MHF Preprint Series

Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with
High Functionality

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MHF 2005-8

(Received February 17, 2005)

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Abstract

With the help of a general methodology of asymptotic expansions for mixing processes, we obtain an asymptotic expansion for a special class of stochastic processes which is partly described by a stationary non-Gaussian Ornstein-Uhlenbeck process (OU process) with an invariant distribution F. Our results include (i) a higher order asymptotics as well as a central limit theorem in Barndorff-Nielsen and Shephard's stochastic volatility model; and also (ii) an asymptotic expansion for a natural estimator for the location of F. The Malliavin calculus formulated by Bichteler, Gravereaux and Jacod for processes with jumps and the exponential mixing property of the OU process play substantial roles, where especially the former ensures a "conditional type Cramér condition" under a truncation. Owing to several inherent properties of OU processes, the regularity conditions for the expansions can be easily verified, and moreover, the coefficients of the expansions up to any order can be explicitly computed.

Keywords: Edgeworth expansion, Lévy process, mixing process, non-Gaussian Ornstein-Uhlenbeck process, stochastic volatility model.

*This is an extended version of the previous version entitled "Asymptotic expansion for Barndorff-Nielsen and Shephard's stochastic volatility model", which has been accepted by *Stoch. Process Appl.* for publication, at 16 February 2005.

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1 Introduction

In this paper, we are concerned with the model $(X, Y) = \{(X_t, Y_t)\}_{t\in\mathbf{R}_+}$ given by

$$\begin{cases} dX_t &= -\lambda X_t dt + dZ_t, \\ dY_t &= (\gamma + \beta X_t)dt + \theta\sqrt{X_t}dw_t + \rho dZ_t, \qquad Y_0 = 0, \end{cases} \tag{1}$$

where $Z = (Z_t)_{t\in\mathbf{R}_+}$ and $w = (w_t)_{t\in\mathbf{R}_+}$ denote a Lévy process and a Wiener process independent of Z, respectively, and $(\lambda, \gamma, \beta, \theta, \rho) \in (0, \infty) \times \mathbf{R}^4$ are constants. The process X is called an Ornstein-Uhlenbeck process (OU process), where the initial variable X_0 is supposed to be independent of (Z, w). The positivity of λ together with a mild condition on the Lévy measure of Z ensure existence of an invariant distribution of X, which is necessarily selfdecomposable; see references given in Section 2.4 for details. The purpose of this paper is to obtain the Edgeworth expansion of an expectation $E[f(T^{-1/2}H_T)]$ as $T \to \infty$, where

$$H_T = Y_T - E[Y_T] \tag{2}$$

and $f : \mathbf{R} \to \mathbf{R}$ is a measurable function at most polynomial growth; see Section 2.1 for rigorous formulation. Within the original model (1), we single out the following two exclusive cases:

Case A. $\theta \neq 0$ and Z is a subordinator (i.e. a strictly increasing Lévy process);

Case B. $\theta = 0$, $\beta \neq 0$ and $\rho\lambda + \beta \neq 0$.

In both cases, we shall suppose that X is strictly stationary with a stationary distribution admitting moments of any order, and also that the generating triplet of Z satisfies mild regularity conditions. There is a substantial difference between these two cases, that is, there is no need for Z being a subordinator in Case B. On the other hand, our proof for Case A essentially relies on the fact that Z is a subordinator, so that the proofs will be given separately. We do not deal with the case where $\beta = \theta = 0$; in this case, Y is merely a Lévy process and the expansion is trivial.

Case A (with $\theta = 1$) corresponds to Barndorff-Nielsen and Shephard's continuous-time stochastic volatility model, in which X stays in $\mathbf{R}_+$ and describes a time-varying volatility:

$$\begin{cases} dX_t &= -\lambda X_t dt + dZ_{\lambda t}, \\ dY_t &= (\gamma + \beta X_t)dt + \sqrt{X_t}dw_t + \rho dZ_{\lambda t}, \qquad Y_0 = 0. \end{cases}$$

Here the unusual timing $dZ_{\lambda t}$ for dZ_t is their custom (then, a given marginal distribution of X is unchanged whatever $\lambda > 0$ is); of course, this is not essential for validity of the expansion, and we do not here employ it in order to make the discussion unified. This model not only captures several stylized features in finance and turbulence, but also offers a great deal of analytic tractability. See Barndorff-Nielsen and Shephard (2001) and Barndorff-Nielsen (1998) for details, and also Barndorff-Nielsen et al. (2002) for a summary of recent developments in this direction. If X is ergodic, then the martingale central limit theorem yields

$$\begin{aligned} T^{-1/2}\left(Y_T - \gamma T - \beta\int_0^T X_s ds - \rho(Z_T - E[Z_T])\right) &= T^{-1/2}\int_0^T \sqrt{X_s}dw_s \\ &\xrightarrow{\mathscr{L}} N_1(0, E[X_0]) \end{aligned} \tag{3}$$

as $T \to \infty$. Barndorff-Nielsen and Shephard call (3) "aggregational Gaussianity", which is recognized as one of important stylized features in turbulence as well as finance: here the ergodicity of X is indeed ensured by our Assumption 1 (cf. Masuda (2004)). In this paper, we consider the term "aggregational Gaussianity" as the central limit effect of $T^{-1/2}H_T$, the log-returns for long time-lags: thus our setup in principle includes (3) with $\gamma = \beta = \rho = 0$. For real market data, it is quite well known that a distribution of log-returns exhibits non-Gaussianity for short time-lags and approximate Gaussianity for long time-lags. For this reason, it is interesting to investigate the higher order asymptotics of $\mathscr{L}(T^{-1/2}H_T)$ as well as its central limit effect (first order result), so that we obtain a result which simultaneously explain non-Gaussianity for small (less of long) T and approximated Gaussianity for large T simultaneously. Our first result (Theorem 1) provides this.

Turning to Case B, the Lévy process Z here may take negative values due to absence of the diffusion coefficient $\theta\sqrt{X_t}$ of H. In this case, the regularity of $\mathscr{L}(X, H)$, which plays an essential role in derivation of expansions for Markov processes, is inferior to that in Case A since we have only one-dimensional random input Z against the two-dimensional objective (X, H); hence it is not obvious that $\mathscr{L}(X, H)$ possesses enough regularity. In particular, for pure-jump Z, this distributional problem is mathematically interesting in its own right. Under rather mild conditions, our second result (Theorem 2) guarantees the expansion even in pure-jump situations.

It will turn out that the restriction $\rho\lambda + \beta \neq 0$ is necessary for non-degeneracy of the limit distribution of $T^{-1/2}H_T$. A simple explanation is as follows. From the expression

$$X_t = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-s)}dZ_s, \tag{4}$$

we have

$$Y_t = \gamma t + (\beta + \rho\lambda)\int_0^t X_s ds + \rho(X_t - X_0). \tag{5}$$

Therefore if $\beta + \rho\lambda = 0$, then

$$T^{-1/2}H_T = \rho T^{-1/2}(X_T - X_0) \xrightarrow{\mathscr{L}} 0$$

as $T \to \infty$, so that the limiting distribution is degenerate and hence the problem becomes meaningless.

Case B includes the following statistical implication. Forget the process Y for a moment, and suppose that we can directly observe $\{X_t : 0 \leq t \leq T\}$. Estimation of $\theta_0 = E[X_0]$ (the mean of the stationary distribution) is a basic problem, and a natural estimator is given by $\hat{\theta}_T = T^{-1}\int_0^T X_s ds$. Then, we easily see that

$$T^{-1/2}H_T = T^{1/2}(\hat{\theta}_T - \theta_0)$$

with $\beta = 1$ and $\gamma = \rho = 0$, hence the consistency, asymptotic normality, and higher order expansion of $\hat{\theta}_T$ are obtained according to Theorem 2.

An important and remarkable feature common to Cases A and B is that we can explicitly write down the coefficients of the asymptotic expansions up to any order, utilizing the relation

$$\int_0^t X_s ds = \eta(\lambda, t)X_0 + \int_0^t \eta(\lambda, t-s)dZ_s, \tag{6}$$

where $\eta(\lambda, u) = \lambda^{-1}(1 - e^{-\lambda u})$: the formula (6) directly follows from the explicit expression (4), or, the affine structure of the process, see Duffie et al. (2003). One can consults Barndorff-Nielsen and Shephard (2003) for a detailed analysis of integrated OU processes.

Our goals will be achieved, applying a general methodology of asymptotic expansions for mixing processes developed in Yoshida (2004) (see also the previous works Kusuoka and Yoshida (2000), Sakamoto and Yoshida (1999), and Uchida and Yoshida (2004) for some statistical applications in this direction), together with the exponential mixing property of OU processes proved in Masuda (2004). The integration-by-parts formula plays a fundamental role to induce the decay of the characteristic function of $T^{-1/2}H_T$. However, as is well-known in this area, the direct validation of the Edgeworth expansion, namely direct estimate of the characteristic function of $T^{-1/2}H_T$, is intractable. Just for reference, let us mention this briefly. Lemma 3.1 below, which is more or less well known, and conditional argument (note that here X and w are independent) enable us to write down the characteristic function of $T^{-1/2}H_T$ as

$$\begin{aligned}\varphi(u; T^{-1/2}H_T) \quad := \quad & \exp\{-iuT^{1/2}(\beta + \lambda\rho)E[X_0]\} \\ & \cdot E\left[\exp\left\{\left(\frac{iu\beta}{T^{1/2}} - \frac{u^2}{2T}\right)\eta(\lambda, T)X_0\right\}\right] \\ & \cdot \exp\left\{\int_0^T \log E[\exp\{K(u,s)Z_1\}]ds\right\}, \qquad (7)\end{aligned}$$

where the complex-valued function K is given by

$$K(u, s) = \frac{iu}{T^{1/2}}\{\rho + \beta\eta(\lambda, s)\} - \frac{u^2}{2T}\eta(\lambda, s),$$

whose real part is negative, so that $E[\exp\{K(u, s)Z_1\}]$ indeed exists since Z is a subordinator; see e.g. Sato [19, Theorem 30.1]. The most direct route to obtain the Edgeworth expansion is estimating $\varphi(u; T^{-1/2}H_T)$ for large $|u|$; this is called the "global approach" recently developed in Yoshida [21, 22] covering processes with jumps. Unfortunately, the expression of $|\varphi(u; T^{-1/2}H_T)|$ involves the following rather intractable term coming from the Lévy-integral part in (7):

$$\left|\exp\left\{\int_0^T \int_{\mathbf{R}_+} \left(\exp\left[z\left\{\frac{iu}{T^{1/2}}(\rho + \beta\eta(\lambda, s)) - \frac{u^2}{2T}\eta(\lambda, s)\right\}\right] - 1\right)\Pi_Z(dz)ds\right\}\right|,$$

where Π_Z denotes the Lévy measure of Z. Hence we shall take another route.

In this paper we are going to look at the "local approach", which is initiated by Götze and Hipp [10] recently extended to continuous-time framework by Yoshida [23]. [See also the previous works Kusuoka and Yoshida [11], Sakamoto and Yoshida [17, 18], and Uchida and Yoshida [23] for some statistical applications in this direction.] According to the Markov property of X as well as its exponential mixing property, this approach will turn out to be tailor-made for our aim. The main task is then to establish the following estimate for some $t^0, B > 0$, which results from the integration-by-parts formula:

$$E\left[\sup_{|u|\geq B}\left|E[\psi e^{iuH_{t^0}}|X_0, X_{t^0}]\right|\right] < 1, \qquad (8)$$

where ψ fulfilling $E[\psi] > 0$ is a truncation functional, which enables us to extract a "nice event". Though (8), called the "conditional type Cramér condition", is generally not easy to verify, the concrete structure of the model (1) considerably simplifies the task. Also, the truncation technique is often inevitable, and this is indeed the case for our goal. In the proof we shall construct ψ in a tangible way in order to avoid the irregular square-root diffusion coefficient of Y, and consequently validate the expansion. See Section 4 for details.

The main results are given in Section 2, and then Section 3 presents the explicit formulae for the asymptotic expansions. The proofs of the main results are given in Section 4.

2 The results

Let $\mathbf{B} = (\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t)_{t\in\mathbf{R}_+}, P)$ be a stochastic basis. Suppose that

— in Case A, $\mathbf{B}$ is endowed with an $\mathbf{F}$-adapted non-trivial subordinator Z and an $\mathbf{F}$-adapted Wiener process w as well as an $\mathscr{F}_0$-measurable random variable X_0 independent of (w, Z);

— or, in Case B, $\mathbf{B}$ is endowed with an $\mathbf{F}$-adapted non-trivial Lévy process Z as well as an $\mathscr{F}_0$-measurable random variable X_0 independent of Z.

Throughout this article, $\varphi(u;\xi)$ stands for the characteristic function of ξ indicating a random variable or a distribution, and we write $\kappa(u;\xi) = \log\varphi(u;\xi)$ for the corresponding cumulant transform. The (partial) differentiation with respect to some variable v will be denoted by ∂_v, or simply by ∂ when there is no confusion.

We denote by (b_Z, C_Z, Π_Z) the generating triplet of Z:

$$\varphi(u;Z_t) = \exp\left\{t\left(ib_Z u - \frac{1}{2}C_Z u^2 + \int_{\mathbf{R}}(e^{iuz} - 1 - iuz1_{\{|z|\le 1\}}(z))\Pi_Z(dz)\right)\right\}, \tag{9}$$

where $b_Z \in \mathbf{R}$, $C_Z \ge 0$, and the Lévy measure Π_Z defined on $\mathbf{R}$ is a σ-finite measure meeting $\Pi_Z(\{0\}) = 0$ and $\int_{0<|z|\le 1} z^2\Pi_Z(dz) < \infty$. If Z is in particular a subordinator, (9) can be written in the form of

$$\varphi(u;Z_t) = \exp\left\{t\left(ib_Z u + \int_{\mathbf{R}_+}(e^{iuz} - 1)\Pi_Z(dz)\right)\right\},$$

where $b_Z \ge 0$, $\mathrm{supp}\Pi_Z \subset \mathbf{R}_+$ and $\int_{0<z\le 1} z\Pi_Z(dz) < \infty$.

2.1 Formulation of the asymptotic expansion

Before stating our results, let us briefly present the formulation of the Edgeworth expansion; see Yoshida (2004) for a more general exposition.

Denote by $\chi_{r,T}(u)$ the r-th cumulant function of $T^{-1/2}H_T$ ($r \in \mathbf{N}$, $r \ge 2$), where H is defined by (2):

$$\chi_{r,T}(u) = \partial_u^r \log E[\exp(iuT^{-1/2}H_T)].$$

Define $\tilde{P}_{r,T}(u)$ by the formal expansion

$$\exp\left(\sum_{r=2}^{\infty}\frac{1}{r!}\chi_{r,T}(u)\right) = \exp\left(\frac{1}{2}\chi_{2,T}(u)\right) + \sum_{r=1}^{\infty} T^{-r/2}\tilde{P}_{r,T}(u).$$

Fix $p \in \mathbf{N}$ ($p \ge 3$), and define $\hat{\Psi}_{p,T}(u)$ by

$$\hat{\Psi}_{p,T}(u) = \exp\left(\frac{1}{2}\chi_{T,2}(u)\right) + \sum_{r=1}^{p-2} T^{-r/2}\tilde{P}_{r,T}(u).$$

Then the $(p-2)$-th Edgeworth expansion, say $\Psi_{p,T}$, is defined by the Fourier inversion of $\hat{\Psi}_{p,T}$. Denote by $\phi(\cdot;\Sigma)$ the one-dimensional Gaussian density with mean zero and variance $\Sigma > 0$, and let $h_r(y;\Sigma)$ stand for the r-th Hermite polynomial associated with $\phi(\cdot;\Sigma)$:

$$h_r(y;\Sigma) = (-1)^r \phi(y;\Sigma)^{-1} \partial_y^r \phi(y;\Sigma).$$

Put $\chi_{r,T} = (-i)^r \chi_{r,T}(0)$ (the r-th cumulant of $T^{-1/2}H_T$) and write $\chi_{2,T}$ as Σ_T for convenience: in our case, $\chi_{r,T} = O(T^{-(r-2)/2})$ for $T \to \infty$. Then the density of $\Psi_{p,T}$ with respect to the Lebesgue measure is given by

$$g_p(y; T^{-1/2}H_T) = \{1 + G_{p,T}(y)\}\phi(y;\Sigma_T),$$

where

$$G_{p,T}(y) = \sum_{k=1}^{p-2} \sum_{l=1}^{k} \sum_{\substack{k_1,\dots,k_l \in \mathbf{N}: \\ k_1+\cdots+k_l=k}} \frac{\chi_{k_1+2,T} \cdot \cdots \cdot \chi_{k_l+2,T}}{l!(k_1+2)! \cdots (k_l+2)!} h_{k+2l}(y;\Sigma_T).$$

For instance, the third-order approximation $g_4(y; T^{-1/2}H_T)$ (corresponding to the second-order Edgeworth expansion) is given by

$$g_4(y; T^{-1/2}H_T) = \phi(y;\Sigma_T)\left\{1 + \sum_{k=1}^{2} B_{k,T}(y)\right\},$$

where

$$\begin{aligned} B_{1,T}(y) &= \frac{\chi_{3,T}}{3!}\left(\frac{y^3}{\Sigma_T^3} - \frac{3y}{\Sigma_T^2}\right), \\ B_{2,T}(y) &= \frac{\chi_{4,T}}{4!}\left(\frac{y^4}{\Sigma_T^4} - \frac{6y^2}{\Sigma_T^3} + \frac{3}{\Sigma_T^2}\right) + \frac{\chi_{3,T}^2}{2!(3!)^2}\left(\frac{y^6}{\Sigma_T^6} - \frac{15y^4}{\Sigma_T^5} + \frac{45y^2}{\Sigma_T^4} - \frac{15}{\Sigma_T^3}\right). \end{aligned}$$

Let $p_0 = 2[p/2]$ and denote by $\mathcal{E}(M,p_0)$ the set of all measurable functions $f : \mathbf{R} \to \mathbf{R}$ satisfying $|f(x)| \le M(1+|x|^{p_0})$ for every $x \in \mathbf{R}$. Put

$$\Delta_{p,T}(f) = |E[f(T^{-1/2}H_T)] - \Psi_{p,T}[f]|,$$

and

$$\omega(f;\delta,\nu) = \int_{\mathbf{R}} \sup_{|y|\le\delta} |f(x+y) - f(x)|\nu(dx)$$

for $\delta > 0$, measurable function f and Borel measure ν on $\mathbf{R}$.

Suppose that $\Sigma_T \to \Sigma > 0$ as $T \to \infty$, and fix any positive constant Σ^0 such that $\Sigma^0 > \Sigma$. We say that "*Estimate (10) holds true for* $T^{-1/2}H_T$" if "*for any* $M, K > 0$, *there exist positive constants* M^* *and* δ^* *such that*

$$\Delta_{p,T}(f) \le M^* \omega(f; T^{-K}, \phi(x;\Sigma^0)dx) + o(T^{-(p-2+\delta^*)/2}) \tag{10}$$

for $T \to \infty$ *uniformly in* $f \in \mathcal{E}(M,p_0)$". Our goal is to show that Estimate (10) holds true for $T^{-1/2}H_T$ in both of Cases A and B. We impose the following moment condition:

Assumption 1. *X is strictly stationary with a non-trivial stationary distribution F admitting moments of any order.*

Remark 1. A natural question is that "given an order of the expansion, is it possible to specify up to what order of F's moments are actually required?". To answer this, apart from $[A2]$ easy to check (see Section 4), we must carefully estimate the moment of the dominating polynomial $\mathscr{P}$ of $|\Psi(\hat{\psi}_{\epsilon,\epsilon'})|$ (see (39) and (40) in the proof), where the function Ψ essentially comes from the integration by parts formula, and hence the specification of the required order is in principle possible. However we do not pursue this problem here because in most applications (cf Barndorff-Nielsen and Shephard (2001,2002,2003), also Barndorff-Nielsen et al. (2002)), Condition [A2] is fulfilled and it is not so constructive to spare the space to count the order.

In the sequel, we denote by $\kappa_\xi^{(k)}$ the k-th cumulant of ξ, a random variable or a distribution.

2.2 Case A

In this case, H satisfies

$$dH_t = \beta(X_t - \kappa_F^{(1)})dt + \theta\sqrt{X_t}dw_t + \rho d\bar{Z}_t, \qquad H_0 = 0, \tag{11}$$

where $\bar{Z}_t = Z_t - E[Z_t] = Z_t - E[Z_1]t$ is the centred Z. Since $\lambda > 0$ and Z is a subordinator, we have $\mathrm{supp}F \subset \mathbf{R}_+$. If the Lévy measure of Z admits moments of any order outside neighborhoods of the origin, then Assumption 1 is satisfied under $\mathscr{L}(X_0) = F$; see Remark 5 below.

Denote by Λ_Z the Poisson random measure associated with jumps of Z, and let it be written as

$$\Lambda_Z(dt,dz) = \mu_Z^\flat(dt,dz) + \mu_Z(dt,dz) \tag{12}$$

for some Poisson random measures $\mu_Z^\flat$ and μ_Z. Correspondingly, write

$$\Pi_Z(dz) = \nu_Z^\flat(dz) + \nu_Z(dz), \tag{13}$$

where $\nu_Z^\flat$ and ν_Z stand for the Lévy measures on $\mathbf{R}_+$ associated with $\mu_Z^\flat$ and μ_Z, respectively. The second assumption here is

Assumption 2. *There exists a non-empty open subset of $\mathbf{R}_+$ on which the Lévy measure ν_Z admits a positive C^3-density with respect to the Lebesgue measure.*

We need the C^3-property of the density of ν_Z for the condition $(\tilde{A}' - 4)$ of Bichteler et al. (1987). The Lévy measure $\nu_Z^\flat$ may be any one as long as Assumption 1 is satisfied; in particular, we may take $\nu_Z^\flat \equiv 0$ if Π_Z admits a sufficiently smooth positive density. Many examples of F treated by Barndorff-Nielsen and Shephard satisfy Assumptions 1 and 2; for instance, generalized inverse Gaussian, tempered stable and selfdecomposable modified stable (cf. Barndorff-Nielsen et al. (2002)).

Theorem 1. *Let H be given by (11). Suppose that Assumptions 1 and 2 are met, and fix any positive number Σ^0 such that*

$$\Sigma^0 > \theta^2\kappa_F^{(1)} + \frac{2}{\lambda}(\beta + \lambda\rho)^2\kappa_F^{(2)}.$$

Then, Estimate (10) holds true for $T^{-1/2}H_T$.

The proof of Theorem 1 is given in Section 4.1.

2.3 Case B

In this case, H satisfies

$$dH_t = \beta(X_t - \kappa_F^{(1)})dt + \rho d\bar{Z}_t, \qquad H_0 = 0, \tag{14}$$

where, differently from Case A, X may take its values in $\mathbf{R}$.

As in Case A, suppose that the Poisson random measure Λ_Z and the Lévy measure Π_Z are of the forms (12) and (13), respectively, except that Z is not necessarily a subordinator. The further assumption here is

Assumption 3. *Either of the following two conditions holds true:*

(i) $C_Z > 0$*;*

(ii) *there exists a non-empty open subset of* $\mathbf{R}\backslash\{0\}$ *on which* ν_Z *admits a positive* C^3*-density with respect to the Lebesgue measure.*

Assumption (i) of B3 requires nothing for jumps of Z; in particular, Z may be a Wiener process.

Theorem 2. *Let* H *be given by (14). Suppose that Assumptions 1 and 3 are met, and fix any positive number* Σ^0 *such that*

$$\Sigma^0 > \frac{2}{\lambda}(\beta + \rho\lambda)^2 \kappa_F^{(2)}.$$

Then, Estimate (10) holds true for $T^{-1/2}H_T$*.*

The proof of Theorem 2 is given in Section 4.2.

2.4 Some remarks concerning OU processes

Let us refer to some convenient previous results concerning an OU process

$$dX_t = -\lambda X_t dt + dZ_t$$

whose solution is explicitly given by (4). See Masuda (2004) and references therein for more information.

Remark 2. Any selfdecomposable distribution can be realized as a possible stationary distribution of an OU process; more precisely, there is one-to-one correspondence between a possible stationary distribution of an OU process and a selfdecomposable distribution. See Sato and Yamazato (1983) for details.

Remark 3. Two theoretical construction of a stationary OU process with a concrete marginal distribution are possible. First, suppose that a selfdecomposable distribution F is given. If $\varphi(u;F)$ is differentiable at $u \neq 0$ and moreover if the function $u \mapsto u\partial_u\kappa(u;F)$ is continuous at $u = 0$, then there exists a stationary OU process X with the marginal distribution F and Z determined by $\kappa(u;Z_1) = \lambda u \partial_u \kappa(u;F)$: see Lemma 3.1 of Barndorff-Nielsen et al. (1998). Secondly, we can determine the stationary distribution F of X via a given generating triplet of Z_1; in this case, the Lévy measure $\Pi_Z(dz)$ of Z must meet

$$\int_{|z|>1} \log|z| \Pi_Z(dz) < \infty.$$

See Sato and Yamazato (1983).

Remark 4. If X is strictly stationary and the Lévy measure of F admits a differentiable density $g_F(x)$ for $x \neq 0$, then the Lévy measure of Z admits a density $g_Z(x)$ given by

$$g_Z(x) = -\lambda^{-1}\{g_F(x) + x\partial g_F(x)\}. \tag{15}$$

The relation (15) is convenient to determine Z, given F; see, e.g., Barndorff-Nielsen (1998).

Remark 5. If F (resp. Z_1) admits the k-th cumulant, then Z_1 (resp. F) admits the k-th cumulant as well and they are related by

$$k\lambda\kappa_F^{(k)} = \kappa_{Z_1}^{(k)}. \tag{16}$$

See Section 2.1 of Barndorff-Nielsen et al. (2001). The formula (16) enables us to write down the coefficients of the asymptotic expansion in terms of only $\kappa_F^{(k)}$ or only $\kappa_{Z_1}^{(k)}$, $k \in \mathbf{N}$. If we use Barndorff-Nielsen and Shephard's custom $dZ_{\lambda t}$ instead of dZ_t, then (16) becomes $k\kappa_F^{(k)} = \kappa_{Z_1}^{(k)}$.

3 Coefficients of the expansion

As already mentioned, the formula (6) is useful for computation of the coefficients of the asymptotic expansions. In both cases, simple but tedious computations give explicit expressions for $\chi_{r,T}$.

3.1 Case A

A minor modification of Theorem 1 of Lukacs (1969) yields the following simple lemma:

Lemma 3.1. *Let Z be a subordinator. Let $h : [0,T] \times \mathbf{R} \to \mathbf{C}$ be continuous in the first component, and suppose that the real part of $h(s,u)$ is non-positive for every (s,u). Then*

$$\log E\left[\exp\left\{\int_0^T h(s,u)dZ_s\right\}\right] = \int_0^T \log E\left[\exp\{h(s,u)Z_1\}\right] ds$$

for every $u \in \mathbf{R}$.

Using (6), conditional argument (note that here X and w are independent), and Lemma 3.1, one can easily obtain that under Assumption 1

$$\chi_{r,T}(u) = \partial_u^r \kappa(a_T(u); F) + \int_0^T \partial_u^r \kappa(b_T(v,u); Z_1)dv, \qquad r \geq 2, \tag{17}$$

where

$$\begin{aligned} a_T(u) &= \left(\frac{u\beta}{\sqrt{T}} + i\frac{\theta^2 u^2}{2T}\right)\eta(\lambda, T), \\ b_T(v,u) &= \frac{u}{\sqrt{T}}\{\beta\eta(\lambda, v) + \rho\} + i\frac{\theta^2 u^2}{2T}\eta(\lambda, v). \end{aligned}$$

The elementary chain rule for differentiations and the above formula readily yield the explicit expressions for $\chi_{r,T}$. Note that if $\beta = \rho = 0$, then all odd-order cumulants vanish since $\mathscr{L}(T^{-1/2}H_T)$ is symmetric and centered at the origin. See Theorem 2.2 of Nicolato and Venardos (2003) for formulation of the Laplace transform of $T^{-1/2}H_T$.

The following formula is convenient for computations of $\chi_{r,T}$ (for the second-term on the right-hand side in (17)):

$$\begin{aligned} J_{k,l}(T) &:= \int_0^T \{\beta\eta(\lambda,v)+\rho\}^k \eta(\lambda,v)^l ds \\ &= \sum_{j=0}^{k} \binom{k}{j} \beta^j \rho^{k-j} I_{l+j}(T) \end{aligned} \tag{18}$$

for $k,l \in \mathbf{N}\cup\{0\}$, where

$$I_m(T) = \int_0^T \{\eta(\lambda,v)\}^m dv, \quad m \in \mathbf{N}\cup\{0\},$$

satisfy the recurrence formula

$$I_k(T) = \lambda^{-1} I_{k-1}(T) - (\lambda k)^{-1}\{\eta(\lambda,T)\}^k, \qquad k \in \mathbf{N},$$

from which we get

$$\begin{cases} I_m(T) &= \lambda^{-m}T - \lambda^{-(m+1)} \sum_{q=1}^m q^{-1}\{\lambda\eta(\lambda,T)\}^q, \quad m \geq 1, \\ I_0(T) &= T. \end{cases} \tag{19}$$

It follows from (18) and (19) that

$$\frac{1}{T} J_{k,l}(T) \to \lambda^{-(k+l)} \sum_{j=0}^{k} \binom{k}{j} \beta^j \rho^{k-j} \lambda^{k-j} = \lambda^{-(l+k)}(\beta+\rho\lambda)^k, \qquad \text{as } T\to\infty.$$

In particular, we obtain that

$$\Sigma_T = \chi_{2,T} \to \theta^2 \kappa_F^{(1)} + \frac{2}{\lambda}(\beta+\lambda\rho)^2 \kappa_F^{(2)} > 0 \text{ as } T\to\infty.$$

For the next two, we obtain that

$$\begin{aligned} \chi_{3,T} &= T^{-1/2}\Big\{\kappa_F^{(3)} T^{-1}\left(\beta^3(\eta(\lambda,T))^3 + 3\lambda J_{3,0}(T)\right) \\ &\qquad +3\theta^2\kappa_F^{(2)} T^{-1}\left(\beta(\eta(\lambda,T))^2 + 2\lambda J_{1,1}(T)\right)\Big\} \\ &\sim T^{-1/2}\left\{3\lambda^{-2}(\beta+\rho\lambda)^3\kappa_F^{(3)} + 6\lambda^{-1}\theta^2(\beta+\rho\lambda)\kappa_F^{(2)}\right\} \end{aligned}$$

and

$$\begin{aligned} \chi_{4,T} &= T^{-1}\Big\{\kappa_F^{(4)} T^{-1}\left(\beta^4(\eta(\lambda,T))^4 + 4\lambda J_{4,0}(T)\right) \\ &\qquad +6\theta^2\kappa_F^{(3)} T^{-1}\left(\beta^2(\eta(\lambda,T))^3 + 3\lambda J_{2,1}(T)\right) \\ &\qquad +3\theta^4\kappa_F^{(2)} T^{-1}\left((\eta(\lambda,T))^2 + 2\lambda I_2(T)\right)\Big\} \\ &\sim T^{-1}\Big\{4\lambda^{-3}(\beta+\rho\lambda)^4\kappa_F^{(4)} + 18\lambda^{-2}\theta^2(\beta+\rho\lambda)^2\kappa_F^{(3)} + 6\lambda^{-1}\theta^4\kappa_F^{(2)}\Big\} \end{aligned}$$

where $F_T \sim G_T$ means that $F_T/G_T \to 1$ as $T\to\infty$.

Remark 6. Barndorff-Nielsen and Shephard (2002) advocated that the tempered stable distribution denoted by $TS(\kappa,\delta,\xi)$, where $0<\kappa<1$, $\delta>0$, and $\xi\ge 0$, is one of good candidates for F when the model is applied to finance; a special case is $IG(\delta,\xi)$ for $\kappa=1/2$. For $TS(\kappa,\delta,\xi)$, we must assume that $\xi>0$ for Assumption 1, and in this case the normal tempered stable distribution (NTS) including the normal inverse Gaussian (NIG) for $\kappa=1/2$ appears as the approximation of the distribution of the instantaneous log-return of a stock price. NTS (also NIG) is known to be able to exhibit skewness and steepness (fat tails) very flexibly and it also possesses reproducing-property. Further, the cumulant generating function of $TS(\kappa,\delta,\xi)$ is simply given by $\delta\{\xi-(\xi^{1/\kappa}-2u)^{\kappa}\}$, from which one easily gets

$$\kappa^{(k)}_{TS(\kappa,\delta,\xi)}=-\delta(-2)^k\xi^{(\kappa-k)/\kappa}\prod_{j=0}^{k-1}(\kappa-j),\qquad k\in\mathbf{N}.$$

3.2 Case B

In this case the cumulants $\chi_{r,T}$ are simpler than Case A. In similar fashion to Case A, we can obtain that, for $r\ge 2$,

$$\chi_{r,T}(u)=\partial_u^r\kappa\left(\beta T^{-1/2}\eta(\lambda,T)u;F\right)+\int_0^T\partial_u^r\kappa\left((\rho+\beta\eta(\lambda,v))T^{-1/2}u;Z_1\right)dv,$$

from which we obtain

$$\chi_{r,T}=T^{-(r-2)/2}\kappa_F^{(r)}\left[T^{-1}\left\{\beta^r(\eta(\lambda,T))^r+\lambda rJ_{r,0}(T)\right\}\right],\quad r\ge 2.$$

For example, we see that under $\beta+\rho\lambda\neq 0$

$$\Sigma_T=\chi_{2,r}\to\frac{2\kappa_F^{(2)}}{\lambda}(\beta+\rho\lambda)^2>0\text{ as }T\to\infty,$$

which says that we actually need $\beta+\lambda\rho\neq 0$ in this case. Higher order cumulants have the same expressions as Case A with $\theta=0$.

4 Proofs

The proof will be carried out essentially through Theorem 4 of Yoshida (2004), which is a reduced version of Theorem 1 of that paper and particularly targets at stochastic differential equations with jumps; the Theorem 1 deals with general (partially) mixing processes. Hence, before entering the proof, let us briefly mention what Theorem 1 of Yoshida (2004) says for reader's convenience.

Building on Markov nature and stationarity of X, *the exponential mixing version* of Theorem 1 of Yoshida (2004) asserts that it suffices to verify the following conditions in order to validate our results:

[$A1$] X is strongly mixing with exponential rate;

[$A2$] for each $T\in\mathbf{R}_+$, $\sup_{t\in[0,T]}\|H_t\|_{L^{p+1}(P)}<\infty$;

$[A3]$ (a version of conditional type Cramér conditions) there exist positive constants t^0, a, a' and B, and a truncation functional $\psi : (\Omega, \mathscr{F}) \to ([0,1], \mathcal{B}([0,1]))$ such that $0 < a, a' < 1$, $4a' < (a-1)^2$ and that the following two conditions are met:

$$E\left[\sup_{|u|\ge B} \left|E[\psi e^{iuH_{t^0}}|X_0, X_{t^0}]\right|\right] < a', \tag{20}$$

$$1 - E[\psi] < a. \tag{21}$$

It is difficult in general to check $[A3]$ directly, however, we can employ infinite dimensional stochastic calculus (Malliavin calculus) with truncation to verify it. This is just what Theorem 4 of Yoshida (2004) provides: $[A3]$ is replaced by another condition called $[A3^Q]$, in which local non-degeneracy of a Malliavin covariance matrix of interest as well as some other regularity conditions is required. Our plan is thus to prove $[A3^Q]$ under our assumptions.

In both of Cases A and B, Assumption 1 ensures the conditions $[A1]$ and $[A2]$. Indeed, X is exponentially β-mixing under Assumption 1: see Theorem 4.3 of Masuda (2004) for details. Turning to $[A2]$, the relation (16) implies that Z_1 as well as F admits moments of any order, and then Jensen's inequality (also Burkholder-Davis-Gundy's inequality in Case A) readily ensures $[A2]$. Hence, in both of Cases A and B, it remains to verify $[A3]$.

In the rest of this section, we write $\tilde{\mu}_Z^\flat(dt,dz) = \mu_Z^\flat(dt,dz) - \nu_Z^\flat(dz)dt$ and $\tilde{\mu}_Z(dt,dz) = \mu_Z(dt,dz) - \nu_Z(dz)dt$; recall (12) and (13).

4.1 Proof of Theorem 1

Suppose that Assumptions 1 and 2 hold true. Without loss of generality, we may set $\theta = 1$. In addition to direct application of Theorem 4 of Yoshida (2004) itself, we shall introduce an auxiliary process $\tilde{H}$ for H, which will turn out to be essential for the condition $(\tilde{A}'-4)$ of Bichteler et al. (1987). Here the condition $(\tilde{A}'-r)$, $r \in \mathbf{N}$, is a series of conditions for smoothness of the coefficients of stochastic differential equations of interest, moreover, it requires polynomial growth rate of the derivatives of the coefficients; consult p.147 of Bichteler et al. (1987) for details. More precisely, we shall circumvent the irregular behavior of the derivatives of H's diffusion coefficient $\sqrt{X_t}$ near the origin, introducing a suitable truncation functional.

4.1.1 Transforming the Poisson random measure in Case A

Under Assumption 1, it follows from the Lévy-Itô decomposition that

$$Z_t = \lambda\kappa_F^{(1)} t + \int_0^t\int_{\mathbf{R}_+} z\tilde{\mu}_Z^\flat(ds,dz) + \int_0^t\int_{\mathbf{R}_+} z\tilde{\mu}_Z(ds,dz)$$

for each $t \in \mathbf{R}_+$. Under Assumption 2, we can find an open set $E_{A,0} = (c_1, c_2)$ with $0 < c_1 < c_2 < \infty$, on which ν_Z admits a C^3-density g_Z such that $\inf_{z\in E_{A,0}} g_Z(z) > 0$.

To begin with, we partly rewrite the stochastic differential equation of (X, H), replacing partial jumps associated with μ_Z corresponding to the region (c_1, c_2) by the uniform Poisson space, so that the resulting compensating measure becomes the Lebesgue measure; this is required for direct application of the theory of Bichteler et al. (1987). Under Assumption 2, this corresponds to the change of variable

$$z^* = z^*(z) = \int_z^{c_2} g_Z(v)dv, \qquad z \in E_{A,0}. \tag{22}$$

Write $g_Z^+(z) = z^*(z)$. Then $g_Z^+(z)$ is strictly decreasing on $E_{A,0}$, hence $g_Z^+(c_1) > g_Z^+(c_2) > 0$. Accordingly, we have

$$\int_0^t\int_{c_1}^{c_2} z\tilde{\mu}_Z(ds,dz) = \int_0^t\int_{g_Z^+(c_2)}^{g_Z^+(c_1)} g_Z^-(z^*)\tilde{\mu}_Z^*(ds,dz^*), \tag{23}$$

where g_Z^- stands for the inverse function of $z \mapsto g_Z^+(z)$, which is also strictly decreasing, and $\tilde{\mu}_Z^*(dt,dz^*) = \mu_Z^*(dt,dz^*) - dtdz^*$ with the integer-valued random measure μ_Z^* defined by

$$\int_0^t\int_{a_1}^{a_2} h(s,z)\mu_Z(ds,dz) = \int_0^t\int_{g_Z^+(a_2)}^{g_Z^+(a_1)} h(s,g_Z^-(z^*))\mu_Z^*(ds,dz^*) \tag{24}$$

for each $t \in \mathbf{R}_+$, $a_1, a_2 \in \mathbf{R}$ such that $a_1 < a_2$, and any measurable function h on $\mathbf{R}_+ \times \mathbf{R}_+$. Put $E_A = (g_Z^+(c_2), g_Z^+(c_1))$. It follows that, for $B \in \mathcal{B}(E_A)$ and $t \in \mathbf{R}_+$, we have

$$E[\mu_Z^*([0,t],B)] = l(B)t,$$

where $l(\cdot)$ stands for the Lebesgue measure. Then the stochastic differential equation of (X,H) becomes

$$\begin{aligned}\begin{pmatrix} dX_t \\ dH_t \end{pmatrix} &= (\kappa_F^{(1)} - X_t)\begin{pmatrix} \lambda \\ -\beta \end{pmatrix}dt + \begin{pmatrix} 0 \\ \sqrt{X_t} \end{pmatrix}dw_t \\ &\quad + \int_{\mathbf{R}_+} z\begin{pmatrix} 1 \\ \rho \end{pmatrix}\{\tilde{\mu}_Z^\flat + 1_{E_{A,0}^c}\tilde{\mu}_Z\}(dt,dz) + \int_{\tilde{E}_A} J_A(z^*)\begin{pmatrix} 1 \\ \rho \end{pmatrix}\tilde{\mu}_Z^*(dt,dz^*),\end{aligned} \tag{25}$$

where $E_{A,0}^c$ denotes the complement of $E_{A,0}$, $\tilde{E}_A = E_A \cup (g_Z^+(c_1), \infty)$, and the function J_A is given by

$$J_A(z^*) = g_Z^-(z^*)1_{E_A}(z^*), \qquad z^* \in \tilde{E}_A.$$

Note that (25) is clearly a graded stochastic differential equation according to the grading $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ of $\mathbf{R}^2$ in the sense of 5-5 of Bichteler et al. (1987). Also, note that for each $t \in \mathbf{R}_+$ the random number $\mu_Z^*([0,t],E_A)$ is a.s. finite, and that the function $z^* \mapsto J_A(z^*)$ is of class C^4 on $\tilde{E}_A$ by virtue of Assumption 2 and the inverse function theorem.

Remark 7. We have presented a (partial) transformation of the Poisson random measure μ_Z demonstratively, however, we should note that it is always possible to extract a uniform Poisson random measure from any Poisson random measure μ on $I \times E \subset \mathbf{R}_+ \times \mathbf{R}$, as soon as μ's Lévy measure at least admits a positive density on E. Of course this is true of the multi-dimensional case.

Let $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$ be the canonical space defined as follows. Let $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{P})$ stand for the canonical product Wiener-Poisson space over a non-empty time-interval $[0,t^0]$, and then define $(\hat{\Omega}, \hat{\mathscr{B}})$ by the product measurable space $(\hat{\Omega}, \hat{\mathscr{B}}) = (\mathbf{R}_+ \times \tilde{\Omega}, \mathcal{B}(\mathbf{R}_+) \otimes \tilde{\mathscr{B}})$. Define a probability measure $\hat{P}$ by $\hat{P} = F \times \tilde{P}$: under $\hat{P}$, the projection to the first space, say $\hat{x}$, yields the same law as F, the canonical projection w is a one-dimensional Wiener process, and that the canonical projections $\mu_Z^\flat + 1_{E_{A,0}^c}\mu_Z$ and μ_Z^* are independent Poisson random measures on $[0,t^0] \times \mathbf{R}_+$ and $[0,t^0] \times \tilde{E}_A$, respectively. Also, $\hat{x}$ and $(w, \mu_Z^\flat + 1_{E_{A,0}^c}\mu_Z, \mu_Z^*)$ are independent under $\hat{P}$. We shall consistently write Z for its distributional equivalent on the space $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$, that is, $\mathscr{L}(Z|P) = \mathscr{L}(Z|\hat{P})$, where $\mathscr{L}(\xi|Q)$ stands for the distribution of a random variable ξ under a probability measure Q: accordingly, we write $\bar{Z}_t = Z_t - \hat{E}[Z_1]t$.

On the space $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$, we consider the flow $(X(t,v), H(t,v))^\top$ associated with (X, H) starting from $v = (x,h)^\top \in \mathbf{R}_+ \times \mathbf{R}$, which of course satisfies

$$\begin{cases} X(t,v) = e^{-\lambda t}x + \int_0^t e^{-\lambda(t-s)}dZ_s, \\ H(t,v) = h + \beta \int_0^t (X(s,v) - \kappa_F^{(1)})ds + \int_0^t \sqrt{X(s,v)}dw_s + \rho \bar{Z}_t. \end{cases} \tag{26}$$

We shall execute the Malliavin calculus for this flow on a suitable event $\{\hat{\psi}_{\epsilon,\epsilon'} > 0\}$, where $\hat{\psi}_{\epsilon,\epsilon'}$ is the truncation functional defined in the next subsection.

4.1.2 Construction of a truncation functional in Case A

Here we concretely construct a truncation functional defined on $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$, say $\hat{\psi}_{\epsilon,\epsilon'}$, in order to extract a "nice event" on which an integration-by-parts formula can be applied: we must show that such an event has positive $\hat{P}$-probability. Here the meaning of the argument (ϵ, ϵ') will be clarified below. The functional $\hat{\psi}_{\epsilon,\epsilon'}$ corresponds to a distributional equivalent of ψ appearing in [$A3$].

Let $\varphi_1 \in C_B^\infty(\mathbf{R}_+; [0,1])$ be a non-increasing function such that $\varphi_1(x) = 1$ if $0 \le x \le 1/2$ and $\varphi_1(x) = 0$ if $x \ge 1$, where $C_B^\infty(\mathbf{R}_+; [0,1])$ denotes the set of all $[0,1]$-valued smooth functions defined on $\mathbf{R}_+$ with bounded derivatives. We shall consider $\hat{\psi}_{\epsilon,\epsilon'}$ of the form

$$\hat{\psi}_{\epsilon,\epsilon'} = \varphi_1(\hat{\xi}_{\epsilon,\epsilon'}) \tag{27}$$

for some $\hat{\xi}_{\epsilon,\epsilon'} \in D_{2,\infty-}^{\mathcal{L}}$, where $D_{2,\infty-}^{\mathcal{L}}$ denotes the domain of the extended Malliavin operator $\mathcal{L}$ employed in Section 5 of Yoshida (2004).

From now on, we construct a "nice event" step by step, and then suitably define $\hat{\xi}_{\epsilon,\epsilon'}$ ((36) below). In what follows, we fix arbitrary positive constants x_0 and t^0, and put $\hat{v} = (\hat{x}, 0)^\top$.

1) Define an auxiliary event $\mathscr{A}_1$ by

$$\mathscr{A}_1 = \{\hat{x} \ge e^{\lambda t^0} x_0\}.$$

Clearly $\hat{P}[\mathscr{A}_1] > 0$ since any non-trivial selfdecomposable distribution possesses an unbounded support. Since Z is a subordinator, we see that from (26)

$$X(t,\hat{v}) \ge e^{-\lambda t}\hat{x} \ge e^{\lambda(t^0 - t)}x_0 \ge x_0$$

for every $t \in [0, t^0]$ on $\mathscr{A}_1$, so that we have

$$\inf_{0 \le t \le t^0} X(t, \hat{v}) \ge x_0$$

uniformly on $\mathscr{A}_1$.

Fix any function $\tau \in C_b^\infty(\mathbf{R}_+; \mathbf{R}_+)$ satisfying the following conditions, where $C_b^\infty(\mathbf{R}_+; \mathbf{R}_+)$ stands for the set of all smooth functions on $\mathbf{R}_+$ with bounded derivatives of order≥ 1:

(τ-1) $\tau(x) = \sqrt{x}$ for $x \ge x_0/7$;

(τ-2) $x \mapsto \tau(x)$ and $x \mapsto \partial\tau(x)$ are globally Lipschitz.

Using this τ, define a process $\tilde{H}(\cdot, v)$ by

$$\tilde{H}(t,v) = h + \beta \int_0^t (X(s,x) - \kappa_F^{(1)})ds + \int_0^t \tau(X(s,x))dw_s + \rho \bar{Z}_t,$$

which is same as H except for the smooth diffusion coefficient. By the previous paragraph, $H(\cdot, v) = \tilde{H}(\cdot, v)$ for $t \in [0, t^0]$ on $\mathscr{A}_1$ paving positive $\hat{P}$-probability.

2) Let c_j' and c_j'' $(j = 1, 2)$ be positive constants such that $0 < c_1 < c_1' < c_1'' < c_2'' < c_2' < c_2 < \infty$, and write $\check{E}_A = (g_Z^+(c_2''), g_Z^+(c_1'')) \Subset E_A$. Let $\eta_A \in C_B^\infty(\mathbf{R}_+; \mathbf{R}_+)$ be any function satisfying $\inf_{z^* \in \check{E}_A} \eta_A(z^*) > 0$, and $\eta_A(z^*) = 0$ for $z^* \notin (g_Z^+(c_2'), g_Z^+(c_1'))$: we shall utilize this η_A as an auxiliary function satisfying 10-1 of Bichteler et al (1987).

Denote by ∇ the differential operator with respect to $v = (x, h)^\top$. Then, taking account of the expression (26), the matrix-valued process $\tilde{K}(\cdot, v) = \nabla (X(\cdot, v), \tilde{H}(\cdot, v))^\top$ is given by

$$\tilde{K}(t,v) = \begin{pmatrix} e^{-\lambda t} & 0 \\ \beta \lambda^{-1}(1 - e^{-\lambda t}) + \partial_x \int_0^t \tau(X(s,v))dw_s & 1 \end{pmatrix}. \tag{28}$$

Obviously

$$\det \tilde{K}_t(v) = e^{-\lambda t}$$

for any $v \in \mathbf{R}_+ \times \mathbf{R}$. Denote by A_t the $(2,1)$-component of the right-hand side of (28). In view of Assumption 1, the definition of τ, and (26), it is clear that $E[\int_0^{t^0} \tau(X(s, \hat{v}))^2 ds] < \infty$ and $E[\int_0^{t^0} \{\partial_x \tau(X(s, \hat{v}))\}^2 ds] < \infty$. Then it is well known that the Lipschitz property (τ-2) ensures existence of a differentiable version of $x \mapsto \int_0^t \tau(X(s,v))dw_s$, so we have

$$\tilde{A}_t := \partial_x \int_0^t \tau(X(s,v))dw_s \bigg|_{x = \hat{x}} = \int_0^t e^{-\lambda s} (\partial \tau) \circ (X(s,v))dw_s \bigg|_{x = \hat{x}}.$$

Fix $t_1 \in (0, t^0)$ and $z_0 \in \check{E}_A$. Take a sufficiently small constant $\epsilon > 0$ so that $I_1^\epsilon := (t_1 - \epsilon, t_1 + \epsilon) \subset (0, t^0)$ and that $E_A^\epsilon := (z_0 - \epsilon, z_0 + \epsilon) \subset \check{E}_A$. Now we define $\mathscr{A}_2^\epsilon$ by

$$\mathscr{A}_2^\epsilon = \{\mu_Z^*(I_1^\epsilon, E_A^\epsilon) = 1\}. \tag{29}$$

Obviously $\hat{P}[\mathscr{A}_2^\epsilon] = 4\epsilon^2 \exp(-4\epsilon^2) > 0$ for any $\epsilon > 0$.

3) Next, for $\epsilon' > 0$, we introduce

$$\mathscr{A}_3^{\epsilon'} = \left\{ \sup_{0 \le t \le t^0} |\tilde{A}_t| < \epsilon' \right\}. \tag{30}$$

Recall that here $x \mapsto \partial \tau(x)$ is supposed to be bounded, so that $\tilde{A}$ is a continuous $\mathbf{F}$-martingale. Enlarging the underlying stochastic basis, we see that there exists a standard Wiener process $B = (B_t)_{t \in \mathbf{R}_+}$ such that $\tilde{A}_t = B_{[\tilde{A}]_t}$ (e.g., Theorem IV 34.11 of Rogers and Williams (1994)), where $[\tilde{A}]_t = \int_0^t e^{-2\lambda s} \{(\partial \tau) \circ (X(s, \hat{v}))\}^2 ds$, and obviously $[\tilde{A}]_{t^0} \le \|\partial \tau\|_\infty^2 t^0$. Therefore, we can estimate as

$$\begin{aligned} \hat{P}[\mathscr{A}_3^{\epsilon'} | \mathscr{A}_2^\epsilon] &= \hat{P}\left[\hat{P}\left[\sup_{0 \le t \le t^0} |B_{[\tilde{A}]_t}| < \epsilon' \middle| \sigma(X, \mu_Z^*) \right] \middle| \mathscr{A}_2^\epsilon \right] \\ &\ge \hat{P}\left[\hat{P}\left[\sup_{0 \le t \le \|\partial \tau\|_\infty^2 t^0} |B_t| < \epsilon' \middle| \sigma(X, \mu_Z^*) \right] \middle| \mathscr{A}_2^\epsilon \right], \end{aligned}$$

where the random number

$$\hat{P}\left[\sup_{0\le t\le \|\partial\tau\|_\infty^2 t^0}|B_t| < \epsilon' \middle| \sigma(X,\mu_Z^*)\right]$$

is a.s. positive for any $t^0, \epsilon' > 0$ (cf. p.97 of Billingsley (1999)), hence we obtain that $\hat{P}[\mathscr{A}_3^{\epsilon'}|\mathscr{A}_2^{\epsilon}] > 0$ a.s. Put $\mathscr{A}^{\epsilon,\epsilon'} = \mathscr{A}_1 \cap \mathscr{A}_2^{\epsilon} \cap \mathscr{A}_3^{\epsilon'}$, then we have

$$\begin{aligned}\hat{P}[\mathscr{A}^{\epsilon,\epsilon'}] &= \hat{P}[\mathscr{A}_1]\hat{P}[\mathscr{A}_2^{\epsilon}\cap\mathscr{A}_3^{\epsilon'}] \\ &= \hat{P}[\mathscr{A}_1]\hat{P}[\mathscr{A}_2^{\epsilon}]\hat{P}[\mathscr{A}_3^{\epsilon'}|\mathscr{A}_2^{\epsilon}] \\ &> 0\end{aligned}$$

for any ϵ and ϵ': here, due to the independence between μ_Z^* and w, it should be noted that we may control ϵ and ϵ' independently.

4) With the smooth modification $\tilde{H}$ introduced before, the Malliavin covariance matrix $U(\cdot,\hat{v})$ associated with the flow $(X(\cdot,\hat{v}),\tilde{H}(\cdot,\hat{v}))^\top$ is well-defined for $t\in[0,t^0]$, and given by

$$U(t,\hat{v}) = \tilde{K}(t,\hat{v})\tilde{S}(t,\hat{v})\tilde{K}(t,\hat{v})^\top, \qquad t\in[0,t^0], \tag{31}$$

where, on $\mathscr{A}^{\epsilon,\epsilon'}$,

$$\begin{aligned}\tilde{S}(t,\hat{v}) = &\int_0^t \tilde{K}(s,\hat{v})^{-1}\begin{pmatrix}0 & 0\\ 0 & X(s,\hat{v})\end{pmatrix}\tilde{K}(s,\hat{v})^{\top\ -1}ds \\ &+ \int_0^t\int_{E_A} V_A(z^*)\tilde{K}(s,\hat{v})^{-1}\begin{pmatrix}1 & \rho\\ \rho & \rho^2\end{pmatrix}\tilde{K}(s,\hat{v})^{\top\ -1}\mu_Z^*(ds,dz^*)\end{aligned} \tag{32}$$

with $V_A(z^*) = \{\partial J_A(z^*)\}^2\eta_A(z^*)$: See Section 10 of Bichteler et al. (1987) for details. Due to (28) and non-negative definiteness of the second term of the right-hand side of (32), we obtain that

$$\begin{aligned}\tilde{S}(t^0,\hat{v}) \ge &\int_0^{t^0} \tilde{K}(s,\hat{v})^{-1}\begin{pmatrix}0 & 0\\ 0 & X(s,\hat{v})\end{pmatrix}\tilde{K}(s,\hat{v})^{\top\ -1}ds \\ &+ \int_{I_1^\epsilon}\int_{E_A^\epsilon} V_A(z^*)\tilde{K}(s,\hat{v})^{-1}\begin{pmatrix}1 & \rho\\ \rho & \rho^2\end{pmatrix}\tilde{K}(s,\hat{v})^{\top\ -1}\mu_Z^*(ds,dz^*) \\ = &\begin{pmatrix}\int_{I_1^\epsilon}\int_{E_A^\epsilon} V_A(z^*)e^{2\lambda s}\mu_Z^*(ds,dz^*) & \text{sym.} \\ \int_{I_1^\epsilon}\int_{E_A^\epsilon} V_A(z^*)e^{\lambda s}(\rho - e^{\lambda s}A_s)\mu_Z^*(ds,dz^*) & \int_{I_1^\epsilon}\int_{E_A^\epsilon} V_A(z^*)(\rho - e^{\lambda s}A_s)^2\mu_Z^*(ds,dz^*) + \int_0^{t^0} X(s,\hat{v})ds\end{pmatrix}\end{aligned}$$

on $\mathscr{A}^{\epsilon,\epsilon'}$, so that we have

$$\begin{aligned}\det\tilde{S}(t^0,\hat{v}) \ge &\left(\int_{I_1^\epsilon}\int_{E_A^\epsilon} V_A(z^*)e^{2\lambda s}\mu_Z^*(ds,dz^*)\right) \\ &\cdot\left(\int_{I_1^\epsilon}\int_{E_A^\epsilon} V_A(z^*)(\rho - e^{\lambda s}A_s)^2\mu_Z^*(ds,dz^*) + \int_0^{t^0} X(s,\hat{v})ds\right) \\ &- \left(\int_{I_1^\epsilon}\int_{E_A^\epsilon} V_A(z^*)e^{\lambda s}(\rho - e^{\lambda s}A_s)\mu_Z^*(ds,dz^*)\right)^2.\end{aligned} \tag{33}$$

Also, clearly we have

$$\det U(t^0,\hat{v}) = e^{-2\lambda t^0}\det\tilde{S}(t^0,\hat{v}) \tag{34}$$

in view of (28) and (31). We shall show that $\det\tilde{S}(t^0,\hat{v}) > 0$ to conclude that $\det U(t^0,\hat{v}) > 0$ uniformly on $\mathscr{A}^{\epsilon,\epsilon'}$, i.e., local non-degeneracy of $U(t^0,\hat{v})$. In the sequel, we use the small order symbol $o''(1)$ for random or non-random variables $R_{\epsilon,\epsilon'}$ such that $R_{\epsilon,\epsilon'} \to 0$ as $\epsilon,\epsilon' \downarrow 0$ uniformly on $\mathscr{A}^{\epsilon,\epsilon'}$.

Under Assumption 2, $z^* \mapsto V_A(z^*)$ is of class C^3 and strictly positive uniformly on E_A^ϵ. Applying the Taylor series around z_0 and t_1, we easily obtain

$$\begin{aligned}
V_A(z^*)e^{2\lambda s} &= V_A(z_0)e^{2\lambda t_1} + o''(1),\\
V_A(z^*)(\rho - e^{\lambda s}A_s)^2 &= V_A(z_0)\left\{\rho - \beta\lambda^{-1}(e^{\lambda t_1}-1)\right\}^2 + o''(1),\\
V_A(z^*)e^{\lambda s}(\rho - e^{\lambda s}A_s) &= V_A(z_0)e^{\lambda t_1}\left\{\rho - \beta\lambda^{-1}(e^{\lambda t_1}-1)\right\} + o''(1).
\end{aligned}$$

Substituting these three displays in (33), we get

$$\begin{aligned}
\det\tilde{S}(t^0,\hat{v}) \;&\geq\; \{V_A(z_0)e^{2\lambda t_1} + o''(1)\}\\
&\quad \cdot\left\{V_A(z_0)\left(\rho + \lambda^{-1}\beta - \lambda^{-1}\beta e^{\lambda t_1}\right)^2 + \int_0^{t^0} X(s,\hat{v})ds + o''(1)\right\}\\
&\quad - \left\{V_A(z_0)e^{\lambda t_1}\left(\rho + \lambda^{-1}\beta - \lambda^{-1}\beta e^{\lambda t_1}\right) + o''(1)\right\}^2\\
&= V_A(z_0)e^{2\lambda t_1}\int_0^{t^0} X(s,\hat{v})ds + o''(1).
\end{aligned} \tag{35}$$

Here we of course used the fact that, on $\mathscr{A}^{\epsilon,\epsilon'}$, $\mu_Z^*(I_1^\epsilon, E_A^\epsilon) = 1$ for any $\epsilon,\epsilon' > 0$. Therefore we see that, from (34) and (35),

$$\begin{aligned}
\det U(t^0,\hat{v}) &= e^{-2\lambda t^0}\det\tilde{S}(t^0,\hat{v})\\
&\geq e^{2\lambda(t_1 - t^0)}V_A(z_0)x_0 t^0 + o''(1)
\end{aligned}$$

on $\mathscr{A}^{\epsilon,\epsilon'}$. Hence, choosing η_A as $\eta_A(z_0)$ is sufficiently large (without loss of generality) and letting ϵ and ϵ' be sufficiently small, we may take $\det U(t^0,\hat{v}) \geq 3$ on $\mathscr{A}^{\epsilon,\epsilon'}$. Fix ϵ and ϵ' like this in the rest of this proof.

5) Now we define a functional $\hat{\xi}_{\epsilon,\epsilon'} \in D_{2,\infty-}^{\mathcal{L}}$ by

$$\hat{\xi}_{\epsilon,\epsilon'} = \frac{1}{1 + \det U(t^0,\hat{v})} + \frac{2}{1 + 7\hat{x}x_0^{-1}e^{-\lambda t^0}}; \tag{36}$$

it is clear that $\hat{\xi}_{\epsilon,\epsilon'} \in \cap_{p<\infty} L^p(\hat{P})$. By the choice of ϵ and ϵ' in the previous step, it follows that

$$\begin{aligned}
0 < \hat{P}[\mathscr{A}^{\epsilon,\epsilon'}] \;&\leq\; \hat{P}[\det U(t^0,\hat{v}) \geq 3,\; \hat{x} \geq e^{\lambda t^0}x_0]\\
&\leq \hat{P}\left[\frac{1}{1 + \det U(t^0,\hat{v})} \leq \frac{1}{4}, \quad \frac{2}{1 + 7\hat{x}x_0^{-1}e^{-\lambda t^0}} \leq \frac{1}{4}\right]\\
&\leq \hat{P}\left[\hat{\xi}_{\epsilon,\epsilon'} \leq \frac{1}{2}\right]
\end{aligned}$$

Consequently, $\det U(t^0,\hat{v}) = 0$ implies $\hat{\psi}_{\epsilon,\epsilon'}\{\det U(t^0,\hat{v})\}^{-1} = 0$ (with the convention $0 \cdot \infty = 0$). We thus end up with

Lemma 4.1. *Let $\hat{\psi}_{\epsilon,\epsilon'}$ be of the form (27). Then there exists $\hat{\xi}_{\epsilon,\epsilon'} \in D_{2,\infty-}^{\mathcal{L}}$ such that $\hat{P}[\hat{\xi}_{\epsilon,\epsilon'} \leq 1/2] > 0$ and that $\hat{\psi}_{\epsilon,\epsilon'}\{\det U(t^0,\hat{v})\}^{-1} \in \cap_{p<\infty} L^p(\hat{P})$ for each $t^0 > 0$.*

4.1.3 On the condition $(\tilde{A}' - 4)$

We must check $(\tilde{A}' - 4)$ of Bichteler et al. (1987) for the flow $(X(\cdot, v), H(\cdot, v))^\top$. Here ϵ and ϵ' are fixed so that the assertion of Lemma 4.1 holds true.

As already mentioned, the diffusion coefficient $\sqrt{X(t,v)}$ of $H(\cdot, v)$ causes trouble for $(\tilde{A}' - 4)$. However, *it is sufficient that we can apply the integration-by-parts formula on the event carved out by the truncation functional $\hat{\psi}_{\epsilon,\epsilon'}$.* Now, let us note that the definition (27) leads to the following inclusive relation:

$$\begin{aligned}
\{\hat{\psi}_{\epsilon,\epsilon'} > 0\} &\subset \{\hat{\xi}_{\epsilon,\epsilon'} \le 1\} \\
&\subset \left\{ \frac{2}{1 + 7\hat{x}x_0^{-1}e^{-\lambda t^0}} \le 1 \right\} \\
&= \left\{ \frac{x_0 e^{\lambda t^0}}{7} \le \hat{x} \right\} \\
&\subset \left\{ \inf_{0 \le s \le t^0} X(s, \hat{v}) \ge \frac{x_0}{7} \right\}.
\end{aligned}$$

Thus, the property (τ-1) implies that $H(t, \hat{v}) = \tilde{H}(t, \hat{v})$ for $t \in [0, t^0]$ on $\{\hat{\psi}_{\epsilon,\epsilon'} > 0\}$: in other word, we have

$$\mathscr{L}\left\{ \left(\psi_{\epsilon,\epsilon'}, 1_{\{\psi_{\epsilon,\epsilon'} > 0\}}(X_{t^0}, H_{t^0}) \right) \middle| P \right\} = \mathscr{L}\left\{ \left(\hat{\psi}_{\epsilon,\epsilon'}, 1_{\{\hat{\psi}_{\epsilon,\epsilon'} > 0\}}(X(t^0, \hat{v}), H(t^0, \hat{v})) \right) \middle| \hat{P} \right\},$$

where $\psi_{\epsilon,\epsilon'}$ and H_{t^0} (both defined on the original probability space $(\Omega, \mathscr{F}, P)$) stand for a distributional equivalent of $\hat{\psi}_{\epsilon,\epsilon'}$ and $\tilde{H}(t^0, 0)$, respectively. On the other hand, it is quite straightforward to verify $(\tilde{A}' - 4)$ for $\{(X(\cdot, \hat{v}), \tilde{H}(\cdot, \hat{v}))^\top\}_{t \in [0,t^0]}$, hence we have seen that

Lemma 4.2. *Under Assumptions 1 and 2, the process* $\left\{ 1_{\{\hat{\psi}_{\epsilon,\epsilon'} > 0\}}(X(\cdot, \hat{v}), H(\cdot, \hat{v}))^\top \right\}_{t \in [0,t^0]}$ *meets* $(\tilde{A}' - 4)$.

4.1.4 An integration-by-parts formula and moment conditions

Here ϵ and ϵ' are still fixed as the assertion of Lemma 4.1 holds true. On $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$, consider the Malliavin operator $(\mathcal{L}, D^{\mathcal{L}}_{2,\infty-})$ used in Section 5 of Yoshida (2004): see Section 9 of Bichteler et al. (1987) for a detailed exposition. Denote by $\Gamma_{\mathcal{L}}$ the bilinear form corresponding to $\mathcal{L}$: namely, for $F, G \in D^{\mathcal{L}}_{2,\infty-}$,

$$\Gamma_{\mathcal{L}}(F, G) = \mathcal{L}(FG) - G\mathcal{L}F - F\mathcal{L}G. \tag{37}$$

Put $\hat{\mathscr{Z}} = (X(t^0, \hat{v}), H(t^0, \hat{v}))$ and $S_1^*[\hat{\psi}_{\epsilon,\epsilon'}; \hat{\mathscr{Z}}] = \{\sigma^{pq}_{\hat{\mathscr{Z}}}, \Delta^{-1}_{\hat{\mathscr{Z}}}\hat{\psi}_{\epsilon,\epsilon'}\}$, where $\sigma_{\hat{\mathscr{Z}}} = (\sigma^{pq}_{\hat{\mathscr{Z}}}) = \Gamma_{\mathcal{L}}(\hat{\mathscr{Z}}, \hat{\mathscr{Z}})$ and $\Delta_{\hat{\mathscr{Z}}} = \det\sigma_{\hat{\mathscr{Z}}}$ (we shall use similar notation for the other variables).

According to the truncation via $\hat{\psi}_{\epsilon,\epsilon'}$, we can now follow the argument in Section 4.2 of Yoshida (2004), except that H's diffusion coefficient is replaced under our truncation, in order to validate the conditional type Cramér condition: to be precise, for any $B > 0$, distributional equivalence and the integration-by-parts formula yield that

$$\begin{aligned} E\Big[\sup_{u:|u|\geq B}\Big|E[\psi_{\epsilon,\epsilon'}e^{iuH_{t^0}}|X_0,X_{t^0}]\Big|\Big] &= E\Big[\sup_{u:|u|\geq B}\Big|E[\psi_{\epsilon,\epsilon'}e^{iu\tilde{H}_{t^0}}|X_0,X_{t^0}]\Big|\Big] \\ &= \hat{E}\Big[\sup_{u:|u|\geq B}\Big|E[\hat{\psi}_{\epsilon,\epsilon'}e^{iu\tilde{H}(t^0,\hat{v})}|X(t^0,\hat{v})]\Big|\Big] \\ &= \hat{E}\Big[\sup_{u:|u|\geq B}\Big|(iu)^{-1}\hat{E}[e^{iu\tilde{H}(t^0,\hat{v})}\Psi(\hat{\psi}_{\epsilon,\epsilon'})|X(t^0,\hat{v})]\Big|\Big], \end{aligned} \tag{38}$$

where $\mathscr{L}\left((\psi_{\epsilon,\epsilon'},\tilde{H}_{t^0})\Big|P\right)=\mathscr{L}\left((\hat{\psi}_{\epsilon,\epsilon'},\tilde{H}(t^0,\hat{v}))\Big|\hat{P}\right)$, and the functional Ψ, which is well-defined on $\{\hat{\psi}_{\epsilon,\epsilon'}>0\}$, is given by

$$\begin{aligned} \Psi(\hat{\psi}_{\epsilon,\epsilon'}) = &\,\mathit{\Gamma}_{\mathcal{L}}\left(X(t^0,\hat{v}),\sigma^{-1}_{X(t^0,\hat{v})}\hat{\psi}_{\epsilon,\epsilon'}\mathit{\Gamma}_{\mathcal{L}}(X(t^0,\hat{v}),\tilde{H}(t^0,\hat{v}))\right) \\ &-\mathit{\Gamma}_{\mathcal{L}}\left(\hat{\psi}_{\epsilon,\epsilon'},\tilde{H}(t^0,\hat{v})\right)-2\hat{\psi}_{\epsilon,\epsilon'}\mathcal{L}\tilde{H}(t^0,\hat{v}) \\ &+2\sigma^{-1}_{X(t^0,\hat{v})}\hat{\psi}_{\epsilon,\epsilon'}\mathit{\Gamma}_{\mathcal{L}}\left(X(t^0,\hat{v}),\tilde{H}(t^0,\hat{v})\right)\mathcal{L}X(t^0,\hat{v}). \end{aligned} \tag{39}$$

It follows from (38) that

$$E\Big[\sup_{u:|u|\geq B}|E[\psi_{\epsilon,\epsilon'}e^{iuH_{t^0}}|X_0,X_{t^0}]|\Big]\leq\frac{1}{B}\hat{E}[|\Psi(\hat{\psi}_{\epsilon,\epsilon'})|].$$

We must show $\Psi(\hat{\psi}_{\epsilon,\epsilon'})\in L^1(\hat{P})$: if this is true, then (20) of [$A3$] follows by letting B be sufficiently large. Note that (21) in [$A3$] holds true with $\psi=\psi_{\epsilon,\epsilon'}$ since $P[\psi_{\epsilon,\epsilon'}>0]=\hat{P}[\hat{\psi}_{\epsilon,\epsilon'}>0]$ and this probability is positive by virtue of Lemma 4.1.

As remarked in Section 5.1 of Yoshida (2004), there exists a polynomial function $\mathscr{P}$ such that

$$|\Psi(\hat{\psi}_{\epsilon,\epsilon'})|\leq\mathscr{P}\left(1_{\{|\hat{\xi}_{\epsilon,\epsilon'}|\leq 1\}}Q(t^0,\hat{v})^{-1},|U(t^0,\hat{v})|,|V(t^0,\hat{v})|,|U^*(t^0,\hat{v})|,|\sigma_{\hat{\xi}_{\epsilon,\epsilon'}}|\right), \tag{40}$$

where $Q(t^0,\hat{v})=\det U(t^0,\hat{v})$, $V(t^0,\hat{v})=\mathcal{L}\hat{\mathscr{Z}}\in\mathbf{R}^2$ and $U^*(t^0,\hat{v})=\mathit{\Gamma}_{\mathcal{L}}(U(t^0,\hat{v}),U(t^0,\hat{v}))\in\mathbf{R}^4\otimes\mathbf{R}^4$.

1) In Lemma 4.1, we have seen that $1_{\{|\hat{\xi}_{\epsilon,\epsilon'}|\leq 1\}}Q(t^0,\hat{v})^{-1}\in\cap_{p<\infty}L^p(\hat{P})$.

2) Since $\hat{\mathscr{Z}}\in D^{\mathcal{L}}_{2,\infty-}$ and $\mathcal{L}$ takes its values in $\cap_{p<\infty}L^p(\hat{P})$, we see that $V(t^0,\hat{v})$ and $U(t^0,\hat{v})$ belong to $\cap_{p<\infty}L^p(\hat{P})$.

3) Applying Theorems 10-3 and 10-17 of Bichteler et al. (1987) repeatedly and then using Theorem 5-10 of the same monograph, it is easy to see that $U^*(t^0,\hat{v})\in\cap_{p<\infty}L^p(\hat{P})$, taking into account that $\tau\in C_b^\infty(\mathbf{R}_+)$.

4) Since $\hat{\xi}_{\epsilon,\epsilon'}\in\cap_{p<\infty}L^p(\hat{P})$, it follows from (37) and the property of $\mathcal{L}$ that $\sigma_{\hat{\xi}_{\epsilon,\epsilon'}}\in\cap_{p<\infty}L^p(\hat{P})$.

Summarizing the above steps yields

Lemma 4.3. *Under Assumptions 1 and 2, we have* $\Psi(\hat{\psi}_{\epsilon,\epsilon'})\in L^1(\hat{P})$ *for* Ψ *of (39).*

Combining Lemmas 4.1, 4.2 and 4.3 guarantees [$A3^Q$] of Yoshida (2004), hence the proof of Theorem 1 is now complete.

4.2 Proof of Theorem 2

Suppose that Assumptions 1 and 3 hold true.

4.2.1 A transformation of the Poisson random measure in Case B

Under Assumption 1, we can write

$$Z_t = \lambda\kappa_F^{(1)} t + \sqrt{C_Z}\tilde{w}_t + \int_0^t\int_{\mathbf{R}} z\tilde{\mu}_Z^{\flat}(ds,dz) + \int_0^t\int_{\mathbf{R}} z\tilde{\mu}_Z(ds,dz)$$

for each $t \in \mathbf{R}_+$, where $\tilde{w}$ stands for a one-dimensional Wiener process. As in Case A, let us consider a transformation of the absolutely continuous part of the Poisson random measure; of course this procedure may be skipped if we know that Z has no jumps and $C_Z > 0$.

Assumption 3 assures the existence of a bounded domain $E_{B,0} = (c_1, c_2) \subset \mathbf{R}\backslash\{0\}$ for which the Lévy density g_Z meets $\inf_{z\in E_{B,0}} g_Z(z) > 0$. Without loss of generality, we may suppose that $c_1, c_2 > 0$: if $\nu_Z(\mathbf{R}_+) \equiv 0$, then regard $-Z$ as Z.

As in Case A, we introduce the change of variables $z^* = z^*(z) = g_Z^+(z)$ for $z \in E_{B,0}$ and transform μ_Z into μ_Z^*, as in (22), (23) and (24), with g_Z^- still denoting the strictly decreasing inverse function defined on $E_B = (g_Z^+(c_2), g_Z^+(c_1))$. Put $\tilde{E}_B = E_B \cup (g_Z^+(c_1), \infty)$. Then, just like (25), (X, H) satisfies

$$\begin{aligned}\begin{pmatrix} dX_t \\ dH_t \end{pmatrix} &= (\kappa_F^{(1)} - X_t)\begin{pmatrix} \lambda \\ -\beta \end{pmatrix} dt + \sqrt{C_Z}\begin{pmatrix} 1 \\ \rho \end{pmatrix} d\tilde{w}_t \\ &\quad + \int_{\mathbf{R}_+} z\begin{pmatrix} 1 \\ \rho \end{pmatrix}\{\tilde{\mu}_Z^{\flat} + 1_{E_{B,0}^c}\tilde{\mu}_Z\}(dt,dz) + \int_{\tilde{E}_B} J_B(z^*)\begin{pmatrix} 1 \\ \rho \end{pmatrix}\tilde{\mu}_Z^*(dt,dz^*),\end{aligned} \tag{41}$$

where

$$J_B(z^*) = g_Z^-(z^*)1_{E_B}(z^*), \qquad z^* \in \tilde{E}_B.$$

Fix any constant $t^0 > 0$ and let $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$ be the canonical space defined as in the proof of Theorem 1, except for some trivial changes of notation. Also define on $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$ the flow $(X(\cdot, v), H(\cdot, v))^\top$ associated with (X, H) of (41) starting from $v = (x, h)^\top \in \mathbf{R}^2$. Let $\hat{x}$ be a random variable such that $\mathscr{L}(\hat{x}|\hat{P}) = F$ and that $\hat{x}$ is independent of $(\tilde{w}, \mu_Z^{\flat} + 1_{E_{B,0}^c}\mu_Z, \mu_Z^*)$. Again put $\hat{v} = (\hat{x}, 0)^\top$. It is easy to see that

$$K(t^0, \hat{v}) := \nabla(X(t^0, \hat{v}), H(t^0, \hat{v}))^\top = e^{tQ},$$

where $Q \in \mathbf{R}^2 \otimes \mathbf{R}^2$ is given by

$$Q = \begin{pmatrix} -\lambda & 0 \\ \beta & 0 \end{pmatrix}.$$

As in Case A, let c_j' and c_j'' $(j = 1, 2)$ be positive constants such that $0 < c_1 < c_1' < c_1'' < c_2'' < c_2' < c_2 < \infty$, and write $\check{E}_B = (g_Z^+(c_2''), g_Z^+(c_1'')) \Subset E_B$. Also let $\eta_B \in C_B^\infty(\mathbf{R}_+; \mathbf{R}_+)$ be any function satisfying $\inf_{z^*\in\check{E}_B} \eta_B(z^*) > 0$, and $\eta_B(z^*) = 0$ for $z^* \notin (g_Z^+(c_2'), g_Z^+(c_1'))$. Under Assumption 3, the flow $(X(\cdot, \hat{v}), H(\cdot, \hat{v}))^\top$ clearly satisfies the condition $(\tilde{A}' - 4)$.

In this case, the process $S(\cdot,\hat{v})$ corresponding to (32) is given by

$$\begin{aligned} S(t,\hat{v}) = C_Z \int_0^t e^{-sQ} \begin{pmatrix} 1 & \rho \\ \rho & \rho^2 \end{pmatrix} e^{-sQ^\top} ds \\ + \int_0^t \int_{E_B} e^{-sQ} \begin{pmatrix} 1 & \rho \\ \rho & \rho^2 \end{pmatrix} e^{-sQ^\top} V_B(z^*) \mu_Z^*(ds, dz^*), \end{aligned} \tag{42}$$

where $V_B(z^*) := \{\partial J_B(z^*)\}^2 \eta_B(z^*)$; $S(\cdot,\hat{v})$ is actually independent of $\hat{v}$. Then the Malliavin covariance matrix $U(t^0,\hat{v})$ of $(X(t^0,\hat{v}), H(t^0,\hat{v}))^\top$ is well-defined and given by $U(t^0,\hat{v}) = e^{t^0 Q} S(t^0,\hat{v}) e^{t^0 Q^\top}$, so that

$$\det U(t^0,\hat{v}) = e^{-2\lambda t^0} \det S(t^0,\hat{v}). \tag{43}$$

4.2.2 Proof of Theorem 2 under Assumption 3 (i)

Suppose that $C_Z > 0$. Since the second term on the right-hand side of (42) is non-negative-definite, we have

$$S(t^0,\hat{v}) \geq C_Z \int_0^{t^0} e^{-sQ} \begin{pmatrix} 1 & \rho \\ \rho & \rho^2 \end{pmatrix} e^{-sQ^\top} ds,$$

and hence elementary computations yield

$$\det S(t^0,\hat{v}) \geq C_Z^2 \lambda^{-4} (\beta + \rho\lambda)^2 \left\{ \frac{\lambda t^0}{2} (e^{2\lambda t^0} - 1) - (e^{\lambda t^0} - 1)^2 \right\}. \tag{44}$$

The right-hand side of (44) is positive whenever $t^0 > 0$, $\beta + \rho\lambda \neq 0$, and $\lambda > 0$. Thus $S(t^0,\hat{v})$ is bounded from below by a positive-definite matrix, so that the non-degeneracy of $U(t^0,\hat{v})$ follows from (43) without any non-trivial truncation functional; simply let $\hat{\psi} \equiv 1$ in [$A3$]. Then the analogous assertions as Lemmas 4.2 and 4.3 can be easily obtained all without essential distinction. Thus the assertion of Theorem 2 has been proved under Assumption 1 and Assumption 3 (i).

4.2.3 Construction of a truncation functional in Case B

It remains to prove Theorem 2 under Assumptions 1 and 3 (ii). We here again construct a truncation functional, which is much simpler than $\hat{\psi}_{\epsilon,\epsilon'}$ used in Case A: differently from Case A, we need no modification of H.

Fix $t^0 > 0$ arbitrarily, and let $t_1, t_2 \in (0, t^0)$ be constants such that $t_1 \neq t_2$. Also fix $z_0 \in \check{E}_B$. Let $\epsilon > 0$ be sufficiently small so that $I_j^\epsilon := (t_j - \epsilon, t_j + \epsilon) \Subset I_j$ $(j = 1, 2)$ and that $E_B^\epsilon := (z_0 - \epsilon, z_0 + \epsilon) \Subset \check{E}_B$. Define $\mathscr{A}^\epsilon$ by

$$\mathscr{A}^\epsilon = \{\mu_Z^*(I_j^\epsilon, E_B^\epsilon) = 1, \text{ for } j = 1, 2.\}. \tag{45}$$

Then we have $\hat{P}[\mathscr{A}^\epsilon] = 16\epsilon^4 \exp(-8\epsilon^2) > 0$ for any $\epsilon > 0$. Define the truncation functional $\hat{\psi}_\epsilon$ by $\hat{\psi}_\epsilon = \varphi_1(\hat{\xi}_\epsilon)$ with φ_1 introduced in Section 4.1.2, where

$$\hat{\xi}_\epsilon = \frac{2}{1 + 3\det U(t^0,\hat{v})}. \tag{46}$$

Using this $\hat{\psi}_\epsilon$, we shall proceed as in the proof of Theorem 1.

Let us show that the Malliavin covariance matrix $U(t^0,\hat{v})$ is non-degenerate on the event $\mathscr{A}^\epsilon$ for any $\epsilon>0$: here we shall in turn utilize the jump part of (42), ignoring the diffusion part. Since

$$e^{-sQ}=\begin{pmatrix} e^{\lambda s} & 0\\ \beta\lambda^{-1}(1-e^{\lambda s}) & 1\end{pmatrix}=\begin{pmatrix} e^{\lambda t_j} & 0\\ \beta\lambda^{-1}(1-e^{\lambda t_j}) & 1\end{pmatrix}+o(1)$$

for $\epsilon\downarrow 0$ (we use $o(1)$ for matrices too), Taylor's expansion around z_0 and t_j $(j=1,2)$ says that, on $\mathscr{A}^\epsilon$,

$$\begin{aligned} S(t^0,\hat{v}) &\geq \sum_{j=1}^{2}\int_{I_j^\epsilon}\int_{E_B^\epsilon} e^{-sQ}\begin{pmatrix} 1 & \rho\\ \rho & \rho^2\end{pmatrix} e^{-sQ^\top} V_B(z^*)\mu_Z^*(ds,dz^*)\\ &= V_B(z_0)\begin{pmatrix} J^{(2)} & \text{sym.}\\ (\rho+\beta\lambda^{-1})J^{(1)}-\beta\lambda^{-1}J^{(2)} & \sum_{j=1}^{2}\{(\rho+\beta\lambda^{-1})-\beta\lambda^{-1}e^{\lambda t_j}\}^2\end{pmatrix}+o(1)\end{aligned}$$

by virtue of (45), where $J^{(1)}:=e^{\lambda t_1}+e^{\lambda t_2}$ and $J^{(2)}:=e^{2\lambda t_1}+e^{2\lambda t_2}$. Therefore we obtain

$$\det S(t^0,\hat{v})=V_B(z_0)^2\lambda^{-2}(\beta+\lambda\rho)^2(e^{\lambda t_1}-e^{\lambda t_2})^2+o(1),$$

which is positive for ϵ sufficiently small whenever $\rho\lambda+\beta\neq 0$ and $t_1\neq t_2$. Note that we may set $V_B(z_0)$ arbitrarily large by choosing the function η_B suitably, so that we can conclude that, recalling (43), $\det U(t^0,\hat{v})\geq 1$ on $\mathscr{A}^\epsilon$ for some $\epsilon>0$. In this case, we have $\hat{P}[\hat{\xi}_\epsilon\leq 1/2]>0$ by the definition (46), and the assertion corresponding to Lemma 4.1 holds true.

Clearly $\hat{\psi}_\epsilon>0$ implies that $1/3\leq \det U(t^0,\hat{v})$, so that the integration-by-parts formula under the truncation is in force as in the proof of Theorem 1. The assertions corresponding to Lemmas 4.2 and 4.3 can be obtained in a similar manner to the case of Theorem 1. All in all, the proof of Theorem 2 is complete.

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