)x_2 + \frac{\partial g_2}{\partial x}(t, x_0)x_1 + g_3(t, x_0), \tag{A.9}$$

respectively. We have to solve the above equations. At first, we denote by $x_0(t, 0, A) = X(t)A$ a solution of the unperturbed part $\dot{x}_0 = Fx_0$, where $X(t) = e^{Ft}$ is the fundamental matrix and $A \in \mathbf{R}^n$ is an initial value. With this x_0, by the similar discussion to Sec.4, a solution of Eq.(A.4) is given by

$$x_1(t, \tau; A) = h_t^{(1)}(A) + X(t)R_1(A)(t - \tau), \tag{A.10}$$

where $h_t^{(1)}(A)$ and $R_1(A)$ are defined by

$$R_1(A) = \lim_{t\to\infty} \frac{1}{t} \int^t X(s)^{-1} G_1(s, X(s)A) ds, \tag{A.11}$$

$$h_t^{(1)}(A) = X(t) \int^t \Big(X(s)^{-1} G_1(s, X(s)A) - R_1(A) \Big) ds, \tag{A.12}$$

respectively. The integral constants of the indefinite integrals in Eq.(A.11),(A.12) and Eq.(A.13),(A.14) below are fixed arbitrary. By choosing these integral constants appropriately, we can reduce the RG equation. This will be done in a forthcoming paper. Note that since $X(t)$ and $G_1(t, x)$ are almost periodic in t, $X(t)^{-1}G_1(t, X(t)A)$ is bounded uniformly in $t \in \mathbf{R}$ and $R_1(A)$ is well-defined (see Lemma 4.1). With this x_0 and x_1, we solve the equation for x_2, as will be shown in Prop.A.1. This process is performed step by step until a solution x_m to Eq.(A.6) is obtained.

Proposition A.1. Define functions $R_i(A)$ and $h_t^{(i)}(A)$, $i = 2, \cdots, m$ by

$$R_i(A) := \lim_{t\to\infty} \frac{1}{t} \int^t \Big(X(s)^{-1} G_i(s, X(s)A, h_s^{(1)}(A), \cdots, h_s^{(i-1)}(A)) - X(s)^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_A R_{i-k}(A) \Big) ds, \tag{A.13}$$

$$h_t^{(i)}(A) := X(t) \int^t \Big(X(s)^{-1} G_i(s, X(s)A, h_s^{(1)}(A), \cdots, h_s^{(i-1)}(A)) - X(s)^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_A R_{i-k}(A) - R_i(A) \Big) ds. \tag{A.14}$$

Then, the curve defined by

$$x_i := x_i(t, \tau; A) = h_t^{(i)}(A) + y_1^{(i)}(t, A)(t-\tau) + y_2^{(i)}(t, A)(t-\tau)^2 + \cdots + y_i^{(i)}(t, A)(t-\tau)^i \tag{A.15}$$

gives a solution to Eq.(A.5) for $i = 1, 2, \cdots, m$, where $y_1^{(i)}, \cdots, y_i^{(i)}$ are defined by

$$y_1^{(i)}(t, A) = X(t) R_i(A) + \sum_{k=1}^{i-1} (Dh_t^{(k)})_A R_{i-k}(A), \tag{A.16}$$

$$y_j^{(i)}(t, A) = \frac{1}{j} \sum_{k=1}^{i-1} \frac{\partial y_{j-1}^{(k)}}{\partial A}(t, A) R_{i-k}(A), \ \ (j = 2, 3, \cdots, i-1), \tag{A.17}$$

$$y_i^{(i)}(t, A) = \frac{1}{i} \sum_{k=1}^{i-1} \frac{\partial y_{i-1}^{(k)}}{\partial A}(t, A) R_{i-k}(A) = \frac{1}{i} \frac{\partial y_{i-1}^{(i-1)}}{\partial A}(t, A) R_1(A), \tag{A.18}$$

$$y_j^{(i)}(t, A) = 0, \ \ (j > i). \tag{A.19}$$

Proof. We prove Prop.A.1 by induction. Assume that $x_1, \cdots, x_{i-1}$ defined by Eq.(A.15) are solutions of Eq.(A.5) for $i = 1, 2, \cdots, i-1$. In order to prove that x_i defined by Eq.(A.15) is a solution of Eq.(A.5), we substitute Eq.(A.15) into Eq.(A.5) to obtain

$$\begin{aligned} & F h_t^{(i)}(A) + G_i(t, X(t)A, h_t^{(1)}(A), \cdots, h_t^{(i-1)}(A)) - \sum_{k=1}^{i-1} (Dh_t^{(k)})_A R_{i-k}(A) - X(t) R_i(t) \\ & + \sum_{k=1}^{i} \dot{y}_k^{(i)}(t, A)(t-\tau)^k + \sum_{k=1}^{i} y_k^{(i)}(t, A) k (t-\tau)^{k-1} \\ = \; & F h_t^{(i)}(A) + F \sum_{k=1}^{i} y_k^{(i)}(t, A)(t-\tau)^k + G_i(t, x_0, x_1, \cdots, x_{i-1}). \end{aligned} \tag{A.20}$$

It is easy to verify that $G_i(t, x_0, x_1, \cdots, x_{i-1})$ with $x_0, x_1, \cdots, x_{i-1}$ defined by Eq.(A.15) is a polynomial in $t-\tau$ whose degree is at most $i-1$. We denote it by

$$G_i(t, x_0, \cdots, x_{i-1}) = \sum_{k=0}^{i-1} \widetilde{G}_i^{(k)}(t, x_0, \cdots, x_{i-1})(t-\tau)^k. \tag{A.21}$$

Note that $\widetilde{G}_i^{(0)}(t, x_0, \cdots, x_{i-1}) = G_i(t, X(t)A, h_t^{(1)}(A), \cdots, h_t^{(i-1)}(A))$. Equating the coefficients of $(t-\tau)^k$ of the both sides of Eq.(A.20) with Eq.(A.21), we obtain equations

$$y_1^{(i)}(t, A) = \sum_{k=1}^{i-1} (Dh_t^{(k)})_A R_{i-k}(A) + X(t)R_i(A), \tag{A.22}$$

$$\dot{y}_k^{(i)}(t, A) + (k+1)y_{k+1}^{(i)}(t, A) = Fy_k^{(i)}(t, A) + \widetilde{G}_i^{(k)}(t, x_0, \cdots, x_{i-1}), \ (k = 1, \cdots, i-1), \tag{A.23}$$

$$\dot{y}_i^{(i)}(t, A) = Fy_i^{(i)}(t, A). \tag{A.24}$$

These equations can determine $y_1^{(i)}, y_2^{(i)}, \cdots, y_i^{(i)}$. Eq.(A.22) gives Eq.(A.16). From Eq.(A.23) for $k = 1$, we obtain

$$\begin{aligned}
2y_2^{(i)}(t, A) &= Fy_1^{(i)}(t, A) - \dot{y}_1^{(i)}(t, A) + \widetilde{G}_i^{(1)}(t, x_0, \cdots, x_{i-1}) \\
&= \sum_{k=1}^{i-1} F(Dh_t^{(k)})_A R_{i-k}(A) + FX(t)R_i(A) \\
&\quad - \sum_{k=1}^{i-1} \frac{\partial}{\partial t}(Dh_t^{(k)})_A R_{i-k}(A) - \frac{\partial}{\partial t} X(t)R_i(A) + \widetilde{G}_i^{(1)}(t, x_0, \cdots, x_{i-1}) \\
&= \sum_{k=1}^{i-1} F(Dh_t^{(k)})_A R_{i-k}(A) - \sum_{k=1}^{i-1} \frac{\partial}{\partial A}\Big(Fh_t^{(k)}(A) + G_k(t, X(t)A, h_t^{(1)}(A), \cdots, h_t^{(k-1)}(A)) \\
&\quad - \sum_{j=1}^{k-1} (Dh_t^{(j)})_A R_{k-j}(A) - X(t)R_k(A)\Big) R_{i-k}(A) + \widetilde{G}_i^{(1)}(t, x_0, \cdots, x_{i-1}) \\
&= \sum_{k=1}^{i-1} \frac{\partial}{\partial A}\left(\sum_{j=1}^{k-1} (Dh_t^{(j)})_A R_{k-j}(A) + X(t)R_k(A)\right) R_{i-k}(A) \\
&\quad + \widetilde{G}_i^{(1)}(t, x_0, \cdots, x_{i-1}) - \sum_{k=1}^{i-1} \frac{\partial}{\partial A} G_k(t, X(t)A, h_t^{(1)}(A), \cdots, h_t^{(k-1)}(A)) R_{i-k}(A) \\
&= \sum_{k=1}^{i-1} \frac{\partial}{\partial A} y_1^{(k)}(t, A) R_{i-k}(A) \\
&\quad + \widetilde{G}_i^{(1)}(t, x_0, \cdots, x_{i-1}) - \sum_{k=1}^{i-1} \frac{\partial}{\partial A} G_k(t, X(t)A, h_t^{(1)}(A), \cdots, h_t^{(k-1)}(A)) R_{i-k}(A).
\end{aligned} \tag{A.25}$$

If the equality

$$\widetilde{G}_i^{(1)}(t, x_0, \cdots, x_{i-1}) = \sum_{k=1}^{i-1} \frac{\partial}{\partial A} G_k(t, X(t)A, h_t^{(1)}(A), \cdots, h_t^{(k-1)}(A)) R_{i-k}(A) \tag{A.26}$$

holds, Eq.(A.17) for $j = 2$ is obtained. The left hand side of the above is calculated as

$$
\begin{aligned}
\widetilde{G}_i^{(1)}(t, x_0, \cdots, x_{i-1}) &= -\frac{\partial}{\partial \tau}\Big|_{\tau=t} G_i(t, x_0, \cdots, x_{i-1}) \\
&= -\sum_{j=1}^{i-1} \lim_{\tau\to t} \frac{\partial G_i}{\partial x_j}(t, x_0, \cdots, x_{i-1}) \frac{\partial}{\partial \tau}\Big|_{\tau=t} x_j(t, \tau; A) \\
&= \sum_{j=1}^{i-1} \lim_{\tau\to t} \frac{\partial G_i}{\partial x_j}(t, x_0, \cdots, x_{i-1}) \Big(X(t) R_j(A) + \sum_{k=1}^{j-1} (Dh_t^{(k)})_A R_{j-k}(A) \Big).
\end{aligned} \tag{A.27}
$$

The right hand side of Eq.(A.26) is calculated as

$$
\begin{aligned}
&\sum_{k=1}^{i-1} \frac{\partial}{\partial A} G_k(t, X(t)A, h_t^{(1)}(A), \cdots, h_t^{(k-1)}(A)) R_{i-k}(A) \\
&= \sum_{k=1}^{i-1} \sum_{j=1}^{k-1} \lim_{\tau\to t} \frac{\partial G_k}{\partial x_j}(t, x_0, \cdots, x_{i-1}) (Dh_t^{(j)})_A R_{i-k}(A) + \sum_{k=1}^{i-1} \lim_{\tau\to t} \frac{\partial G_k}{\partial x_0}(t, x_0, \cdots, x_{i-1}) X(t) R_{i-k}(A).
\end{aligned} \tag{A.28}
$$

Now we need a simple lemma.

Lemma A.2. For integers i, j with $i > j$, the equality

$$
\frac{\partial G_i}{\partial x_j} = \frac{\partial G_{i-1}}{\partial x_{j-1}} = \cdots = \frac{\partial G_{i-j}}{\partial x_0} \tag{A.29}
$$

holds.

We will prove this lemma after the proof of Prop.A.1. is completed. According to Lemma A.2, Eq.(A.27) and Eq.(A.28) are brought into

$$
\widetilde{G}_i^{(1)}(t, x_0, \cdots, x_{i-1}) = \sum_{j=1}^{i-1} \lim_{\tau\to t} \frac{\partial G_{i-j}}{\partial x_0}(t, x_0, \cdots, x_{i-1}) \Big(X(t) R_j(A) + \sum_{k=1}^{j-1} (Dh_t^{(k)})_A R_{j-k}(A) \Big), \tag{A.30}
$$

and

$$
\begin{aligned}
&\sum_{k=1}^{i-1} \frac{\partial}{\partial A} G_k(t, X(t)A, h_t^{(1)}(A), \cdots, h_t^{(k-1)}(A)) R_{i-k}(A) \\
&= \sum_{k=1}^{i-1} \sum_{j=1}^{k-1} \lim_{\tau\to t} \frac{\partial G_{k-j}}{\partial x_0}(t, x_0, \cdots, x_{i-1}) (Dh_t^{(j)})_A R_{i-k}(A) + \sum_{k=1}^{i-1} \lim_{\tau\to t} \frac{\partial G_k}{\partial x_0}(t, x_0, \cdots, x_{i-1}) X(t) R_{i-k}(A),
\end{aligned} \tag{A.31}
$$

respectively. This proves Eq.(A.26), and Eq.(A.17) for $j = 2$ is verified.

By using Eq.(A.23), $y_3^{(i)}, \cdots, y_{i-1}^{(i)}, y_i^{(i)}$ are calculated in the same way as above, and Eq.(A.17) and Eq.(A.18) are proved, but we omit the detailed calculation here. Next, we have to show that $y_i^{(i)}$ given by Eq.(A.18) satisfies

Eq.(A.24). To show this, according to $y_1^{(1)}(t,A) = X(t)R_1(t)$, we rewrite Eq.(A.18) as

$$
\begin{aligned}
y_i^{(i)}(t,A) &= \frac{1}{i}\frac{\partial y_{i-1}^{(i-1)}}{\partial A}R_1(A) \\
&= \frac{1}{i(i-1)}\frac{\partial}{\partial A}\left(\frac{\partial y_{i-2}^{(i-2)}}{\partial A}R_1(A)\right)R_1(A) \\
&= \ \vdots \\
&= \frac{1}{i!}\frac{\partial}{\partial A}\left(\frac{\partial}{\partial A}\Big(\cdots\frac{\partial}{\partial A}\Big(\frac{\partial y_1^{(1)}}{\partial A}R_1(A)\Big)\cdots\Big)R_1(A)\right)R_1(A) \\
&= X(t)\frac{1}{i!}\frac{\partial}{\partial A}\left(\frac{\partial}{\partial A}\Big(\cdots\frac{\partial}{\partial A}\Big(\frac{\partial R_1}{\partial A}R_1(A)\Big)\cdots\Big)R_1(A)\right)R_1(A).
\end{aligned}
$$

Since $X(t)$ is the fundamental matrix of the equation $\dot{y} = Fy$, $y_i^{(i)}$ satisfies Eq.(A.24). Therefore x_i defined by Eq.(A.15) satisfies Eq.(A.5). This ends the proof of Prop.A.1. ■

Proof of Lemma A.2. By definition, $G_i(t, x_0, \cdots, x_{i-1})$ is written as

$$
G_i(t, x_0, \cdots, x_{i-1}) = \sum_{k=1}^{i-1}\frac{1}{k!}\frac{d^k}{d\varepsilon^k}\Big|_{\varepsilon=0} g_{i-k}(t, \textstyle\sum_{l=0}^m \varepsilon^l x_l) + g_i(t, x_0).
$$

On the other hand, $G_{i-1}(t, x_0, \cdots, x_{i-2})$ is rewritten as

$$
\begin{aligned}
G_{i-1}(t, x_0, \cdots, x_{i-2}) &= \sum_{k=0}^{i-2}\frac{1}{k!}\frac{d^k}{d\varepsilon^k}\Big|_{\varepsilon=0} g_{i-k-1}(t, \textstyle\sum_{l=0}^m \varepsilon^l x_l) \\
&= \sum_{k=1}^{i-1}\frac{1}{(k-1)!}\frac{d^{k-1}}{d\varepsilon^{k-1}}\Big|_{\varepsilon=0} g_{i-k}(t, \textstyle\sum_{l=0}^m \varepsilon^l x_l).
\end{aligned}
$$

To show the equality $\partial G_i/\partial x_j = \partial G_{i-1}/\partial x_{j-1}$, it is sufficient to prove the equality

$$
\frac{\partial}{\partial x_j}\frac{1}{k!}\frac{d^k}{d\varepsilon^k}\Big|_{\varepsilon=0} g_{i-k}(t, \textstyle\sum_{l=0}^m \varepsilon^l x_l) = \frac{\partial}{\partial x_{j-1}}\frac{1}{(k-1)!}\frac{d^{k-1}}{d\varepsilon^{k-1}}\Big|_{\varepsilon=0} g_{i-k}(t, \textstyle\sum_{l=0}^m \varepsilon^l x_l), \qquad \text{(A.32)}
$$

for $k = 1, 2, \cdots, i-1$. For simplicity, we denote $g_{i-k}(t,x)$ by $g(x)$. Consider the trivial equality

$$
\frac{\partial}{\partial x_j} g(\textstyle\sum_{l=0}^m \varepsilon^l x_l) = \varepsilon\frac{\partial}{\partial x_{j-1}} g(\textstyle\sum_{l=0}^m \varepsilon^l x_l), \quad j = 1, \cdots, m. \qquad \text{(A.33)}
$$

Expanding the both sides of the above equation with respect to ε, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial x_j}\left(\sum_{p=0}^k \frac{\varepsilon^p}{p!}\frac{d^p}{d\varepsilon^p}\Big|_{\varepsilon=0} g(\textstyle\sum_{l=0}^m \varepsilon^l x_l) + \widetilde{R}(\varepsilon, x_0, \cdots, x_m)\right) \\
&= \varepsilon\frac{\partial}{\partial x_{j-1}}\left(\sum_{p=0}^k \frac{\varepsilon^p}{p!}\frac{d^p}{d\varepsilon^p}\Big|_{\varepsilon=0} g(\textstyle\sum_{l=0}^m \varepsilon^l x_l) + \widetilde{R}(\varepsilon, x_0, \cdots, x_m)\right),
\end{aligned}
$$

where $\widetilde{R}$ is some function satisfying $\widetilde{R} \sim o(|\varepsilon|^{k+1})$. Equating the coefficients of ε^k of the both sides of the above, we obtain Eq.(A.32). This ends the proof of Lemma A.2. ■

Remark A.3. Prop.A.1 also holds for a time-dependent matrix $F(t)$ as long as the fundamental matrix $X(t)$ of

$F(t)$ is periodic in t. Further, for Prop.A.1, we do not need to assume that functions g_i in Eq.(A.1) are polynomial in x. These assumptions are used to prove statements below.

Lemma A.4. For Eq.(A.1), functions $h_t^{(i)}(A)$ with $i = 1, 2, \cdots, m$ defined by (A.12) and (A.14) are bounded uniformly in t.

To prove this lemma, we need a theory of almost periodic functions. Indeed, we can show that functions $h_t^{(i)}(A)$ are almost periodic functions. This fact also holds even if $g_i(t, x)$'s in Eq.(A.1) are not periodic in t but almost periodic in t as long as the set of Fourier exponents of $g_i(t, x)$'s does not have accumulation points in $\mathbf{R}$. See Fink [13] for the definitions and basic facts of almost periodic functions.

Proof of Lem.A.4. We prove the proposition by induction. At first, note that $G_1(t, x_0)$ defined by Eq.(A.7) is almost periodic uniformly in x_0 because it is periodic in t and polynomial in x_0. Therefore, a function $X(t)^{-1}G_1(t, X(t)A)$ included in Eq.(A.12) is almost periodic uniformly in A (see Thm.2.11 of Fink [13]). Each components of the vector-valued function $X(t)^{-1}G_1(t, X(t)A)$ is of the form $\sum_{k=1}^{p} b_k(t)e^{i\xi_k t}$, where $b_k(t)$ are some periodic functions and $\xi_k \in \mathbf{R}$ are some constants. Since each $b_k(t)$ can be expanded as a Fourier series in ordinary sense, the set of Fourier exponents of $\sum_{k=1}^{p} b_k(t)e^{i\xi_k t}$ does not have accumulation points on $\mathbf{R}$. Since the Fourier coefficient corresponding to the zero Fourier exponent, if it exists, is $R_1(A)$ defined by Eq.(A.11), $X(t)^{-1}G_1(t, X(t)A) - R_1(A)$ does not have a zero as a Fourier exponent. Therefore $\int^t (X(s)^{-1}G_1(s, X(s)A) - R_1(A))ds$ is almost periodic (we use Thm.4.12 of Fink [13]) and this proves the lemma A.4 for $h_t^{(1)}(A)$.

Suppose that Lem.A.4 holds for $h_t^{(1)}(A), \cdots, h_t^{(i-1)}(A)$. Like the above, the integrand in Eq.(A.14) is almost periodic uniformly in A because $G_1(t, x_0, \cdots, x_{i-1})$ is periodic in t and polynomial in $x_0, \cdots, x_{i-1}$. Since $X(s)A, h_s^{(1)}(A), \cdots, h_s^{(i-1)}(A)$ are almost periodic the set of whose Fourier exponents has no accumulation points by the assumption of induction, the set of Fourier exponents of the function

$$p(s, A) := X(s)^{-1}G_i(s, X(s)A, h_s^{(1)}(A), \cdots, h_s^{(i-1)}(A)) - X(s)^{-1}\sum_{k=1}^{i-1}(Dh_s^{(k)})_A R_{i-k}(A)$$

included in Eq.(A.14) does not have accumulation points. Since $R_i(A)$ defined by Eq.(A.13) gives the Fourier coefficient corresponding to the zero Fourier exponent of $p(s, A)$, if it exists, there exists $M > 0$ such that all Fourier exponents λ of the integrand in Eq.(A.14) satisfies $|\lambda| \geq M$. Then Thm.4.12 of Fink [13] proves that $h_t^{(i)}(A)$ is almost periodic. ■

Definition A.5. Along with $R_1(A), \cdots, R_m(A)$ defined by Eq.(A.11) and (A.13), we define the *m-th order RG equation* for Eq.(A.1) by

$$\dot{A} = \varepsilon R_1(A) + \varepsilon^2 R_2(A) + \cdots + \varepsilon^m R_m(A), \quad A \in \mathbf{R}^n, \tag{A.34}$$

and we call $\varepsilon R_1(A) + \cdots + \varepsilon^m R_m(A)$ the *m-th order RG vector field* for Eq.(A.1). We denote by $\varphi_t^{(m)}$ the flow generated by the m-th order RG vector field.

Fix an open set $U \subset \mathbf{R}^n$ such that $\overline{U}$ is compact. Define the map α_t to be

$$\alpha_t(A) := X(t)A + \varepsilon h_t^{(1)}(A) + \varepsilon^2 h_t^{(2)}(A) + \cdots + \varepsilon^m h_t^{(m)}(A), \tag{A.35}$$

for all $t \in \mathbf{R}$. Now we are in a position to restate Thm.4.4 in the present situation.

Theorem A.6. Let $\varphi_t^{(m)}$ be the flow of the m-th order RG equation for Eq.(A.1) and α_t the map defined by Eq.(A.35). Then, there exists $\varepsilon_0 > 0$ such that the following holds for all $|\varepsilon| < \varepsilon_0$: A map

$$\Phi_{t,t_0} := \alpha_t \circ \varphi_{t-t_0}^{(m)} \circ \alpha_{t_0}^{-1} : \alpha_{t_0}(U) \to \mathbf{R}^n \tag{A.36}$$

defines a flow on $U_\varepsilon := \{(t,x)\,|\,t \in \mathbf{R},\ x \in \alpha_t(U)\}$ associated with a time-dependent vector field

$$F_\varepsilon(t,x) := \frac{d}{da}\Big|_{a=t} \Phi_{a,t}(x). \tag{A.37}$$

Further, there exists a vector field $\widetilde{F}_\varepsilon(t,x)$, which is bounded in t and bounded as $\varepsilon \to 0$, satisfying

$$F_\varepsilon(t,x) = Fx + \varepsilon g_1(t,x) + \cdots + \varepsilon^m g_m(t,x) + \varepsilon^{m+1}\widetilde{F}_\varepsilon(t,x). \tag{A.38}$$

Proof. The proof of the fact that the map Φ_{t,t_0} defines a flow is the same as that of Thm.4.4 (i). We prove Eq.(A.38). The vector field defined by Eq.(A.37) is calculated as

$$\begin{aligned}
F_\varepsilon(t,x) &= \frac{d}{da}\Big|_{a=t} \alpha_a \circ \varphi_{a-t}^{(m)} \circ \alpha_t^{-1}(x) \\
&= \frac{d}{da}\Big|_{a=t} \Big(x_0(a,0,\alpha_t^{-1}(x)) + \varepsilon x_1(a,a;\alpha_t^{-1}(x)) + \cdots + \varepsilon^m x_m(a,a;\alpha_t^{-1}(x))\Big) \\
&\quad + \Big(X(t) + \varepsilon (Dh_t^{(1)})_{\alpha_t^{-1}(x)} + \cdots + \varepsilon^m (Dh_t^{(m)})_{\alpha_t^{-1}(x)}\Big) \circ \Big(\varepsilon R_1(\alpha_t^{-1}(x)) + \cdots + \varepsilon^m R_m(\alpha_t^{-1}(x))\Big) \\
&= \frac{d}{da}\Big|_{a=t} \Big(x_0(a,0,\alpha_t^{-1}(x)) + \varepsilon x_1(a,t;\alpha_t^{-1}(x)) + \cdots + \varepsilon^m x_m(a,t;\alpha_t^{-1}(x))\Big) \\
&\quad + \frac{d}{da}\Big|_{a=t} \Big(\varepsilon x_1(t,a;\alpha_t^{-1}(x)) + \cdots + \varepsilon^m x_m(t,a;\alpha_t^{-1}(x))\Big) \\
&\quad + \Big(X(t) + \varepsilon (Dh_t^{(1)})_{\alpha_t^{-1}(x)} + \cdots + \varepsilon^m (Dh_t^{(m)})_{\alpha_t^{-1}(x)}\Big) \circ \Big(\varepsilon R_1(\alpha_t^{-1}(x)) + \cdots + \varepsilon^m R_m(\alpha_t^{-1}(x))\Big).
\end{aligned} \tag{A.39}$$

Since $x_i(a,t;\alpha_t^{-1}(x))$ is a solution of Eq.(A.5), it satisfies

$$\begin{aligned}
&\frac{d}{da}\Big|_{a=t} x_i(a,t;\alpha_t^{-1}(x)) = Fx_i(t,t;\alpha_t^{-1}(x)) + G_i(t, x_0(t,0,\alpha_t^{-1}(x)), \cdots, x_{i-1}(t,t;\alpha_t^{-1}(x))) \\
&= Fh_t^{(i)}(\alpha_t^{-1}(x)) + G_i(t, x_0, h_t^{(1)}(\alpha_t^{-1}(x)), \cdots, h_t^{(i-1)}(\alpha_t^{-1}(x))).
\end{aligned} \tag{A.40}$$

And according to Eq.(A.15) and (A.16), the equality

$$\frac{d}{da}\Big|_{a=t} x_i(t,a;\alpha_t^{-1}(x)) = -y_1^{(i)}(t,\alpha_t^{-1}(x)) = -X(t)R_i(\alpha_t^{-1}(x)) - \sum_{k=1}^{i-1} (Dh_t^{(k)})_{\alpha_t^{-1}(x)} R_{i-k}(\alpha_t^{-1}(x)) \tag{A.41}$$

holds. Substituting Eq.(A.40) and Eq.(A.41) into Eq.(A.39), we obtain

$$\begin{aligned}
&F_\varepsilon(t,x) \\
&= Fx_0(t,0,\alpha_t^{-1}(x)) + \sum_{k=1}^{m} \varepsilon^k \Big(Fh_t^{(k)}(\alpha_t^{-1}(x)) + G_k(t, x_0, h_t^{(1)}(\alpha_t^{-1}(x)), \cdots, h_t^{(k-1)}(\alpha_t^{-1}(x)))\Big) + O(\varepsilon^{m+1}) \\
&= Fx + \varepsilon g_1(t,x) + \cdots + \varepsilon^m g_m(t,x) + O(\varepsilon^{m+1}).
\end{aligned} \tag{A.42}$$

It is hard to write out the term $O(\varepsilon^{m+1})$ explicitly. However, it is easy to prove that the term $O(\varepsilon^{m+1})$ is bounded uniformly in t, because it consists of the almost periodic functions $X(t), g_i(t,x),\ h_t^{(i)},\ \alpha_t^{-1}$. This ends the proof of

Thm.A.6. ∎

Theorem 6.1 follows immediately as a corollary of the next theorem.

Theorem A.7. Consider an autonomous equation

$$\dot{x} = Fx + \varepsilon g_1(x) + \cdots + \varepsilon^m g_m(x), \quad x \in \mathbf{R}^n, \tag{A.43}$$

where F is a diagonalizable $n \times n$ constant matrix all of whose eigenvalues lie on the imaginary axis, and where $g_1, \cdots, g_m$ are polynomial vector fields on $\mathbf{R}^n$. Suppose that its m-th order RG vector field satisfies

$$R_1(A) = \cdots = R_{k-1}(A) = 0, \; R_k(A) \neq 0, \; k \leq 2m. \tag{A.44}$$

If the vector field $R_k(A)$ has a compact normally hyperbolic invariant manifold N, then Eq.(A.43) also has a normally hyperbolic invariant manifold N_ε for sufficiently small $\varepsilon > 0$. The N_ε is diffeomorphic to N and its stability coincides with that of N.

Proof. Before proving the theorem, we point out that the condition $k \leq 2m$ is not essential because we can take $m \in \mathbf{N}$ sufficiently large. Let us denote by $F_\varepsilon(t, x)$ the approximate vector field for Eq.(A.43) defined by (A.37). From Thm A.6, we can rewrite Eq.(A.43) as

$$\dot{x} = F_\varepsilon(t, x) - \varepsilon^{m+1} \widetilde{F}_\varepsilon(t, x). \tag{A.45}$$

On account of Eq.(A.36), the RG vector field $\varepsilon^k R_k(x) + \cdots + \varepsilon^m R_m(x)$ satisfies the equation

$$\begin{aligned} \varepsilon^k R_k(x) + \cdots + \varepsilon^m R_m(x) &= \frac{d}{da}\Big|_{a=t} \alpha_a^{-1} \circ \Phi_{a,t} \circ \alpha_t(x) \\ &= \frac{d\alpha_a^{-1}}{da}\Big|_{a=t} (\alpha_t(x)) + (D\alpha_t^{-1})_{\alpha_t(x)} \frac{d}{da}\Big|_{a=t} \Phi_{a,t} \circ \alpha_t(x) \\ &= -(D\alpha_t)_x^{-1} \frac{d\alpha_t}{dt}(x) + (D\alpha_t)_x^{-1} F_\varepsilon(t, \alpha_t(x)). \end{aligned} \tag{A.46}$$

Introducing a new function $y(t)$ by $x(t) = \alpha_t \circ y(t)$ and substituting it into Eq.(A.45), we obtain

$$\frac{d\alpha_t}{dt}(y(t)) + (D\alpha_t)_{y(t)} \dot{y}(t) = F_\varepsilon(t, \alpha_t(y(t))) - \varepsilon^{m+1} \widetilde{F}_\varepsilon(t, \alpha_t(y(t))).$$

This equation is put together with (A.46) to yield

$$\dot{y} = \varepsilon^k R_k(y) + \cdots + \varepsilon^m R_m(y) - \varepsilon^{m+1} (D\alpha_t)_y^{-1} \circ \widetilde{F}_\varepsilon(t, \alpha_t(y)). \tag{A.47}$$

We introduce a new scaled time s by $t = s/\varepsilon^k$. Then the above equation is rewritten as

$$\frac{dy}{ds} = R_k(y) + \varepsilon R_{k+1}(y) + \cdots + \varepsilon^{m-k} R_m(y) - \varepsilon^{m-k+1} (D\alpha_{s/\varepsilon^k})_y^{-1} \circ \widetilde{F}_{\varepsilon^k}(s/\varepsilon^k, \alpha_{s/\varepsilon^k}(y)). \tag{A.48}$$

Since $\alpha_t, (D\alpha_t)_y$ and $\widetilde{F}_\varepsilon(t, y)$ are bounded uniformly in $t \in \mathbf{R}$, $(D\alpha_{s/\varepsilon^k})_y^{-1} \circ \widetilde{F}_{\varepsilon^k}(s/\varepsilon^k, \alpha_{s/\varepsilon^k}(y))$ is also bounded as $s \to \pm\infty$ and $\varepsilon \to 0$. Therefore the time-dependent vector field $H(s, y)$ defined by the right hand side of the above equation is sufficiently close to the vector field $R_k(y)$ in C^1 topology if $\varepsilon > 0$ is sufficiently small.

Now we use Fenichel's theorem. We regard the vector field $R_k(y)$ on $\mathbf{R}^n$ as a vector field on $\mathbf{R} \times \mathbf{R}^n$ by putting $R_k(t, y) := R_k(y)$. If $R_k(y)$ has a normally hyperbolic invariant manifold N, $R_k(t, y)$ has a normally hyperbolic invariant manifold $\mathbf{R} \times N$ in (t, y) space. Since $H(s, y)$ is sufficiently close to $R_k(t, y)$ as an vector field on $\mathbf{R} \times \mathbf{R}^n$

in C^1 topology, $H(s,y)$ also has a normally hyperbolic invariant manifold $\widetilde{N}_\varepsilon$ which is diffeomorphic to $\mathbf{R}\times N$. Since $x(t) = \alpha_t \circ y(t)$ and since $D\alpha_t$ is bounded, Eq.(A.43) for $x(t)$ has a normally hyperbolic invariant manifold $\hat{N}_\varepsilon$ which is diffeomorphic to $\mathbf{R}\times N$ in (t,x) space.

Since Eq.(A.43) is autonomous, the manifold $\hat{N}_\varepsilon$ must be straight along the time axis (see the Fig.4). Consequently, Eq.(A.43) has a normally hyperbolic invariant manifold on $\mathbf{R}^n$ which is diffeomorphic to N. ■

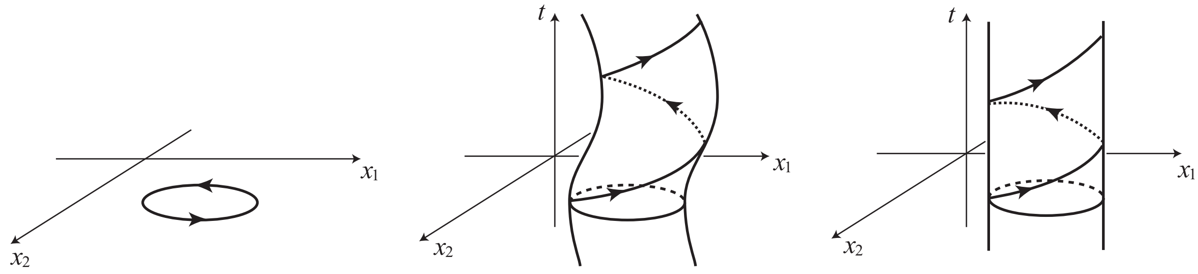


Fig. 4: The case that $R_k(x)$ has an invariant circle. In this case, the flows of Eqs.(A.48, 43) have invariant cylinders in the (t,x) space.

Let $A(t)$ be a solution of the m-th order RG equation (A.34) for Eq.(A.1) and define the curve $\tilde{x}(t)$ to be

$$\tilde{x}(t) := \alpha_t(A(t)) = X(t)A(t) + \varepsilon h_t^{(1)}(A(t)) + \cdots + \varepsilon^m h_t^{(m)}(A(t)). \tag{A.49}$$

Then, $\tilde{x}(t)$ is an integral curve of the approximate vector field $F_\varepsilon(t,x)$ defined by Eq.(A.37) and it gives an approximate solution for Eq.(A.1).

Theorem A.8. There exist positive constants ε_0, C, T, and a compact subset $V = V(\varepsilon) \subset \mathbf{R}^n$ including the origin such that for all $|\varepsilon| < \varepsilon_0$, every solution $x(t)$ of Eq.(A.1) and $\widetilde{x}(t)$ defined by Eq.(A.49) with $x(0) = \widetilde{x}(0) \in V$ satisfy the inequality

$$||x(t) - \widetilde{x}(t)|| < C\varepsilon^m, \quad \text{for } 0 \leq t \leq T/\varepsilon. \tag{A.50}$$

Proof of Thm.A.8. Suppose that $||x(0)|| < K$. At first, we show that there exists $T > 0$ such that $||x(t)|| < 2K$ for $0 \leq t \leq T/\varepsilon$. We rewrite Eq.(A.1) as the integral equation

$$x(t) = e^{Ft}x(0) + e^{Ft}\int_0^t e^{-Fs}\varepsilon g(s, x(s), \varepsilon)ds, \tag{A.51}$$

where $g(t,x,\varepsilon) := g_1(t,x) + \varepsilon g_2(t,x) + \cdots + \varepsilon^{m-1}g_m(t,x)$. Choose $t \geq 0$ so that $||x(s)|| < 2K$ if $0 \leq s \leq t$. Then, there exists a positive constant $K' > 0$ such that $||g(s, x(s), \varepsilon)|| < K'$ and the inequality

$$\begin{aligned} ||x(t)|| &\leq ||x(0)|| + \int_0^t \varepsilon ||g(s, x(s), \varepsilon)||ds \\ &\leq K + \int_0^t \varepsilon K' ds = K\left(1 + \frac{K'}{K}\varepsilon t\right) \end{aligned}$$

holds. When $0 \leq t \leq K/(K'\varepsilon)$, we have $||x(t)|| < 2K$, so that we put $T := K/K'$ for the existence of T.

By Thm.A.6, an approximate solution $\widetilde{x}(t)$ satisfies an ODE

$$\dot{\widetilde{x}}(t) = F_\varepsilon(t,\widetilde{x}) = F\widetilde{x} + \varepsilon g_1(t,\widetilde{x}) + \cdots + \varepsilon^m g_m(t,\widetilde{x}) + \varepsilon^{m+1}\widetilde{F}_\varepsilon(t,\widetilde{x}). \tag{A.52}$$

Fix a positive number K such that the closed ball B_{2K} of radius $2K$ centered at the origin is included in the open set $\alpha_t(U)$, where U is an open set on which α_t is a diffeomorphism. Then, we can verify that $\|\widetilde{x}(t)\| < 2K$ if $\|\widetilde{x}(0)\| < K$ and if $0 \leq t \leq T/\varepsilon$ in the same way as above.

For $x(t)$ and $\widetilde{x}(t)$ such that $x(0) = \widetilde{x}(0), \|x(0)\| < K$, we put $\xi(t) = \alpha_t^{-1} \circ x(t)$, $\eta(t) = \alpha_t^{-1} \circ \widetilde{x}(t)$. They satisfy respective ODEs

$$\dot{\xi}(t) = \varepsilon R_1(\xi) + \varepsilon^2 R_2(\xi) + \cdots + \varepsilon^m R_m(\xi) + \varepsilon^{m+1}\widetilde{G}_\varepsilon(t,\xi), \tag{A.53}$$

$$\dot{\eta}(t) = \varepsilon R_1(\eta) + \varepsilon^2 R_2(\eta) + \cdots + \varepsilon^m R_m(\eta), \tag{A.54}$$

where $\widetilde{G}_\varepsilon$ is a smooth function which is bounded uniformly in $t \in \mathbf{R}$ and bounded as $\varepsilon \to 0$ for each $\xi \in \mathbf{R}^n$. Let W be the image of the closed ball B_{2k} under the map α_t^{-1}. Then $\xi(t)$ and $\eta(t)$ are sitting in the compact set W if $0 \leq t \leq T/\varepsilon$. Let $L_1 > 0$ be a Lipschitz constant for $R_1(\xi) + \varepsilon R_2(\xi) + \cdots + \varepsilon^{m-1}R_m(\xi)$ on W and suppose that $\sup_{t\in\mathbf{R},\xi\in W} \|\widetilde{G}_\varepsilon(t,\xi)\| < L_2$. Then, for $0 \leq t \leq T/\varepsilon$, the inequality

$$\|\xi(t) - \eta(t)\| \leq \varepsilon L_1 \int_0^t \|\xi(s) - \eta(s)\| ds + \varepsilon^{m+1} L_2 t \tag{A.55}$$

holds. Then, the Gronwall inequality implies that

$$\|\xi(t) - \eta(t)\| \leq \frac{L_2}{L_1}\varepsilon^m (e^{\varepsilon L_1 t} - 1) \leq \frac{L_2}{L_1}\varepsilon^m (e^{L_1 T} - 1), \quad 0 \leq t \leq T/\varepsilon. \tag{A.56}$$

This shows that there exists a positive constant C such that $\|x(t) - \widetilde{x}(t)\| = \|\alpha_t \circ \xi(t) - \alpha_t \circ \eta(t)\| \leq C\varepsilon^m$ holds if $0 \leq t \leq T/\varepsilon$. ■

The next theorem is simple extension of the Propositions 5.1 and 5.2.

Theorem A.9. Consider an autonomous equation (A.43).

(i) If vector fields Fx and $g_1(x), g_2(x), \cdots$ are invariant under the action of a Lie group G, then the m-th order RG equation is also invariant under the action of G.

(ii) The m-th order RG equation commutes with the linear vector field Fx with respect to Lie bracket product. Equivalently, each $R_i(A)$, $i = 1, 2, \cdots$, satisfies

$$X(t)R_i(A) = R_i(X(t)A), \quad A \in \mathbf{R}^n. \tag{A.57}$$

Proof of Thm.A.9. Recall that G_i in Eq.(A.5) is independent of t since Eq.(A.43) is autonomous.

(i) We prove by induction that $R_i(A)$ and $h_t^{(i)}(A)$, $i = 1, 2, \cdots$, are invariant under the action of a Lie group G. Since $aX(t)A = X(t)aA$ and $ag_1(x) = g_1(ax)$ hold for all $a \in G$, $R_1(aA)$ is brought into the form

$$\begin{aligned} R_1(aA) &= \lim_{t\to\infty} \frac{1}{t} \int^t X(s)^{-1} G_1(X(s)aA) ds \\ &= a \lim_{t\to\infty} \frac{1}{t} \int^t X(s)^{-1} G_1(X(s)A) ds = aR_1(A). \end{aligned}$$

And the invariance of $h_t^{(1)}$, $h_t^{(1)}(aA) = ah_t^{(1)}(A)$, is verified in a similar way. Suppose that $R_k(aA) = aR_k(A)$ and $h_t^{(k)}(aA) = ah_t^{(k)}(A)$ hold for $k = 1, 2, \cdots, i-1$. Then, it is easy to verify that

$$(Dh_t^{(k)})_{aA} = a(Dh_t^{(k)})_A a^{-1}, \tag{A.58}$$
$$G_k(X(t)aA, h_t^{(1)}(aA), \cdots, h_t^{(k-1)}(aA)) = aG_k(X(t)A, h_t^{(1)}(A), \cdots, h_t^{(k-1)}(A)) \tag{A.59}$$

for $k = 1, 2, \cdots, i-1$. This and Eqs.(A.13), (A.14) implies that $R_i(aA) = aR_i(A)$ and $h_t^{(i)}(aA) = ah_t^{(i)}(A)$.

(ii) We prove by induction that $R_i(X(t)A) = X(t)R_i(A)$ and $h_t^{(i)}(X(t')A) = h_{t+t'}^{(i)}(A)$ hold for $i = 1, 2, \cdots$. For all $s' \in \mathbf{R}$, $R_1(X(s')A)$ takes the form

$$\begin{aligned} R_1(X(s')A) &= \lim_{t\to\infty} \frac{1}{t} \int^t X(s)^{-1} G_1(X(s)X(s')A) ds \\ &= X(s') \lim_{t\to\infty} \frac{1}{t} \int^t X(s+s')^{-1} G_1(X(s+s')A) ds. \end{aligned}$$

Putting $s + s' = s''$, we verify that

$$\begin{aligned} R_1(X(s')A) &= X(s') \lim_{t\to\infty} \frac{1}{t} \int^{t+s'} X(s'')^{-1} G_1(X(s'')A) ds'' \\ &= X(s')R_1(A) + X(s') \lim_{t\to\infty} \frac{1}{t} \int_t^{t+s'} X(s'')^{-1} G_1(X(s'')A) ds'' \\ &= X(s')R_1(A). \end{aligned}$$

Next, $h_t^{(1)}(X(s')A)$ is calculated as

$$\begin{aligned} h_t^{(1)}(X(s')A) &= X(t) \int^t \Big(X(s)^{-1} G_1(X(s)X(s')A) - R_1(X(s')A) \Big) ds \\ &= X(t)X(s') \int^t \Big(X(s')^{-1} X(s)^{-1} G_1(X(s)X(s')A) - R_1(A) \Big) ds \\ &= X(t+s') \int^t \Big(X(s+s')^{-1} G_1(X(s+s')A) - R_1(A) \Big) ds. \end{aligned}$$

Putting $s + s' = s''$ provides

$$h_t^{(1)}(X(s')A) = X(t+s') \int^{t+s'} \Big(X(s'')^{-1} G_1(X(s'')A) - R_1(A) \Big) ds'' = h_{t+s'}^{(1)}(A). \tag{A.60}$$

Suppose that $R_k(X(t)A) = X(t)R_i(A)$ and $h_t^{(k)}(X(t')A) = h_{t+t'}^{(k)}(A)$ hold for $k = 1, 2, \cdots, i-1$. Then, $R_i(X(s')A)$ is calculated as

$$\begin{aligned} R_i(X(s')A) = \lim_{t\to\infty} \frac{1}{t} \int^t & \Big(X(s)^{-1} G_i(X(s)X(s')A, h_s^{(1)}(X(s')A), \cdots, h_s^{(i-1)}(X(s')A)) \\ & - X(s)^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_{X(s')A} R_{i-k}(X(s')A) \Big) ds \\ = X(s') \lim_{t\to\infty} \frac{1}{t} \int^t & \Big(X(s+s')^{-1} G_i(X(s+s')A, h_{s+s'}^{(1)}(A), \cdots, h_{s+s'}^{(i-1)}(A)) \\ & - X(s+s')^{-1} \sum_{k=1}^{i-1} (Dh_{s+s'}^{(k)})_A R_{i-k}(A) \Big) ds. \end{aligned}$$

Putting $s + s' = s''$ provides

$$\begin{aligned}
R_i(X(s')A) &= X(s') \lim_{t\to\infty} \frac{1}{t} \int^{t+s'} \Big(X(s'')^{-1} G_i(X(s'')A, h^{(1)}_{s''}(A), \cdots, h^{(i-1)}_{s''}(A)) \\
&\qquad - X(s'')^{-1} \sum_{k=1}^{i-1} (Dh^{(k)}_{s''})_A R_{i-k}(A) \Big) ds'' \\
&= X(s')R_i(A) + X(s') \lim_{t\to\infty} \frac{1}{t} \int_t^{t+s'} \Big(X(s'')^{-1} G_i(X(s'')A, h^{(1)}_{s''}(A), \cdots, h^{(i-1)}_{s''}(A)) \\
&\qquad - X(s'')^{-1} \sum_{k=1}^{i-1} (Dh^{(k)}_{s''})_A R_{i-k}(A) \Big) ds'' \\
&= X(s')R_i(A).
\end{aligned}$$

We can show that $h^{(i)}_t(X(t')A) = h^{(i)}_{t+t'}(A)$ in a similar way. ■

Reference

[1] L. Y. CHEN, N. GOLDENFELD, Y. OONO, Renormalization group theory for global asymptotic analysis, Phys. Rev. Lett., 73(1994), no.10, pp. 1311-1315.

[2] L. Y. CHEN, N .GOLDENFELD, Y. OONO, Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory, Phys. Rev. E, 54(1996), pp. 376-394.

[3] T. KUNIHIRO, A geometrical formulation of the renormalization group method for global analysis, Progr. Theoret. Phys., 94(1995), no.4, pp. 503-514.

[4] T. KUNIHIRO, The renormalization-group method applied to asymptotic analysis of vector fields, Progr. Theoret. Phys., 97(1997), no.2, pp. 179-200.

[5] K. NOZAKI, Y. OONO, Renormalization-group theoretical reduction, Phys. Rev. E, 63(2001), 046101

[6] S. GOTO, Y. MASUTOMI, K. NOZAKI, Lie-Group Approach to Perturbative Renormalization Group Method, Progr. Theoret. Phys., vol.102, No.3.(1999), pp. 471-497.

[7] S. EI, K. FUJII, T. KUNIHIRO, Renormalization-group method for reduction of evolution equations; invariant manifolds and envelopes, Ann. Physics, 280(2000),no. 2, pp. 236-298.

[8] M. ZIANE, On a certain renormalization group method, J. Math. Phys., 41(2000), no.5, pp. 3290-3299.

[9] R. E. LEE DEVILLE, A. HARKIN, M. HOLZER, K. JOSIĆ, T. KAPER, Analysis of a renormalization group method and normal form theory for perturbed ordinary differential equations, Physica D, (2008)

[10] N. FENICHEL, Persistence and smoothness of invariant manifolds for flows, Indiana Univ. Math. J., 21(1971), pp. 193-226.

[11] M. W. HIRSCH, C. C. PUGH, M. SHUB, Invariant manifolds, Springer-Verlag, 1977, Lec. Notes in Math., 583.

[12] S. WIGGINS, Normally hyperbolic invariant manifolds in dynamical systems , Springer-Verlag, 1994.

[13] A. M. FINK, Almost Periodic Differential Equations, Springer-Verlag, Lec. Notes in Math., 377, 1974.

[14] N. N. BOGOLIUBOV, Y. A. MITROPOLSKI, Asymptotic Methods in the Theory of Non-Linear Oscillations, Gordon and Breach, New York, 1961.