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Notes on Estimating Inverse-Gaussian and Gamma Subordinators under High-frequency Sampling *

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Abstract

We study joint efficient estimation of two parameters dominating gamma and inverse-Gaussian subordinators, based on discrete observations sampled at $(t_i^n)_{i=1}^n$ satisfying $h_n := \max_{i\leq n}(t_i^n - t_{i-1}^n) \to 0$ as $n \to \infty$. Under the condition that $T_n := t_n^n \to \infty$ as $n \to \infty$ we have two kinds of optimal rates, $\sqrt{n}$ and $\sqrt{T_n}$, and especially. Moreover, as in estimation of diffusion coefficient of a Wiener process the $\sqrt{n}$-consistent component of the estimator is effectively workable even when T_n does not tend to infinity. Simulation experiments are given under several h_n's behaviors.

1 Introduction

In this article we shall present two case studies of estimating a subordinator based on a kind of high-frequency discrete data. A subordinator $Z = (Z_t)_{t\in\mathbb{R}_+}$ is a one-dimensional non-decreasing càdlàg (right continuous and having left hand side limits) process a.s. starting from the origin with independent and stationary increments. For any subordinator without drift, there corresponds a Lévy measure ν satisfying $\int_0^1 |z|\nu(dz) < \infty$ and supported by $\mathbb{R}_+$ for which

$$\varphi_{Z_t}(u) = \exp\left\{t\int(e^{iuz}-1)\nu(dz)\right\}, \quad u \in \mathbb{R},\ t \in \mathbb{R}_+. \tag{1}$$

This is a special case of the so called Lévy-Khintchine formula. Here and in the sequel $u \mapsto \varphi_\xi(u)$ stands for the characteristic function of ξ, a random variable or a distribution. Given a subordinator Z the law at time 1, $\mathcal{L}(Z_1)$, is uniquely associated with an infinitely divisible distribution whose support is contained in $\mathbb{R}_+ = [0,\infty)$. See, e.g., Bertoin [3, Chapter III] for a systematic account of subordinators.

We shall consider statistical inference for two specific subordinators, $\mathcal{L}(Z_1) = IG(\delta,\gamma)$ and $\Gamma(\delta,\gamma)$, admitting the density (w.r.t. the Lebesgue measure)

$$p(x;\delta,\gamma) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}x^{-3/2}\exp\left\{-\frac{1}{2}\left(\gamma^2 x + \frac{\delta^2}{x}\right)\right\}\mathbf{1}_{\mathbb{R}_+}(x), \tag{2}$$

$$p(x;\delta,\gamma) = \frac{\gamma^\delta}{\Gamma(\delta)}x^{\delta-1}\exp(-\gamma x)\mathbf{1}_{\mathbb{R}_+}(x), \tag{3}$$

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respectively, where δ and γ are positive constants; both of the inverse-Gaussian and gamma subordinators have infinitely many jumps over each finite time interval as the Lévy measure ν in the formula (1) is given by $g(z;\delta,\gamma)=\delta(2\pi)^{-1/2}z^{-3/2}\exp(-\gamma^2 z/2)\mathbf{1}_{\mathbb{R}_+}(z)$ and $g(z;\delta,\gamma)=\delta z^{-1}\exp(-\gamma z)\mathbf{1}_{\mathbb{R}_+}(z)$, respectively.

We are interested in estimating $\theta=(\delta,\gamma)$ when available data is

$$Z_{t_0^n}, Z_{t_1^n}, \dots, Z_{t_n^n},$$

where $(t_i^n)_{i=0}^n$ is a nonrandom positive sequence satisfying

$$0 \equiv t_0^n < t_1^n < \cdots < t_n^n =: T_n$$

for each $n\in\mathbb{N}$. Throughout this article we suppose

$$\begin{cases} h_n := \max_{1\le i\le n}(t_i^n - t_{i-1}^n) \to 0, \\ T_n \asymp n h_n, \end{cases} \tag{4}$$

as $n\to\infty$, where $a_n \asymp b_n$ means that there exists a constant $c>0$ such that $c^{-1}\le a_n/b_n \le c$ for every n large enough. For joint estimation of δ and γ we shall additionally suppose $T_n\to\infty$; then the sampling scheme is asymptotically the same as in the case where Z is continuously observed over $[0,T_n]$ with $T_n\to\infty$. If we can observe continuous data $(Z_t)_{t\in[0,T]}$, the likelihood theory has been already established: see, e.g., Akritas [1] and Akritas and Johnson [2]. Denote by P_θ^T the law of a sample path $(X_t)_{t\in[0,T]}$ on the Skorohod space (i.e., the space of càdlàg processes endowed with the Skorohod topology), and fix any $\theta^i=(\delta^i,\gamma^i)$, $i=1,2$, and $T>0$. Then, in both of the inverse-Gaussian and gamma cases, $P_{\theta^1}^T$ and $P_{\theta^2}^T$ fail to be mutually absolutely continuous as soon as $\delta^1\neq\delta^2$ (see, e.g., Akritas and Johnson [2, Theorem 4.1]), so that we *cannot* consider the joint likelihood estimation of δ and γ from a continuous record while it makes sense in discrete-observation cases. If $t_i^n - t_{i-1}^n \equiv h > 0$ in particular, then the situation is nothing but the classical iid framework, and the convergence rate of the MLE is of course $\sqrt{n}I_2$ for both component, and we can explicitly write down the corresponding Fisher information matrices (depending on h in this case): Woerner [9] systematically studied such cases for much more general classes of Lévy processes.

Our main goal is to derive uniform asymptotic behaviors of the corresponding maximum-likelihood estimators (MLE), say $\hat{\theta}_n=(\hat{\delta}_n,\hat{\gamma}_n)$. In both cases we presuppose:

> that the parameter space $\Theta\subset(0,\infty)^2$ is a bounded domain whose closure is contained in $(0,\infty)^2$ and that there is a true parameter which lies in Θ.

Denote by P_θ^n the image measure of $(Z_{t_i^n})_{i=0}^n$ associated with θ. We shall derive the local asymptotic normality (LAN) as well as the asymptotic normality of the MLEs with rate $\mathrm{diag}(\sqrt{n},\sqrt{T_n})$, both uniform in Θ. Here the asymptotic normality with rate $\mathrm{diag}(\sqrt{n},\sqrt{T_n})$ means that the convergence in law

$$\left(\sqrt{n}(\hat{\delta}_n-\delta), \sqrt{T_n}(\hat{\gamma}_n-\gamma)\right) \Rightarrow \mathcal{N}_2(0, I(\theta)^{-1})$$

under (P_θ^n)-sequence of distributions, see (9) and (13) for specified expressions of $I(\theta)$: this is a similar phenomenon to the well known case where Z is a Wiener process such that $\mathcal{L}(Z_1)=\mathcal{N}_1(\gamma,\delta)$, or, more generally, a diffusion process. Under our sampling scheme (4), we shall see that a remarkably different feature from the case of $t_i^n - t_{i-1}^n \equiv h > 0$ will arise. Precisely, the forms of the asymptotic Fisher information matrices are different, and especially they are diagonal, which implies that the joint ML estimation of δ and γ is asymptotically mutually independent. See the expressions (9) and (13) below. We shall also derive an efficient estimator of δ even when (T_n) is bounded in n; in this case γ may be unknown, hence a nuisance parameter.

Recall that, for general Lévy processes the likelihood function can be written down only up to the Fourier inversion formula. Jongbloed and van der Meulen [5] studied the parametric estimation of a subordinator based on the empirical characteristic function, where that $t_i^n - t_{i-1}^n \equiv h > 0$ is supposed and the efficiency issue is not discussed. Restricting the model structure, we shall give sharper results than theirs. As a matter of fact, specification of the parametric optimal rates in estimating a general Lévy process seems to be an intricate problem. For example, the author [6] previously studied the LAN property for discretely observed (non-Gaussian) stable Lévy processes, where various optimal rates were found for each component: there the scaling property, which is inherent in the stable case among general Lévy processes, was fully utilized, and it has been shown that the Fisher information matrix is always degenerate as long as joint estimation of scale and index parameters are concerned. Such a phenomenon does not arise in the present context.

In the rest of this article we shall present our asymptotic results in Section 2, then some simulation results in Section 3, and finally the proofs in Section 4.

2 Results

We use asymptotic symbols for $n \to \infty$ unless otherwise stated. Write $\Delta_i^n = t_i^n - t_{i-1}^n$, $\Delta_i^n Z = Z_{t_i^n} - Z_{t_{i-1}^n}$, $\theta := (\delta, \gamma)$, and $\partial_\theta = \partial/\partial\theta$. Note that the sequence $(\Delta_i^n Z)_{i=1}^n$ forms a rowwise independent triangular array fulfilling

$$\mathcal{L}(\Delta_i^n Z) = \mathcal{L}(Z_{\Delta_i^n t}) \tag{5}$$

for each $i \in \{1, 2, \ldots, n\}$. For any $\sigma(Z_{t_i^n} : i \le n)$-measurable random variables $X_n(\theta)$, $n \in \mathbb{N}$, and a constant $X(\theta)$ appearing in the sequel, we write:

(i) "$X_n(\theta) \Rightarrow_u^{P_\theta^n} X(\theta)$" if

$$\sup_{\theta \in \Theta^-} |P^{X_n(\theta)} f - P^{X(\theta)} f| \to 0$$

for every bounded continuous function f, where P^ξ denotes the law of ξ;

(ii) "$X_n(\theta) \to_u^{P_\theta^n} X(\theta)$" if for every $\epsilon > 0$ we have

$$\sup_{\theta \in \Theta^-} P_\theta^n \big[|X_n(\theta) - X(\theta)| > \epsilon \big] \to 0.$$

Given a log-likelihood function $\theta \mapsto \ell_n(\theta)$, we write

$$\mathcal{S}_n(\theta) = \partial_\theta \ell_n(\theta) \quad \text{and} \quad \mathcal{I}_n(\theta) = -\partial_\theta^2 \ell_n(\theta),$$

the score function and the observed information matrix, respectively.

With the above-mentioned notation, we formulate the uniform LAN property in our context as follows. Write

$$A_n = \mathrm{diag}(\sqrt{n}, \sqrt{T_n}), \tag{6}$$

so that $A_n^{-1} \to 0$. The experiment $\{P_\theta^n : n \in \mathbb{N}\}_{\theta \in \Theta}$ is called "uniformly A_n-LAN with the Fisher informations $\{I(\theta)\}_{\theta \in \Theta}$" if:

[U1] $\ell_n(\theta + A_n^{-1\top} u_n) - \ell_n(\theta) - u_n^\top \mathcal{S}_n(\theta) + \frac{1}{2} u_n^\top \mathcal{I}_n(\theta) u_n \to_u^{P_\theta^n} 0$ for any nonrandom bounded sequence $(u_n) \subset \mathbb{R}^2$ such that $u_n \to u$;

[U2] $A_n^{-1} \mathcal{S}_n(\theta) \Rightarrow_u^{P_\theta^n} \mathcal{N}_2(0, I(\theta))$ with $I(\theta)$ being positive definite for any $\theta \in \Theta^-$;

[U3] $A_n^{-1} \mathcal{I}_n(\theta) A_n^{-1\top} \to_u^{P_\theta^n} I(\theta)$, with the same $I(\theta)$ as in [U2].

The forthcoming asymptotic normalities reveal that the MLEs are asymptotically efficient in both of inverse-Gaussian and gamma cases. If $\hat{\theta}_n \in \Theta^-$ is not well-defined, we may assign any number $\theta \in \Theta$ to $\hat{\theta}_n$.

2.1 Inverse-Gaussian case

Let $\mathcal{L}(Z_1) = IG(\delta, \gamma)$ whose density is given by (2). Since

$$\varphi_{IG(\delta,\gamma)}(u) = \exp\{\delta(\gamma - \sqrt{\gamma^2 - 2iu})\},$$

we have $\mathcal{L}(Z_t) = IG(\delta t, \gamma)$ for each $t > 0$. On account of (5), the target log-likelihood function of $(Z_{t_i^n})_{i=0}^n$ is given by

$$\ell_n(\theta) = \sum_{i=1}^n \left\{ \log \delta + \delta\gamma\Delta_i^n t - \frac{1}{2}\left(\frac{\delta^2(\Delta_i^n t)^2}{\Delta_i^n Z} + \gamma^2 \Delta_i^n Z \right) \right\}. \tag{7}$$

Solving $\partial_\theta \ell_n(\theta) = 0$, we get the explicit MLE:

$$\hat{\delta}_n = \left[\frac{1}{n} \left\{ \sum_{i=1}^n \frac{(\Delta_i^n t)^2}{\Delta_i^n Z} - \frac{T_n^2}{Z_{T_n}} \right\} \right]^{-1/2}, \quad \hat{\gamma}_n = \frac{T_n \hat{\delta}_n}{Z_{T_n}}. \tag{8}$$

For the joint estimation we have the following.

Theorem 2.1 (Unbounded-domain asymptotics)**.** *Let Z be a subordinator such that $\mathcal{L}(Z_1) = IG(\delta, \gamma)$, let $\ell_n(\theta)$ and $\hat{\theta}_n = (\hat{\delta}_n, \hat{\gamma}_n)$ be as in (7) and (8), respectively, and suppose (4) and $T_n \to \infty$. Then $\{P_\theta^n : n \in \mathbb{N}\}_{\theta\in\Theta}$ is uniformly A_n-LAN with the Fisher informations*

$$I_{IG}(\theta) = \begin{pmatrix} 2/\delta^2 & 0 \\ 0 & \delta/\gamma \end{pmatrix}, \quad \theta \in \Theta, \tag{9}$$

and we have $A_n(\hat{\theta}_n - \theta) \Rightarrow_u^{P_\theta^n} \mathcal{N}_2(0, I_{IG}(\theta)^{-1})$.

If T_n does not tends to infinity, then the observed information associated with γ is bounded in n, and this is the case also for the gamma Lévy process; see (16) and (17). Therefore no consistent estimation procedure of γ is possible. But this is not the case for estimating δ, and actually we may use the same estimate as in (8).

Corollary 2.2 (Bounded-domain asymptotics)**.** *Let Z be a subordinator such that $\mathcal{L}(Z_1) = IG(\delta, \gamma)$, where $\gamma > 0$ is fixed while it may be unknown, let $\hat{\delta}_n$ be given by (8), and suppose (4) and $T_n = O(1)$. Moreover suppose that the true value of δ lies in some interval (a, b) such that $0 < a < b < \infty$. Then $\{P_\delta^n : n \in \mathbb{N}\}_{\delta\in(a,b)}$ is uniformly $\sqrt{n}$-LAN with the Fisher informations $2/\delta^2$, and $\hat{\delta}_n$ fulfils $\sqrt{n}(\hat{\delta}_n - \delta) \Rightarrow_u^{P_\delta^n} \mathcal{N}_1(0, \delta^2/2)$.*

2.2 Gamma case

Next we set $\mathcal{L}(Z_1) = \Gamma(\delta, \gamma)$ whose density is given by (3), so that

$$\varphi_{\Gamma(\delta,\gamma)}(u) = (1 - iu/\gamma)^{-\delta}$$

and hence $\mathcal{L}(Z_t) = \Gamma(\delta t, \gamma)$ for each $t > 0$. Thus the log-likelihood function of $(Z_{t_i^n})_{i=0}^n$ is given by

$$\ell_n(\theta) = \sum_{i=1}^n \left\{ \delta\Delta_i^n t \log\gamma - \log\Gamma(\delta\Delta_i^n t) + \delta\Delta_i^n t \log(\Delta_i^n Z) - \gamma\Delta_i^n Z \right\}. \tag{10}$$

The corresponding MLE solves

$$\sum_{i=1}^n (\Delta_i^n t)\{\log(\delta\Delta_i^n t) - \psi(\delta\Delta_i^n t)\} = T_n \log\left(\frac{Z_{T_n}}{T_n}\right) - \sum_{i=1}^n (\Delta_i^n t) \log\left(\frac{\Delta_i^n Z}{\Delta_i^n t}\right), \tag{11}$$

$$\gamma = \delta \frac{T_n}{Z_{T_n}}, \tag{12}$$

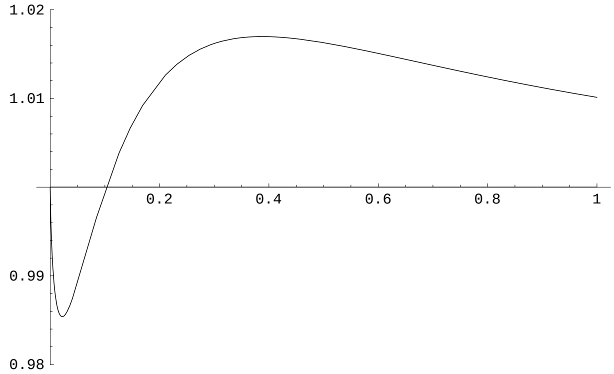

Figure 1: The plot of the bounded function $(0, \infty) \ni x \mapsto \{\log x - \psi(x)\}/\{(3x+1)/(6x^2+x)\}$, which is strictly increasing to 1 (resp. decreasing to 0) as $x \searrow 0$ (resp. as $x \nearrow \infty$).

where ψ denotes the digamma function, $\psi(x) := \partial_x \Gamma(x)/\Gamma(x)$.

For each $n \in \mathbb{N}$ the left-hand side of (12), say $f_n(\delta)$, is a smooth, positive, and strictly decreasing function of $\delta \in (0, \infty)$: $f_n(x) \searrow 0$ (resp. $\nearrow \infty$) as $x \nearrow \infty$ (resp. $x \searrow 0$). So (11) admits a unique root $\hat{\delta}_n$ a.s. on the event where the right-hand side of (11) is positive, and we can simply apply, e.g., the bisection search in order to find the root of (11) readily.

Theorem 2.3 (Unbounded-domain asymptotics). *Let Z be a subordinator such that $\mathcal{L}(Z_1) = \Gamma(\delta, \gamma)$, and let $\ell_n(\theta)$ and $\hat{\theta}_n = (\hat{\delta}_n, \hat{\gamma}_n)$ be as in (10) and the solution to (11) and (12), respectively, and suppose (4) and $T_n \to \infty$. Then $\{P_\theta^n : n \in \mathbb{N}\}_{\theta \in \Theta}$ is uniformly A_n-LAN with the Fisher informations*

$$I_\Gamma(\theta) = \begin{pmatrix} 1/\delta^2 & 0 \\ 0 & \delta/\gamma^2 \end{pmatrix}, \quad \theta \in \Theta, \tag{13}$$

and we have $A_n(\hat{\theta}_n - \theta) \Rightarrow_u^{P_\theta^n} \mathcal{N}_2(0, I_\Gamma(\theta)^{-1})$.

We also have an analogue to Corollary 2.2.

Corollary 2.4 (Bounded-domain asymptotics). *Let Z be a subordinator such that $\mathcal{L}(Z_1) = \Gamma(\delta, \gamma)$, where $\gamma > 0$ is fixed while it may be unknown, let $\hat{\delta}_n$ be a solution of (11), and suppose (4) and $T_n = O(1)$. Moreover, suppose that the true value of δ lies in (a, b) for some $0 < a < b < \infty$. Then $\{P_\delta^n : n \in \mathbb{N}\}_{\delta \in (a,b)}$ is uniformly $\sqrt{n}$-LAN with the Fisher informations $1/\delta^2$, and $\hat{\delta}_n$ fulfils $\hat{\delta}_n$ fulfils $\sqrt{n}(\hat{\delta}_n - \delta) \Rightarrow_u^{P_\theta^n} \mathcal{N}_1(0, \delta^2)$.*

Here is a remark for finding the root of (11) in the equidistant-sampling case, $h_n = t_i^n - t_{i-1}^n$ for each n: in this case (11) can be rewritten as

$$\log(\delta h_n) - \psi(\delta h_n) = \log\left(\frac{1}{n}\sum_{i=1}^n \Delta_i^n Z\right) - \frac{1}{n}\sum_{i=1}^n \log(\Delta_i^n Z).$$

Write this right-hand side as Y_n. We can show that $h_n Y_n \to_u^{P_\theta^n} \delta^{-1} > 0$ (use Genon-Catalot and Jacod [4, Lemma 9]), hence Y_n becomes positive with P_θ^n-probability tending to 1. Now, using the approximation (see Figure 1)

$$\log(x) - \psi(x) \sim \frac{3x+1}{6x^2+x}, \quad x \to 0,$$

and taking the positivity of δ into account, we get the approximate MLE $\tilde{\delta}_n$ of δ given by

$$\tilde{\delta}_n = \frac{3 - Y_n}{12 h_n Y_n} + \left\{\left(\frac{3 - Y_n}{12 h_n Y_n}\right)^2 + \frac{1}{6 h_n Y_n}\right\}^{1/2}, \tag{14}$$

n	T_n	$\hat{\delta}_n^{IG}$-mean (s.d.)	$\hat{\gamma}_n^{IG}$-mean (s.d.)	$\tilde{\delta}_n^{\Gamma}$-mean (s.d.)	$\tilde{\gamma}_n^{\Gamma}$-mean (s.d.)
50	3.23	3.0910 (0.3195)	2.1514 (0.5217)	3.0674 (0.4832)	2.2386 (0.8505)
100	3.98	3.0276 (0.2243)	2.0865 (0.4657)	3.0604 (0.3249)	2.2331 (0.7651)
300	5.54	3.0142 (0.1233)	2.0560 (0.3547)	3.0499 (0.1798)	2.1650 (0.6089)
500	6.45	3.0021 (0.0929)	2.0657 (0.3410)	3.0426 (0.1321)	2.1241 (0.5064)

Table 1: *Simulation 1.* Means and standard deviations (s.d.) of the estimate based on 1000 independent trajectories. We set $h_n = n^{-0.7}$, i.e. $T_n = n^{0.3} \to \infty$, and $(\delta, \gamma) = (3, 2)$.

which, together with $\tilde{\gamma}_n := \tilde{\delta}_n T_n / Z_{T_n}$ in cases of $T_n \to \infty$, enables us to bypass the numerical optimization procedure.

Remark 2.5. As a familiar naive estimator, one may consider moment estimator based on the first and second sample moments, utilizing the convergences

$$\frac{1}{T_n}\sum_{i=1}^n \Delta_i^n Z \to_u^{P_\theta^n} \frac{\delta}{\gamma} \quad \text{and} \quad \frac{1}{T_n}\sum_{i=1}^n (\Delta_i^n Z)^2 \to_u^{P_\theta^n} \frac{\delta}{\gamma^2};$$

again, this can be proved by means of [4, Lemma 9]. However, the asymptotic behavior of the moment estimator $\hat{\theta}_{M,n}$ obtained from the above relations are far from that of $\hat{\theta}_n$: using the delta method we can check

$$\sqrt{T_n}(\hat{\theta}_{M,n} - \theta) \Rightarrow \mathcal{N}_2\left(0, \begin{pmatrix} 2\delta & 2\gamma \\ 2\gamma & 3\gamma^2/\delta \end{pmatrix}\right).$$

This reveals that we *cannot* use $\hat{\theta}_{M,n}$ even as an initial estimate in applying Fisher's scoring or one-step improvement.

3 Simulation experiments

We here report some numerical results. For simplicity we carried out equidistant sampling cases, $\Delta_i^n t = h_n$ for each $i \leq n$. For gamma cases, we utilized the approximate MLE $\tilde{\theta}_n := (\tilde{\delta}_n, \tilde{\gamma}_n)$; the results will show that even $\tilde{\theta}_n$ performs well. In each simulation we simulated 1000 independent discrete sample paths of Z, and then computed mean and standard deviation (s.d.) of the obtained 1000 estimates. Throughout the true value is $(\delta, \gamma) = (3, 2)$. For generating pseudorandom-$\Gamma(p, q)$ numbers with $p \in (0, 1)$, we used the algorithm of Michael et al. [7].

In Tables 1 and 2, we distinguish inverse-Gaussian and gamma cases by the superscripts "IG" and "Γ".

Simulation 1. Next we set $h_n = n^{-0.7}$, so that $T_n = n^{0.3} \to \infty$ and the jointly consistent estimation of δ and γ can be done. The results are given in Table 1.

Simulation 2. Next we set $h_n = n^{-0.3}$, so that $T_n = n^{0.7} \to \infty$; the total observation-time domain diverges faster than the previous case. It is observed that: (i) accuracy of estimating δ is slightly worse than Simulation 1; and (ii) performance of estimating γ is much better than Simulation 1, because of the faster increase of T_n. The results are given in Table 2.

Finally, let us look at a case of $T_n = O(1)$ in the inverse-Gaussian case.

Simulation 3. Set $h_n = 1/n$, so that $T_n \equiv 1$. In this case only δ can be consistently estimated. The results are given in Table 3: just for reference we also give estimates $\hat{\gamma}_n$, which badly behaved and have severe inevitable bias, as was expected.

n	T_n	$\hat{\delta}_n^{IG}$-mean (s.d)	$\hat{\gamma}_n^{IG}$-mean (s.d.)	$\tilde{\delta}_n^{\Gamma}$-mean (s.d)	$\tilde{\gamma}_n^{\Gamma}$-mean (s.d.)
50	15.46	3.0812 (0.3190)	2.0630 (0.2941)	3.1331 (0.5908)	2.1335 (0.5235)
100	25.12	3.0384 (0.2152)	2.0487 (0.2248)	3.0319 (0.3988)	2.0470 (0.3557)
300	54.20	3.0058 (0.1224)	2.0095 (0.1392)	2.9866 (0.2059)	2.0030 (0.2130)
500	77.50	3.0115 (0.0933)	2.0087 (0.1138)	2.9713 (0.1534)	1.9911 (0.1741)

Table 2: *Simulation 2.* Means and standard deviations (s.d.) of the estimate based on 1000 independent trajectories. We set $h_n = n^{-0.3}$, i.e. $T_n = n^{0.7} \to \infty$, $(\delta, \gamma) = (3, 2)$.

n	$\hat{\delta}_n^{IG}$-mean (s.d)	$\hat{\gamma}_n^{IG}$-mean (s.d.)
50	3.0760 (0.3220)	2.3424 (0.9575)
100	3.0431 (0.2200)	2.4193 (0.9441)
300	3.0211 (0.1230)	2.3300 (0.9481)
500	3.0114 (0.0935)	2.3123 (0.9044)
1000	3.0011 (0.0674)	2.3603 (0.9735)

Table 3: *Simulation 3.* Means and standard deviations (s.d.) of the estimate based on 1000 independent trajectories. We set $h_n = 1/n$, i.e. $T_n \equiv 1$.

4 Proofs

4.1 Preliminary

Put $H_n(\theta) = [H_n^{kl}(\theta)]_{k,l=1}^n = A_n^{-1}\mathcal{I}_n(\theta)A_n^{-1\top}$, the normalized observed information matrix; recall (6). First, utilizing the results of Sweeting [8, Theorems 1 and 2], we shall observe that each proof of Theorems 2.1 and 2.3 reduces to the verification of [U3].

Put $\mathcal{I}_n(\theta) = [\mathcal{I}_n^{kl}(\theta)]_{k,l=1}^2$. Given $\theta^k = (\delta^k, \gamma^k)$, $k = 1, 2$, we introduce the notation

$$\mathcal{I}_n(\theta^1, \theta^2) := [\mathcal{I}_n^{kl}(\theta^k)]_{k,l=1}^2,$$

and, for a constant $a > 0$,

$$F_n^a(\theta) = \sup_{\theta^k : |A_n(\theta^k - \theta)| \le a, k=1,2} |A_n^{-1}\{\mathcal{I}_n(\theta^1, \theta^2) - \mathcal{I}_n(\theta)\}A_n^{-1}| :$$

these quantities appear in the assumptions of [8] in relation to deriving asymptotic results through Taylor's formula. Now suppose

$$H_n(\theta) \to_u^{P_\theta^n} H(\theta) \tag{15}$$

for some $H(\theta)$, which is positive definite for every $\theta \in \Theta^-$ and continuous in $\theta \in \Theta$. Then we can easily see as follows that the condition C2(ii) of [8] is fulfilled. Write $A_n = \mathrm{diag}(A_{1n}, A_{2n})$ and observe that for each $\theta^k \in \Theta$ we have

$$\begin{aligned}
&|A_n^{-1}\{\mathcal{I}_n(\theta^1, \theta^2) - \mathcal{I}_n(\theta)\}A_n^{-1}| \\
&\quad \lesssim \sum_{k,l=1}^2 |A_{kn}^{-1}A_{ln}^{-1}\{\mathcal{I}_n^{kl}(\theta^k) - \mathcal{I}_n^{kl}(\theta)\}| \\
&\quad \le \sum_{k,l=1}^2 \left\{|H_n^{kl}(\theta^k) - H^{kl}(\theta^k)| + |H_n^{kl}(\theta) - H^{kl}(\theta)\}| + |H^{kl}(\theta^k) - H^{kl}(\theta)|\right\},
\end{aligned}$$

and therefore, on account of (15), we get $F_n^a(\theta) \to_u^{P_\theta^n} 0$ for every $a > 0$ as desired. In particular, this leads to [U1]. Since the conditions C1 and C2(i) of [8] are automatic under (15), Theorems 1 and 2 of the paper can apply under (15), so that we get [U2], the existence of a local root of $\partial_\theta \ell_n(\theta)=0$, and the uniform asymptotic normality $\{A_n^{-1}\mathcal{I}_n(\theta)A_n^{-1\top}\}^{1/2}A_n(\hat{\theta}_n-\theta) \Rightarrow_u^{P_\theta^n} \mathcal{N}_2(0,I_2)$, where I_2 denotes the 2-dimensional identity matrix.

Thus, in order to prove Theorem 2.1 (resp. Theorem 2.3), it suffices to show (15), namely $H_n(\theta) \to_u^{P_\theta^n} I_{IG}(\theta)$ (resp. $H_n(\theta) \to_u^{P_\theta^n} I_\Gamma(\theta)$).

4.2 Inverse-Gaussian case

Proof of Theorem 2.1. Direct computations yield

$$H_n(\theta) = \begin{pmatrix} \dfrac{1}{\delta^2} + \dfrac{1}{n}\displaystyle\sum_{i=1}^n \frac{(\Delta_i^n t)^2}{\Delta_i^n Z} & -\sqrt{\dfrac{T_n}{n}} \\ \text{sym.} & \dfrac{1}{T_n}\displaystyle\sum_{i=1}^n \Delta_i^n Z \end{pmatrix}. \tag{16}$$

The task here is to show $H_n(\theta) \to_u^{P_\theta^n} I_{IG}(\theta)$ (recall (9)), but in view of (4) it is clear that $H_n^{12}(\theta)=H_n^{21}(\theta) \to_u 0$ (the ordinary convergence). As for the diagonal elements, we shall first utilize [4, Lemma 9] to deduce the θ-pointwise convergence. Fix any $\theta\in\Theta$ and observe that

$$\begin{aligned} E_\theta^n[\Delta_i^n Z] &= \delta\Delta_i^n t/\gamma, \\ E_\theta^n[(\Delta_i^n Z)^2] &= \delta\Delta_i^n t/\gamma^3 + (\delta\Delta_i^n t/\gamma)^2, \\ E_\theta^n[(\Delta_i^n Z)^{-1}] &= 1/\{\delta\Delta_i^n t\}^2 + \gamma/(\delta\Delta_i^n t), \\ E_\theta^n[(\Delta_i^n Z)^{-2}] &= 1/\{\delta\Delta_i^n t\}^4 + 3\gamma/\{\delta\Delta_i^n t\}^3 + (2+\gamma^2)/\{\delta\Delta_i^n t\}^2. \end{aligned}$$

Now consider $H_n^{11}(\theta)$. Under (4) we have

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^n (\Delta_i^n t)^2 E_\theta^n[(\Delta_i^n Z)^{-1}] &= \frac{1}{\delta^2} + \frac{\gamma}{\delta}\frac{1}{n}\sum_{i=1}^n (\Delta_i^n t)^2 = \frac{1}{\delta^2} + o(1), \\ \frac{1}{n^2}\sum_{i=1}^n (\Delta_i^n t)^4 E_\theta^n[(\Delta_i^n Z)^{-2}] &= \frac{1}{n^2}\sum_{i=1}^n O(1) = o(1). \end{aligned}$$

Thus [4, Lemma 9] yields $H_n^{11}(\theta) \to^{P_\theta^n} 2/\delta^2$. Similarly, as for $H_n^{22}(\theta)$ we see

$$\begin{aligned} \frac{1}{T_n}\sum_{i=1}^n E_\theta^n[\Delta_i^n Z] &= \delta/\gamma, \\ \frac{1}{T_n^2}\sum_{i=1}^n E_\theta^n[(\Delta_i^n Z)^2] &= \frac{1}{T_n^2}\sum_{i=1}^n \{O(1)(\Delta_i^n t)^2 + O(1)\Delta_i^n t\} = o(1), \end{aligned}$$

so that $H_n^{22}(\theta) \to^{P_\theta^n} \delta/\gamma$. The uniformity of the convergence of $H_n(\theta)$ directly follows on account of the continuity of $I_{IG}(\theta)$ and the boundedness of Θ. Thus we obtain $H_n(\theta) \to_u^{P_\theta^n} I_{IG}(\theta)$ as desired. □

Proof of Corollary 2.2. Note that the condition $T_n\to\infty$ was not used for $H_n^{11}(\theta)$ in the proof of Theorem 2.1. Hence the claim directly follows from Theorem 2.1 and the continuous mapping theorem. □

4.3 Gamma case

Proof of Theorem 2.3. We proceed along with the inverse-Gaussian case. We see that the observed information is nonrandom and $H_n(\theta)$ is given by

$$H_n(\theta) = \begin{pmatrix} \dfrac{1}{\delta^2 n}\displaystyle\sum_{i=1}^{n}(\delta\Delta_i^n t)^2\psi'(\delta\Delta_i^n t) & -\sqrt{\dfrac{T_n}{\gamma n}} \\ \text{sym.} & \dfrac{\delta}{\gamma^2} \end{pmatrix}, \tag{17}$$

where ψ' denotes the derivative of ψ. It suffices to consider $H_n^{11}(\theta)$. We note that $x^2\psi'(x) \to 1$ as $x \searrow 0$. Put $C = \sup_\delta \delta^{-2}$, which is finite by means of the assumptions on Θ^-. Given any $\epsilon > 0$, from the boundedness of $(\Delta_i^n t)_{i=1}^n$ we can find $n_0 \in \mathbb{N}$ such that

$$\sup_{\theta\in\Theta}\sup_{n\geq n_0}\sup_{i\leq n}|(\delta\Delta_i^n t)^2\psi'(\delta\Delta_i^n t) - 1| \leq \epsilon/C.$$

Hence for every $n \geq n_0$ we have

$$\sup_{\theta\in\Theta}|H_n^{11}(\theta) - \delta^{-2}| \leq \sup_\delta\left\{\delta^{-2}\frac{1}{n}\sum_{i=1}^{n}|(\delta\Delta_i^n t)^2\psi'(\delta\Delta_i^n t) - 1|\right\} \leq C\cdot\epsilon/C = \epsilon,$$

yielding that $H_n^{11}(\theta) \to_u 1/\delta^2$. This complete the proof. □

Proof of Corollary 2.4. This follows every bit as Corollary 2.2. □

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