

# Continuous limit and the moments system for the globally coupled phase oscillators

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## Abstract

The Kuramoto model, which describes synchronization phenomena, is a system of ordinary differential equations on  $N$ -torus defined as coupled harmonic oscillators. The order parameter is often used to measure the degree of synchronization. In this paper, the moments systems are introduced for both of the Kuramoto model and its continuous model. It is shown that the moments systems for both systems take the same form. This fact allows one to prove that the order parameter of the  $N$ -dimensional Kuramoto model converges to that of the continuous model as  $N \rightarrow \infty$ .

## 1 Introduction

Collective synchronization phenomena are observed in a variety of areas such as chemical reactions, engineering circuits and biological populations [17]. In order to investigate such a phenomenon, Kuramoto [10, 11] proposed the system of ordinary differential equations

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad i = 1, \dots, N, \quad (1.1)$$

where  $\theta_i \in [0, 2\pi)$  denotes the phase of an  $i$ -th oscillator on a circle,  $\omega_i \in \mathbf{R}$  denotes its natural frequency,  $K > 0$  is the coupling strength, and where  $N$  is the number of oscillators. Eq.(1.1) is derived by means of the averaging method from coupled dynamical systems having limit cycles, and now it is called the *Kuramoto model*.

It is obvious that when  $K = 0$ ,  $\theta_i(t)$  and  $\theta_j(t)$  rotate on a circle at different velocities unless  $\omega_i$  is equal to  $\omega_j$ , and it is true for sufficiently small  $K > 0$ . On the other hand, if  $K$  is sufficiently large, it is numerically observed that some of oscillators or all of them tend to rotate at the same velocity on average, which is called the *synchronization* [17, 20]. If  $N$  is small, such a transition from de-synchronization to synchronization may be well revealed by means of the bifurcation theory [5, 12, 13]. However, if  $N$  is large, it is difficult to investigate the transition from the view point of the bifurcation theory and it is still far from understood.

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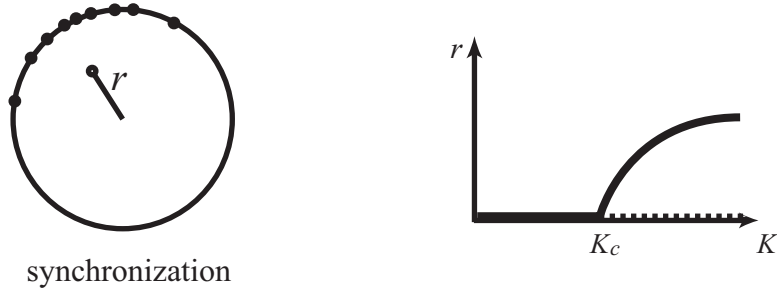


Fig. 1: A bifurcation diagram of the order parameter.

In order to evaluate whether synchronization occurs or not, Kuramoto introduced the *order parameter*  $r_N(t)$  by

$$r_N(t) := \frac{1}{N} \sum_{j=1}^N e^{\sqrt{-1}\theta_j(t)}, \quad (1.2)$$

which gives the centroid of oscillators. It seems that if synchronous state is formed,  $|r_N(t)|$  takes a positive number, while if de-synchronization is stable,  $|r_N(t)|$  is zero on time average. Indeed, based on some formal calculations, Kuramoto assumed a bifurcation diagram of the order parameter: Suppose  $N \rightarrow \infty$ . If  $g(\omega)$ , a distribution function for  $\omega_i$ 's, is an even and unimodal function such that  $g''(0) \neq 0$ , then the bifurcation diagram of  $|r_\infty|$  is given as in Fig.1. In other words, if the coupling strength  $K$  is smaller than  $K_c := 2/(\pi g(0))$ , then  $r_\infty \equiv 0$  is asymptotically stable. If  $K$  exceeds  $K_c$ , then a stable synchronous state emerges. Near the transition point  $K_c$ ,  $|r_\infty|$  is of order  $O((K - K_c)^{1/2})$ . See [20] for Kuramoto's discussion. In order to state his conjecture clearly, let us introduce the continuous model.

The infinite-dimensional version (the continuous model) of the Kuramoto model has been well investigated to reveal a bifurcation diagram of the order parameter (see [1, 3, 4, 6, 14, 15, 20, 21, 22] and references therein). The continuous model is defined as the equation of continuity of the form

$$\frac{\partial \rho_t}{\partial t} + \frac{\partial}{\partial \theta}(\rho_t v) = 0, \quad (1.3)$$

where the unknown function  $\rho_t = \rho_t(\theta, \omega)$  is a probability measure on  $[0, 2\pi) \times \mathbf{R}$  parameterized by  $t \in \mathbf{R}$ . Roughly speaking,  $\rho_t(\theta, \omega)$  denotes a probability that an oscillator having a natural frequency  $\omega$  is placed at a position  $\theta$ . See the next section for the definition of the vector field  $v$ . The continuous version of the order parameter is defined to be

$$r_\infty(t) := \int_{\mathbf{R}} \int_0^{2\pi} e^{\sqrt{-1}\theta} d\rho_t. \quad (1.4)$$

Such a system is rather tractable because the order parameter for the infinite-dimensional version can be constant in time, while the order parameter for the finite dimensional Ku-

ramoto model is not constant in general because solutions fluctuate due to effects of finiteness [20]. Recently, the Kuramoto's conjecture for the continuous model is rigorously proved by Chiba [4]; The bifurcation diagram of the continuous version of the order parameter  $r_\infty$  is given like as Fig.1.

Now the questions arise : How close is the order parameter of the infinite-dimensional version to that of the finite-dimensional Kuramoto model? What is the influence of finite size effects? This issue has been studied by many authors, see a reference paper [1] by Acebron *et al.* In particular, Daido [7] found the scaling law  $|r_\infty - \langle r_N(t) \rangle| \sim (K_c - K)^{-1/2} N^{-1/2}$  for  $K < K_c$  and  $|r_\infty - \langle r_N(t) \rangle| \sim (K - K_c)^{-1/8} N^{-1/2}$  for  $K > K_c$ , although his analysis is not rigorous from a view point of mathematics, where  $r_\infty$  is assumed to be in a steady state, and  $\langle \rangle$  denotes the time average.

In this paper, it is proved that the order parameter of the  $N$ -dimensional Kuramoto model converges to that of the continuous model in the sense of probability, and their difference is of  $O(N^{-1/2})$  as  $N \rightarrow \infty$  for each  $t$  (note that we do not take time average). To prove this, the  $(m, k)$ -th moments are defined for both of the continuous model and the finite dimensional model. In particular,  $(0, 1)$ -th moment is the Kuramoto's order parameter. It is remarkable that both of the continuous model and the  $N$ -dimensional model become the same evolution equation, called the *moments system*, if they are rewritten by using the moments. It means that any solutions of the continuous model and the  $N$ -dimensional model for any  $N$  are embedded in the *same* phase space of the moments system. This fact allows us to measure the distance between solutions of the continuous model and that of the  $N$ -dimensional model in the same phase space. These results and the central limit theorem prove that the difference between the order parameter of the  $N$ -dimensional model and that of the continuous model is of order  $O(N^{-1/2})$  for each  $t$ , provided that initial values and natural frequencies for the  $N$ -dimensional model are independent and identically distributed according to a suitable probability measure.

The strategy of the proof is as follows: Let  $Z_k^m(t)$  and  $\hat{Z}_k^m(t)$  be  $(m, k)$ -th moments of the continuous model and the  $N$ -dimensional model, respectively (in particular,  $Z_1^0 = r_\infty$  and  $\hat{Z}_1^0 = r_N$ ). Note that  $Z_k^m(t)$  is determined by  $\rho_t(\theta, \omega)$  and  $\hat{Z}_k^m(t)$  is determined by  $\{(\omega_j, \theta_j(t))\}_{j=1}^N$ . Let  $h(\theta, \omega) = \rho_0(\theta, \omega)$  be an initial measure for the continuous model. Under a certain condition, there is a one-to-one correspondence between  $\rho_t(\theta, \omega)$  and the set of moments  $\{Z_k^m(t)\}_{m,k}$  for each  $t \geq 0$ . If initial values and natural frequencies for the  $N$ -dimensional model are independent and identically distributed according to a measure  $h(\theta, \omega)$ , the law of large number proves that  $\hat{Z}_k^m(0) \rightarrow Z_k^m(0)$  as  $N \rightarrow \infty$ . However, this argument is no longer applicable for a positive  $t$  because  $\theta_j(t)$ 's are not independent and identically distributed when  $t$  is positive. Now we use the fact that  $\{Z_k^m(t)\}_{m,k}$  and  $\{\hat{Z}_k^m(t)\}_{m,k}$  are governed by the same differential equation called the *moments system*. Then, the continuity of solutions of the moments system with respect to initial conditions immediately proves that if  $\hat{Z}_k^m(t)$  and  $Z_k^m(t)$  are sufficiently closed to one another for  $t = 0$ , the same is

true for positive  $t > 0$ . See the diagram below.

$$\begin{array}{ccccc}
h(\theta, \omega) & \longleftarrow & \{Z_k^m(0)\}_{m,k} & \xleftarrow{N \rightarrow \infty} & \{\hat{Z}_k^m(0)\}_{m,k} \\
\downarrow & & \downarrow & & \downarrow \\
\rho_t(\theta, \omega) & \longleftarrow & \{Z_k^m(t)\}_{m,k} & \xleftarrow{N \rightarrow \infty} & \{\hat{Z}_k^m(t)\}_{m,k}
\end{array}$$

Since the Kuramoto's conjecture for the continuous model is proved in [4], we obtain the following result as a corollary:

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} r_N(t) = \begin{cases} 0 & (0 < K < K_c) \\ O(\sqrt{K - K_c}) & (K > K_c), \end{cases} \quad (1.5)$$

although behavior of another limit  $\lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} r_N$  is still open.

More generally, a globally coupled phase oscillators defined to be

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^N f(\theta_j - \theta_i), \quad i = 1, \dots, N, \quad (1.6)$$

is called the *Kuramoto-Daido model* [8, 6], where the  $2\pi$ -periodic function  $f : \mathbf{R} \rightarrow \mathbf{R}$  is called the *coupling function*. The results in this paper are easily extended to the Kuramoto-Daido model.

## 2 Continuous model

In this section, we introduce a continuous model of the Kuramoto model and show existence, uniqueness and other properties of solutions of the model used in a later section.

Let us consider the Kuramoto model (1.1). Following Kuramoto, we introduce the *order parameter*  $\hat{Z}_1^0$  by

$$\hat{Z}_1^0(t) = \frac{1}{N} \sum_{j=1}^N e^{\sqrt{-1}\theta_j(t)}. \quad (2.1)$$

The quantities  $\hat{Z}_k^m$  will be defined in the next section. By using it, Eq.(1.1) is rewritten as

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{2\sqrt{-1}} (\hat{Z}_1^0(t) e^{-\sqrt{-1}\theta_i} - \overline{\hat{Z}_1^0(t)} e^{\sqrt{-1}\theta_i}), \quad (2.2)$$

where  $\overline{\hat{Z}_1^0}$  denotes the complex conjugate of  $\hat{Z}_1^0$ . Motivated by these equations, we introduce a continuous model of the Kuramoto model, which is an evolution equation of a

probability measure  $\rho_t = \rho_t(\theta, \omega)$  on  $S^1 \times \mathbf{R}$  parameterized by  $t \in \mathbf{R}$ , as

$$\begin{cases} \frac{\partial \rho_t}{\partial t} + \frac{\partial}{\partial \theta} \left( \left( \omega + \frac{K}{2\sqrt{-1}} (Z_1^0(t) e^{-\sqrt{-1}\theta} - \overline{Z_1^0(t)} e^{\sqrt{-1}\theta}) \right) \rho_t \right) = 0, \\ Z_1^0(t) := \int_{\mathbf{R}} \int_0^{2\pi} e^{\sqrt{-1}\theta} d\rho_t, \\ \rho_0(\theta, \omega) = h(\theta, \omega), \end{cases} \quad (2.3)$$

where  $h(\theta, \omega)$  is an initial measure. The  $Z_1^0(t)$  is a continuous version of  $\hat{Z}_1^0(t)$ , and we also call it the *order parameter*. If we regard

$$v_t := \omega + \frac{K}{2\sqrt{-1}} (Z_1^0(t) e^{-\sqrt{-1}\theta} - \overline{Z_1^0(t)} e^{\sqrt{-1}\theta})$$

as a velocity field, Eq.(2.3) provides an equation of continuity  $\partial \rho_t / \partial t + \partial(\rho_t v_t) / \partial \theta = 0$  known in fluid dynamics. It is easy to prove the law of conservation of mass:

$$\int_{\mathbf{R}} \int_0^{2\pi} \chi_E(\omega) d\rho_t = \int_{\mathbf{R}} \int_0^{2\pi} \chi_E(\omega) dh =: g(E), \quad (2.4)$$

where  $E$  is any Borel set on  $\mathbf{R}$  and  $\chi_E(\omega)$  is the characteristic function on  $E$ . A function  $g$  defined as above gives a probability measure for natural frequencies  $\omega \in \mathbf{R}$  such that  $\int_{\mathbf{R}} dg = 1$ . In particular  $\int_{\mathbf{R}} \int_0^{2\pi} d\rho_t = 1$  if  $\int_{\mathbf{R}} \int_0^{2\pi} dh = 1$ .

By using the characteristic curve method, Eq.(2.3) is formally integrated as follows: Consider the equation

$$\frac{dx}{dt} = \omega + \frac{K}{2\sqrt{-1}} (Z_1^0(t) e^{-\sqrt{-1}x} - \overline{Z_1^0(t)} e^{\sqrt{-1}x}), \quad x \in S^1, \quad (2.5)$$

which defines a characteristic curve. Let  $x = x(t, s; \theta, \omega)$  be a solution of Eq.(2.5) satisfying the initial condition  $x(s, s; \theta, \omega) = \theta$ . Along the characteristic curve  $x(t) := x(t, 0, \tilde{\theta}, \omega)$ ,  $\rho_t(x(t), \omega)$  is differentiated as

$$\begin{aligned} \frac{d}{dt} \rho_t(x(t), \omega) &= \frac{\partial \rho_t}{\partial t}(x(t), \omega) + \frac{\partial \rho_t}{\partial \theta} \Big|_{\theta=x(t)}(\theta, \omega) \cdot \frac{dx}{dt}(t) \\ &= \frac{\partial \rho_t}{\partial t}(x(t), \omega) + \frac{\partial \rho_t}{\partial \theta} \Big|_{\theta=x(t)}(\theta, \omega) \cdot v_t \Big|_{\theta=x(t)} \\ &= \frac{\partial \rho_t}{\partial t}(x(t), \omega) + \frac{\partial}{\partial \theta} \Big|_{\theta=x(t)}(v_t \rho_t) - \rho_t(x(t), \omega) \frac{\partial}{\partial \theta} \Big|_{\theta=x(t)} v_t. \end{aligned}$$

Eq.(2.3) is used to yield

$$\begin{aligned} \frac{d}{dt} \rho_t(x(t), \omega) &= -\rho_t(x(t), \omega) \frac{\partial}{\partial \theta} \Big|_{\theta=x(t)} v_t \\ &= \rho_t(x(t), \omega) \cdot \frac{K}{2} \left( Z_1^0(t) e^{-\sqrt{-1}x(t)} + \overline{Z_1^0(t)} e^{\sqrt{-1}x(t)} \right). \end{aligned}$$

Hence, we obtain

$$\rho_t(x(t), \omega) = h(\tilde{\theta}, \omega) \cdot \exp\left[\frac{K}{2} \int_0^t (Z_1^0(s)e^{-\sqrt{-1}x(s)} + \overline{Z_1^0(s)}e^{\sqrt{-1}x(s)})ds\right],$$

which is true for any characteristic curve  $x(t) = x(t, 0, \tilde{\theta}, \omega)$ . Now we substitute  $\tilde{\theta} = x(0, t; \theta, \omega)$ . Due to the flow property, we have

$$x(s, 0; x(0, t; \theta, \omega), \omega) = x(s, t; \theta, \omega), \quad x(t, 0; x(0, t; \theta, \omega), \omega) = \theta.$$

Therefore, we obtain

$$\rho_t(\theta, \omega) = h(x(0, t; \theta, \omega), \omega) \exp\left[\frac{K}{2} \int_0^t (Z_1^0(s)e^{-\sqrt{-1}x(s,t;\theta,\omega)} + \overline{Z_1^0(s)}e^{\sqrt{-1}x(s,t;\theta,\omega)})ds\right], \quad (2.6)$$

which gives a weak solution of (2.3). By using Eq.(2.6), it is easy to show the equality

$$\int_{\mathbf{R}} \int_0^{2\pi} a(\theta, \omega) d\rho_t = \int_{\mathbf{R}} \int_0^{2\pi} a(x(t, 0; \theta, \omega), \omega) dh, \quad (2.7)$$

for any measurable function  $a(\theta, \omega)$ . In particular, the order parameter  $Z_1^0(t)$  are rewritten as

$$Z_1^0(t) = \int_{\mathbf{R}} \int_0^{2\pi} e^{\sqrt{-1}x(t,0;\theta,\omega)} dh. \quad (2.8)$$

Substituting it into Eqs.(2.5), (2.6), we obtain

$$\frac{d}{dt}x(t, s; \theta, \omega) = \omega + K \int_{\mathbf{R}} \int_0^{2\pi} \sin(x(t, 0; \theta', \omega') - x(t, s; \theta, \omega)) dh(\theta', \omega'), \quad (2.9)$$

and

$$\rho_t(\theta, \omega) = h(x(0, t; \theta, \omega), \omega) \exp\left[K \int_0^t ds \cdot \int_{\mathbf{R}} \int_0^{2\pi} \cos(x(s, 0; \theta', \omega') - x(s, t; \theta, \omega)) dh(\theta', \omega')\right]. \quad (2.10)$$

Even if  $h(\theta, \omega)$  is not differentiable, we consider Eq.(2.10) to be a weak solution of Eq.(2.3). Indeed, even if  $h$  and  $\rho_t$  are not differentiable, the right hand side of (2.7) is differentiable with respect to  $t$  when  $a(\theta, \omega)$  is differentiable.

**Theorem 2.1.** (i) There exists a unique weak solution  $\rho_t$  of the initial value problem (2.3) for any  $t \geq 0$ .

(ii) Solutions of (2.3) depend continuously on initial measures with respect to the weak topology in the sense that for any numbers  $T, \varepsilon > 0$  and for any continuous function  $a(\theta, \omega)$  on  $S^1 \times \mathbf{R}$ , there exist numbers  $M(b) > 0$  and  $\delta = \delta(T, \varepsilon, a) > 0$  such that if initial measures  $h_1, h_2$  satisfy

$$\left| \int_{\mathbf{R}} \int_0^{2\pi} b(\theta, \omega) (dh_1 - dh_2) \right| < M(b)\delta, \quad (2.11)$$

for any continuous function  $b(\theta, \omega)$ , then solutions  $\rho_{t,1}$  and  $\rho_{t,2}$  with  $\rho_{0,1} = h_1$  and  $\rho_{0,2} = h_2$  satisfy

$$\left| \int_{\mathbf{R}} \int_0^{2\pi} a(\theta, \omega) (d\rho_{t,1} - d\rho_{t,2}) \right| < \varepsilon, \quad (2.12)$$

for  $0 \leq t \leq T$ . In particular if  $a$  is Lipschitz continuous, then  $\varepsilon \sim O(\delta)$  as  $\delta \rightarrow 0$ .

**Proof of (i).** It is sufficient to prove that the integro-ODE (2.9) has a unique solution  $x(t, s; \theta, \omega)$  satisfying  $x(s, s; \theta, \omega) = \theta$  for any  $t, s \geq 0$  and  $\theta \in S^1$ . Let us define a sequence  $\{x_n(t, 0; \theta, \omega)\}_{n=0}^\infty$  to be

$$x_{n+1}(t, 0; \theta, \omega) = x_0(t, 0; \theta, \omega) + K \int_0^t d\tau \cdot \int_{\mathbf{R}} \int_0^{2\pi} f(x_n(\tau, 0; \theta', \omega') - x_n(\tau, 0; \theta, \omega)) dh(\theta', \omega') \quad (2.13)$$

and  $x_0(t, 0; \theta, \omega) = \theta + \omega t$ , where  $f(\theta) = \sin \theta$  (since we prove the theorem for any  $C^1$  function  $f(\theta)$ , the theorem is also true for the continuous model for the Kuramoto-Daido model (1.6)). We estimate  $|x_{n+1}(t, 0; \theta, \omega) - x_n(t, 0; \theta, \omega)|$  as

$$\begin{aligned} & |x_{n+1}(t, 0; \theta, \omega) - x_n(t, 0; \theta, \omega)| \\ & \leq K \int_0^t d\tau \cdot \int_{\mathbf{R}} \int_0^{2\pi} \left| f(x_n(\tau, 0; \theta', \omega') - x_n(\tau, 0; \theta, \omega)) \right. \\ & \quad \left. - f(x_{n-1}(\tau, 0; \theta', \omega') - x_{n-1}(\tau, 0; \theta, \omega)) \right| dh(\theta', \omega') \\ & \leq KL \int_0^t d\tau \cdot \int_{\mathbf{R}} \int_0^{2\pi} \left( |x_n(\tau, 0; \theta', \omega') - x_{n-1}(\tau, 0; \theta', \omega')| \right. \\ & \quad \left. + |x_n(\tau, 0; \theta, \omega) - x_{n-1}(\tau, 0; \theta, \omega)| \right) dh(\theta', \omega'), \end{aligned}$$

where  $L > 0$  is the Lipschitz constant of the function  $f(\theta)$ . When  $n = 0$ , we obtain

$$\begin{aligned} |x_1(t, 0; \theta, \omega) - x_0(t, 0; \theta, \omega)| & \leq K \int_0^t d\tau \cdot \int_{\mathbf{R}} \int_0^{2\pi} |f(x_0(\tau, 0; \theta', \omega') - x_0(\tau, 0; \theta, \omega))| dh(\theta', \omega') \\ & \leq KMt, \end{aligned}$$

where  $M = \max |f(\theta)|$ . Thus we can show by induction that

$$|x_n(t, 0; \theta, \omega) - x_{n-1}(t, 0; \theta, \omega)| \leq 2^{n-1} L^{n-1} K^n M \frac{t^n}{n!}. \quad (2.14)$$

This proves that  $x_n(t, 0; \theta, \omega)$  converges to a solution of the equation

$$\frac{dx}{dt}(t, 0; \theta, \omega) = \omega + K \int_{\mathbf{R}} \int_0^{2\pi} f(x(t, 0; \theta', \omega') - x(t, 0; \theta, \omega)) dh(\theta', \omega'),$$

as  $n \rightarrow \infty$  for small  $t \geq 0$ . Existence of global solutions are easily obtained by a standard way because the phase space  $S^1$  is compact; that is, solutions are extended for any  $t > 0$ .

Uniqueness of solutions is also proved in a standard way and the detail is omitted. With this  $x(t, 0; \theta, \omega)$ , we define a sequence  $\{x_n(t, s; \theta, \omega)\}_{n=0}^{\infty}$  to be

$$x_{n+1}(t, s; \theta, \omega) = x_0(t, s; \theta, \omega) + K \int_0^t d\tau \cdot \int_{\mathbf{R}} \int_0^{2\pi} f(x(\tau, 0; \theta', \omega') - x_n(\tau, s; \theta, \omega)) dh(\theta', \omega') \quad (2.15)$$

and  $x_0(t, s; \theta, \omega) = \theta + \omega(t - s)$ . Then, existence and uniqueness of global solutions  $x(t, s; \theta, \omega)$  is proved in the same way as above. For this  $x(t, s; \theta, \omega)$ , Eq.(2.10) gives a (weak) solution of Eq.(2.3).

**Proof of (ii).** Suppose that initial measures  $h_1, h_2$  satisfy Eq.(2.11). Let  $\rho_{t,1}$  and  $\rho_{t,2}$  be solutions of Eq.(2.3) satisfying  $\rho_{0,1} = h_1$  and  $\rho_{0,2} = h_2$ . Let  $x_i = x_i(t, 0; \theta, \omega)$ , ( $i = 1, 2$ ) be solutions of

$$\frac{dx_i}{dt} = \omega + K \int_{\mathbf{R}} \int_0^{2\pi} f(x_i(t, 0; \theta', \omega') - x_i(t, 0; \theta, \omega)) dh_i(\theta', \omega'), \quad x_i \in S^1, \quad (2.16)$$

satisfying  $x_i(0, 0; \theta, \omega) = \theta$ , respectively. Then we obtain

$$\begin{aligned} & \frac{d}{dt} (x_1(t, 0; \theta, \omega) - x_2(t, 0; \theta, \omega)) \\ &= K \int_{\mathbf{R}} \int_0^{2\pi} f(x_1(t, 0; \theta', \omega') - x_1(t, 0; \theta, \omega)) dh_1(\theta', \omega') \\ & \quad - K \int_{\mathbf{R}} \int_0^{2\pi} f(x_2(t, 0; \theta', \omega') - x_2(t, 0; \theta, \omega)) dh_2(\theta', \omega') \\ &= K \int_{\mathbf{R}} \int_0^{2\pi} f(x_1(t, 0; \theta', \omega') - x_1(t, 0; \theta, \omega)) (dh_1(\theta', \omega') - dh_2(\theta', \omega')) \\ & \quad + K \int_{\mathbf{R}} \int_0^{2\pi} (f(x_1(t, 0; \theta', \omega') - x_1(t, 0; \theta, \omega)) \\ & \quad \quad - f(x_2(t, 0; \theta', \omega') - x_2(t, 0; \theta, \omega))) dh_2(\theta', \omega'). \end{aligned} \quad (2.17)$$

Integrating it yields

$$\begin{aligned} & |x_1(t, 0; \theta, \omega) - x_2(t, 0; \theta, \omega)| \\ &\leq \int_0^t \left| K \int_{\mathbf{R}} \int_0^{2\pi} f(x_1(s, 0; \theta', \omega') - x_1(s, 0; \theta, \omega)) (dh_1(\theta', \omega') - dh_2(\theta', \omega')) \right| ds \\ & \quad + \int_0^t K \int_{\mathbf{R}} \int_0^{2\pi} \left| f(x_1(s, 0; \theta', \omega') - x_1(s, 0; \theta, \omega)) \right. \\ & \quad \quad \left. - f(x_2(s, 0; \theta', \omega') - x_2(s, 0; \theta, \omega)) \right| dh_2(\theta', \omega') ds \\ &\leq KM\delta t + KL \int_0^t \int_{\mathbf{R}} \int_0^{2\pi} (|x_1(s, 0; \theta', \omega') - x_2(s, 0; \theta', \omega')| \\ & \quad + |x_1(s, 0; \theta, \omega) - x_2(s, 0; \theta, \omega)|) dh_2(\theta', \omega') ds, \end{aligned} \quad (2.18)$$



where  $L$  is the Lipschitz constant of  $f$  and  $M$  is a constant arising from Eq.(2.11). If we put

$$F(t, \theta, \omega) = |x_1(t, 0; \theta, \omega) - x_2(t, 0; \theta, \omega)|,$$

then (2.18) provides

$$\int_{\mathbf{R}} \int_0^{2\pi} F(t, \theta, \omega) dh_2 \leq KM\delta t + 2KL \int_0^t \int_{\mathbf{R}} \int_0^{2\pi} F(s, \theta, \omega) dh_2 ds.$$

Now the Gronwall inequality proves

$$\int_{\mathbf{R}} \int_0^{2\pi} F(t, \theta, \omega) dh_2 \leq \frac{M\delta}{2L} (e^{2KLt} - 1).$$

Substituting it into (2.18) yields

$$\begin{aligned} & |x_1(t, 0; \theta, \omega) - x_2(t, 0; \theta, \omega)| \\ & \leq KM\delta t + \frac{KM\delta}{2} \int_0^t (e^{2KLS} - 1) ds + KL \int_0^t |x_1(s, 0; \theta, \omega) - x_2(s, 0; \theta, \omega)| ds \\ & \leq \frac{KM\delta t}{2} + \frac{M\delta}{4L} (e^{2KLt} - 1) + KL \int_0^t |x_1(s, 0; \theta, \omega) - x_2(s, 0; \theta, \omega)| ds. \end{aligned}$$

The Gronwall's inequality is applied again to obtain

$$|x_1(t, 0; \theta, \omega) - x_2(t, 0; \theta, \omega)| \leq \frac{M\delta}{2L} (e^{2KLt} - 1). \quad (2.19)$$

Finally, the left hand side of Eq.(2.12) is estimated as

$$\begin{aligned} & \left| \int_{\mathbf{R}} \int_0^{2\pi} a(\theta, \omega) (d\rho_{t,1} - d\rho_{t,2}) \right| \\ & = \left| \int_{\mathbf{R}} \int_0^{2\pi} (a(x_1(t, 0; \theta, \omega), \omega) dh_1 - a(x_2(t, 0; \theta, \omega), \omega) dh_2) \right| \\ & \leq \int_{\mathbf{R}} \int_0^{2\pi} |a(x_1(t, 0; \theta, \omega), \omega) - a(x_2(t, 0; \theta, \omega), \omega)| dh_2 \\ & \quad + \left| \int_{\mathbf{R}} \int_0^{2\pi} a(x_1(t, 0; \theta, \omega), \omega) (dh_1 - dh_2) \right|. \end{aligned} \quad (2.20)$$

Since  $a(\theta, \omega)$  is continuous and since Eq.(2.19) holds, the first term in the right hand side of the above is less than  $\varepsilon/2$  for  $0 \leq t \leq T$  if  $\delta$  is sufficiently small. The second term is also less than  $\varepsilon/2$  if  $\delta$  is sufficiently small because of Eq.(2.11). This proves Eq.(2.12). It is easy to see by Eq.(2.20) that if  $a(\theta, \omega)$  is Lipschitz continuous, then  $\varepsilon$  is of order  $O(\delta)$ .

■

### 3 Moments system

In this section, we introduce a moments system to transform the finite-dimensional Kuramoto model (1.1) and its continuous model (2.3) into the same system. We prove by using the moments system that the order parameter (1.2) for the Kuramoto model converges to the order parameter  $Z_1^0(t)$  for the continuous model as  $N \rightarrow \infty$  under appropriate assumptions.

For a given probability measure  $h(\theta, \omega)$  on  $S^1 \times \mathbf{R}$ , suppose that absolute moments

$$M_k^n := \int_{\mathbf{R}} \int_0^{2\pi} |\omega^n e^{\sqrt{-1}k\theta}| dh \quad (3.1)$$

exist for  $k = 0, \pm 1, \dots$  and  $m = 0, 1, \dots$ . Then, the moments  $m_k^n$  are defined to be

$$m_k^n := \int_{\mathbf{R}} \int_0^{2\pi} \omega^n e^{\sqrt{-1}k\theta} dh. \quad (3.2)$$

Conversely, if there exists a unique probability measure  $h$  for a given sequence of numbers  $\{m_k^n\}_{n,k}$  such that Eq.(3.2) holds, then  $h$  is called *M-determinate*. In this case, we also say that moments  $\{m_k^n\}_{n,k}$  is M-determinate. Many conditions for which  $h$  is M-determinate have been well studied as the moment problem [2, 18, 9, 19]. For example, one of the most convenient conditions is that if  $h$  has all absolute moments  $M_k^n$  and they satisfy  $\sum_{n=1}^{\infty} (M_0^n + 1)^{-1/n} = \infty$  (*Carleman's condition*), then  $h$  is M-determinate.

**Example 3.1.** If  $h$  has compact support, then  $h$  is M-determinate. Suppose that  $h$  has a probability density function of the form  $\hat{h}(\theta)\hat{g}(\omega)$ . If  $\hat{g}(\omega)$  is the Gaussian distribution, then  $h$  is M-determinate. If  $\hat{g}(\omega) = 1/(\pi(1 + \omega^2))$  is the Lorentzian distribution,  $h$  is *not* M-determinate because  $h$  does not have all moments.

In what follows, we suppose that an initial measure  $h(\theta, \omega)$  for the initial value problem (2.3) has all absolute moments and is M-determinate. Recall that a probability measure  $g$  for the natural frequency  $\omega$  is defined through Eq.(2.4). Since  $h$  has absolute moments

$$M_k^n = \int_{\mathbf{R}} \int_0^{2\pi} |\omega^n e^{\sqrt{-1}k\theta}| dh = \int_{\mathbf{R}} |\omega|^n dg < \infty, \quad (3.3)$$

$g$  also has all moments  $\mu_n := \int_{\mathbf{R}} \omega^n dg$ ,  $n = 0, 1, 2, \dots$ . Consider the Lebesgue space  $L^2(\mathbf{R}, dg)$ . Since all moments  $\mu_m$  of  $g$  exist, we can construct a complete orthonormal system  $\{P_m(\omega)\}_{m=0}^{\infty}$  on  $L^2(\mathbf{R}, dg)$ , by using the Gram-Schmidt orthogonalization from  $\{\omega^m\}_{m=0}^{\infty}$ , such that

$$(P_n, P_m) = \int_{\mathbf{R}} P_n(\omega)P_m(\omega)dg = \begin{cases} 1 & (n = m) \\ 0 & (n \neq m), \end{cases} \quad (3.4)$$

where  $(\cdot, \cdot)$  denotes the inner product on  $L^2(\mathbf{R}, dg)$  and  $P_n(\omega)$  is a polynomial of degree  $n$ . In particular,  $P_0(\omega) \equiv 1$ . It is well known that  $P_n(\omega)$  satisfies the relation

$$\omega P_n(\omega) = b_n P_{n+1}(\omega) + a_n P_n(\omega) + b_{n-1} P_{n-1}(\omega) \quad (3.5)$$

for  $n = 0, 1, 2, \dots$ , where  $a_n$  and  $b_n$  are real constants determined by  $g$ . The matrix  $\mathcal{M}$  defined as

$$\mathcal{M} = \begin{pmatrix} a_0 & b_0 & & & \\ b_0 & a_1 & b_1 & & \\ & b_1 & a_2 & b_2 & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \quad (3.6)$$

is called the *Jacobi matrix* for  $g$ . Eq.(3.5) shows that the Jacobi matrix gives the  $l^2(\mathbf{Z}_{\geq 0})$  representation of the multiplication operator

$$\mathcal{M} : p(\omega) \mapsto \omega p(\omega) \quad (3.7)$$

on  $L^2(\mathbf{R}, dg)$ , where  $l^2(\mathbf{Z}_{\geq 0}) = \{\{x_n\}_{n=0}^{\infty} \mid \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$ .

If an initial measure  $h$  is  $\mathcal{M}$ -determinate, so is a solution  $\rho_t$  of the continuous model (2.3) because of Eq.(2.10). Let us define the  $(m, k)$ -th moments  $Z_k^m$  for  $\rho_t$  to be

$$Z_k^m(t) = \int_{\mathbf{R}} \int_0^{2\pi} P_m(\omega) e^{\sqrt{-1}k\theta} d\rho_t, \quad (3.8)$$

for  $m = 0, 1, 2, \dots$  and  $k = 0, \pm 1, \pm 2, \dots$ . In particular  $Z_1^0(t)$  is the order parameter given in Eq.(2.3), and  $Z_{-k}^m(t) = \overline{Z_k^m(t)}$ . Note that

$$Z_0^m(t) = \begin{cases} 1 & (m = 0) \\ 0 & (m \neq 0) \end{cases} \quad (3.9)$$

are constants. It is easy to verify that

$$|Z_k^m(t)| \leq 1 \quad (3.10)$$

by using the Schwarz inequality. By using Eq.(2.7), an evolution equation for  $Z_k^m(t)$  is

derived as

$$\begin{aligned}
\frac{dZ_k^m}{dt} &= \frac{\partial}{\partial t} \int_{\mathbf{R}} \int_0^{2\pi} P_m(\omega) e^{\sqrt{-1}kx(t,0;\theta,\omega)} dh \\
&= \int_{\mathbf{R}} \int_0^{2\pi} P_m(\omega) \sqrt{-1}k \frac{\partial x}{\partial t}(t,0;\theta,\omega) e^{\sqrt{-1}kx(t,0;\theta,\omega)} dh \\
&= \int_{\mathbf{R}} \int_0^{2\pi} P_m(\omega) \sqrt{-1}k \left( \omega + \frac{K}{2\sqrt{-1}} (Z_1^0(t) e^{-\sqrt{-1}x(t,0;\theta,\omega)} - Z_{-1}^0(t) e^{\sqrt{-1}x(t,0;\theta,\omega)}) \right) e^{\sqrt{-1}kx(t,0;\theta,\omega)} dh \\
&= \sqrt{-1}k \int_{\mathbf{R}} \int_0^{2\pi} \omega P_m(\omega) e^{\sqrt{-1}kx(t,0;\theta,\omega)} dh \\
&\quad + \frac{kK}{2} \int_{\mathbf{R}} \int_0^{2\pi} P_m(\omega) (Z_1^0(t) e^{\sqrt{-1}(k-1)x(t,0;\theta,\omega)} - Z_{-1}^0(t) e^{\sqrt{-1}(k+1)x(t,0;\theta,\omega)}) dh \\
&= \sqrt{-1}k \int_{\mathbf{R}} \int_0^{2\pi} (b_m P_{m+1}(\omega) + a_m P_m(\omega) + b_{m-1} P_{m-1}(\omega)) e^{\sqrt{-1}k\theta} d\rho_t \\
&\quad + \frac{kK}{2} \int_{\mathbf{R}} \int_0^{2\pi} P_m(\omega) (Z_1^0(t) e^{\sqrt{-1}(k-1)\theta} - Z_{-1}^0(t) e^{\sqrt{-1}(k+1)\theta}) d\rho_t \\
&= \sqrt{-1}k (b_m Z_k^{m+1} + a_m Z_k^m + b_{m-1} Z_k^{m-1}) + \frac{kK}{2} (Z_1^0 Z_{k-1}^m - Z_{-1}^0 Z_{k+1}^m). \tag{3.11}
\end{aligned}$$

Put  $Z_k = (Z_k^0, Z_k^1, Z_k^2, \dots)^T$ , where  $T$  denotes the transpose. Define the Jacobi matrix  $\mathcal{M}$  and the projection matrix  $\mathcal{P}$  to be Eq.(3.6) and

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}, \tag{3.12}$$

respectively. Then, Eq.(3.11) is rewritten as

$$\frac{d}{dt} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sqrt{-1}\mathcal{M} + \frac{K}{2}\mathcal{P} & & & \\ & 2\sqrt{-1}\mathcal{M} & & \\ & & 3\sqrt{-1}\mathcal{M} & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \end{pmatrix} + \frac{K}{2} \begin{pmatrix} -Z_{-1}^0 Z_2 \\ 2(Z_1^0 Z_1 - Z_{-1}^0 Z_3) \\ 3(Z_1^0 Z_2 - Z_{-1}^0 Z_4) \\ \vdots \end{pmatrix}. \tag{3.13}$$

Note that equations for  $Z_{-1}, Z_{-2}, \dots$  are omitted because  $Z_{-k} = \bar{Z}_k$ . The first term is a linear term and the second is a nonlinear term. We call Eq.(3.11) or Eq.(3.13) the *moments system*. The dynamics of the system is investigated in [4].

Let  $M_D$  be the set of M-determinate sequences  $\{Z_k^m\}_{m,k}$  in the sense that if  $\{Z_k^m\}_{m,k} \in M_D$ , then there exists a unique measure  $h$  on  $S^1 \times \mathbf{R}$  such that  $Z_k^m = \int_{\mathbf{R}} \int_0^{2\pi} P_m(\omega) e^{\sqrt{-1}k\theta} dh$ . Since there is a one-to-one correspondence between elements of  $M_D$  and M-determinate measures, Thm.2.1 is restated as follows.

**Theorem 3.2.** (i) There exists a unique solution  $\{Z_k^m(t)\}_{m,k} \in M_D$  of the moments system if an initial condition  $\{Z_k^m(0)\}_{m,k}$  is in  $M_D$ .

(ii) Let  $\{Z_k^m(t)\}$  and  $\{\tilde{Z}_k^m(t)\}$  be solutions of the moments system with initial conditions  $\{Z_k^m(0)\}, \{\tilde{Z}_k^m(0)\} \in M_D$ , respectively. For any positive numbers  $T$  and  $\varepsilon$ , there exist positive numbers  $C_{m,k}$  and  $\delta = \delta(T, \varepsilon)$  such that if

$$|Z_k^m(0) - \tilde{Z}_k^m(0)| < C_{m,k}\delta, \quad (3.14)$$

for any  $m, k$ , then the inequality

$$|Z_k^m(t) - \tilde{Z}_k^m(t)| < \varepsilon \quad (3.15)$$

holds for  $0 \leq t \leq T$ . In particular  $\varepsilon \sim O(\delta)$  as  $\delta \rightarrow 0$ .

For the  $N$ -dimensional Kuramoto model (1.1), we define the  $(m, k)$ -th moments to be

$$\hat{Z}_k^m(t) := \frac{1}{N} \sum_{j=1}^N P_m(\omega_j) e^{\sqrt{-1}k\theta_j(t)}, \quad (3.16)$$

for  $m = 0, 1, 2, \dots$  and  $k = 0, \pm 1, \pm 2, \dots$ . In particular  $\hat{Z}_1^0$  is the order parameter defined in Eq.(2.1). By using Eq.(1.1), it is easy to verify that  $\hat{Z}_k^m(t)$ 's satisfy a system of differential equations

$$\frac{d\hat{Z}_k^m(t)}{dt} = \sqrt{-1}k \left( b_m \hat{Z}_k^{m+1} + a_m \hat{Z}_k^m + b_{m-1} \hat{Z}_k^{m-1} \right) + \frac{kK}{2} (\hat{Z}_1^0 \hat{Z}_{k-1}^m - \hat{Z}_{-1}^0 \hat{Z}_{k+1}^m). \quad (3.17)$$

It is remarkable that Eq.(3.17) has the same form as Eq.(3.11). This means that all solutions of the Kuramoto model for any  $N$  are embedded in the phase space of the moments system (3.11). This fact allows us to prove Theorem 3.3 below. Originally the moments system for the Kuramoto model was introduced by Perez and Ritort [16], although their definition of the moments is  $H_k^m := 1/N \cdot \sum_{j=1}^N \omega_j^m e^{\sqrt{-1}k\theta_j(t)}$ . Since we adopt orthogonal polynomials  $\{P_m(\omega)\}_{m=0}^\infty$  to define moments (3.16), our moments system is more suitable for mathematical analysis.

Now we are in a position to show the main theorem in this paper, which states that differences between moments  $Z_k^m(t)$  and  $\hat{Z}_k^m(t)$  are of  $O(1/\sqrt{N})$  and thus the continuous model (2.3) is proper to investigate the Kuramoto model (1.1) for large  $N$ .

**Theorem 3.3.** Let  $\rho_t$  be a solution of the continuous model (2.3) such that an initial measure  $h(\theta, \omega)$  is M-determinate. Suppose that for the  $N$ -dimensional Kuramoto model (1.1), pairs  $(\theta_j(0), \omega_j)$  of initial values  $\theta_j(0)$ ,  $j = 1, \dots, N$  and natural frequencies  $\omega_j$ ,  $j = 1, \dots, N$  are independent and identically distributed according to the probability measure  $h(\theta, \omega)$ . Then, moments  $Z_k^m(t)$  and  $\hat{Z}_k^m(t)$  defined by Eqs.(3.8) and (3.16) satisfy

$$|Z_k^m(t) - \hat{Z}_k^m(t)| \rightarrow 0, \quad a.s. \ (N \rightarrow \infty), \quad (3.18)$$

for any  $m, k$  and  $t$  (*a.s.* denotes “almost surely”). Further, for any positive number  $\delta$ , there exists a number  $C = C(m, k, t, \delta) > 0$  such that

$$P(|Z_k^m(t) - \hat{Z}_k^m(t)| < C/\sqrt{N}) \rightarrow 1 - \delta, \quad (N \rightarrow \infty), \quad (3.19)$$

where  $P(A)$  is the probability that an event  $A$  will occur.

**Proof.** Since  $\omega_j$ 's and  $\theta_j(0)$ 's are independent and identically distributed, the average of  $\hat{Z}_k^m(0)$  is calculated as

$$\begin{aligned} E[\hat{Z}_k^m(0)] &= E\left[\frac{1}{N} \sum_{j=1}^N P_m(\omega_j) e^{\sqrt{-1}k\theta_j(0)}\right] \\ &= E[P_m(\omega_j) e^{\sqrt{-1}k\theta_j(0)}] \\ &= \int_{\mathbf{R}} \int_0^{2\pi} P_m(\omega) e^{\sqrt{-1}k\theta} dh = Z_k^m(0). \end{aligned} \quad (3.20)$$

Thus Eqs.(3.18) and (3.19) for  $t = 0$  immediately follow from the strong law of large number and the central limit theorem, respectively. Note that the strong law of large number and the central limit theorem are no longer applicable for  $t > 0$  because  $\theta_j(t)$ 's are not independent and identically distributed when  $t$  is positive. However, since  $Z_k^m(t)$  and  $\hat{Z}_k^m(t)$  satisfy the same moments system, and since solutions of the moments system are continuous with respect to initial values (Thm.3.2 (ii)), Eqs.(3.18),(3.19) hold for each positive  $t$  if they are true for  $t = 0$ ; if initial states satisfy  $|Z_k^m(0) - \hat{Z}_k^m(0)| \rightarrow 0$  as  $N \rightarrow \infty$ , then  $|Z_k^m(t) - \hat{Z}_k^m(t)| \rightarrow 0$  holds for any  $t > 0$ , and if they satisfy  $|Z_k^m(0) - \hat{Z}_k^m(0)| < C_0/\sqrt{N}$  for a positive constant  $C_0$ , then  $|Z_k^m(t) - \hat{Z}_k^m(t)| < C_t/\sqrt{N}$  holds for some  $C_t > 0$ . ■

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