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# Point configurations, Cremona transformations and the elliptic difference Painlevé equations


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# POINT CONFIGURATIONS, CREMONA TRANSFORMATIONS AND THE ELLIPTIC DIFFERENCE PAINLEVÉ EQUATION

*by*

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***Abstract.*** — A theoretical foundation for a generalization of the elliptic difference Painlevé equation to higher dimensions is provided in the framework of birational Weyl group action on the space of point configurations in general position in a projective space. By introducing an elliptic parametrization of point configurations, a realization of the Weyl group is proposed as a group of Cremona transformations containing elliptic functions in the coefficients. For this elliptic Cremona system, a theory of $\tau$-functions is developed to translate it into a system of bilinear equations of Hirota-Miwa type for the $\tau$-functions on the lattice. Application of this approach is also discussed to the elliptic difference Painlevé equation.

***Résumé*** **(Configurations de points, transformations de Cremona et équation de Painlevé aux différences elliptique)**

Dans le cadre de l'action birationnelle du groupe de Weyl sur l'espace des configurations de points en position générale dans un espace projectif on établit des fondements théoriques en vue d'une généralisation aux dimensions supérieures de l'équation de Painlevé aux différences elliptique. On réalise le groupe de Weyl comme un groupe de transformations de Cremona à coefficients fonctions elliptiques grâce à une paramétrisation elliptique des configurations de points. Une théorie des fonctions $\tau$ permet de traduire ce système de Cremona en un système d'équations bilinéaires de type Hirota-Miwa pour les fonctions $\tau$ sur le réseau. On en donne une application à l'équation de Painlevé aux différences elliptique.

## 1. Introduction

The main purpose of this paper is to provide a theoretical foundation for a generalization of the elliptic difference Painlevé equation to higher dimensions in the framework of birational Weyl group actions on the spaces of point configurations in general position in projective spaces.

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Since the pioneering work of Grammaticos, Ramani, Papageorgiou and Hietarinta [**5**], discrete Painlevé equations have been studied from various viewpoints. A large class of second order discrete Painlevé equations, as well as their generalizations, has been discovered through the studies of singularity confinement property, bilinear equations, affine Weyl group symmetries and spaces of initial conditions (see [**18**],[**19**],[**14**],[**20**],...). For historical aspects of discrete Painlevé equations, we refer the reader to the review of Grammaticos-Ramani [**4**].

Among many others, we mention here the geometric approach proposed by Sakai [**20**] for a class of discrete Painlevé equations arising from rational surfaces. Each equation in this class is defined by the group of Cremona transformations on a certain family of surfaces obtained from the projective plane $\mathbb{P}^2(\mathbb{C})$ by blowing-up. According to the types of rational surfaces, those discrete Painlevé equations are classified in terms of affine root systems. Also, their symmetries are described by means of affine Weyl groups. The *elliptic difference Painlevé equation*, which is regarded as the master equation for all discrete Painlevé equations of this class, is a discrete dynamical system defined on a family of surfaces parametrized by the 9-point configurations in general position in $\mathbb{P}^2(\mathbb{C})$; the corresponding group of Cremona transformations is the affine Weyl group of type $E_8^{(1)}$. As we have shown in [**7**], this system of difference equations can be transformed into the eight-parameter discrete Painlevé equation of Ohta-Ramani-Grammaticos [**15**], constructed from a completely different viewpoint of bilinear equations for the $\tau$-functions on the $E_8$ lattice. It is also known by [**7**] that the elliptic difference Painlevé equation has special Riccati type solutions obtained by linearization to the *elliptic difference hypergeometric equation*. This gives a new perspective of nonlinear special functions to the elliptic hypergeometric functions which have been studied for instance by Frenkel-Turaev [**3**] in the context of elliptic 6-$j$ symbols and by Spiridonov-Zhedanov [**21**] in the theory of biorthogonal rational functions on elliptic grids.

Generalizing the geometric approach to the elliptic difference Painlevé equations, in this paper we investigate the configuration space $\mathbb{X}_{m,n}$ of $n$ points $p_1, \dots, p_n$ in general position in the projective space $\mathbb{P}^{m-1}(\mathbb{C})$. It is well-known [**2**] that the Weyl group $W_{m,n}$ associated with the tree $T_{2,m,n-m}$ can be realized as a group of birational transformations on the configuration space $\mathbb{X}_{m,n}$. Through the $W_{m,n}$-equivariant projection $\mathbb{X}_{m,n+1} \to \mathbb{X}_{m,n}$ that maps $[p_1, \dots, p_n, q]$ to $[p_1, \dots, p_n]$, from the birational action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ we obtain a realization of the Weyl group $W_{m,n}$ as a group of Cremona transformations on $q \in \mathbb{P}^{m-1}(\mathbb{C})$ parameterized by the configuration space $\mathbb{X}_{m,n}$. Note that in the case when $(m,n) = (3,9), (4,8)$ or $(6,9)$, the Weyl group $W_{m,n}$ is the affine Weyl group of type $E_8^{(1)}$, $E_7^{(1)}$ or $E_8^{(1)}$, respectively; this group $W_{m,n} = W(E_l^{(1)})$ decomposes into the semidirect product of the root lattice $Q(E_l)$ and the finite Weyl group $W(E_l)$. In each of the three cases, through the birational

action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$, the lattice part of the affine Weyl group provides a *discrete Painlevé system* on $\mathbb{P}^{m-1}(\mathbb{C})$ with parameter space $\mathbb{X}_{m,n}$. The discrete Painlevé system of type $(3,9)$ thus obtained contains the three discrete Painlevé equations, elliptic, trigonometric and rational, with $W(E_8^{(1)})$ symmetry in Sakai's table.

In this framework of configuration spaces, in Section 4 we construct a $W_{m,n}$-equivariant meromorphic mapping $\varphi_{m,n} : \mathfrak{h}_{m,n} \dashrightarrow \mathbb{X}_{m,n}$ by means of elliptic functions, where $\mathfrak{h}_{m,n}$ denotes the Cartan subalgebra of the Kac-Moody Lie algebra associated with the tree $T_{2,m,n-m}$. If we regard the birational $W_{m,n}$-action on $\mathbb{X}_{m,n}$ as a system of functional equations for the coordinate functions, a 'canonical' elliptic solution is provided by the meromorphic mapping $\varphi_{m,n}$. Its image also specifies a $W_{m,n}$-stable class of $n$-point configurations in $\mathbb{P}^{m-1}(\mathbb{C})$ in which the $n$ points are on an elliptic curve. By restricting the point configurations to this class, from the birational Weyl group action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ we obtain a realization of $W_{m,n}$ as a group of Cremona transformations on $\mathbb{P}^{m-1}(\mathbb{C})$ parametrized by elliptic functions, which we call the *elliptic Cremona system* of type $(m,n)$. In Section 5 we develop a theory of $\tau$-functions for this elliptic Cremona system of type $(m,n)$, and show that it is translated into a system of bilinear equations of Hirota-Miwa type for the $\tau$-functions on the lattice. After that we reconsider the case of the elliptic difference Painlevé system of type $(3,9)$ in the scope of the general setting of this paper. There we give explicit description for some of the discrete time evolutions, in terms of homogeneous coordinates in Section 6, and in the language of geometry of plane curves in Section 7.

The $\tau$-function approach developed in this paper can be applied effectively to the study of special hypergeometric solutions of the elliptic Painlevé equation and its degenerations. Also, it is an important problem to *complete* the framework of $\mathbb{X}_{m,n}$ of point configurations in general position, so that it should contain all reasonable degenerate configurations as in Sakai's table. These subjects will be investigated in our subsequent papers.

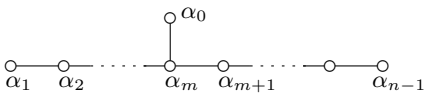
## 2. Point configurations and Cremona transformations

Let $\mathbb{X}_{m,n}$ be the configuration space of $n$ points in general position in $\mathbb{P}^{m-1}(\mathbb{C})$ $(n > m > 1)$. We say that an $n$-tuple of points $(p_1, \dots, p_n)$ in $\mathbb{P}^{m-1}(\mathbb{C})$ is *in general position* if $p_1, \dots, p_n$ are mutually distinct, and $\#(H \cap \{p_1, \dots, p_n\}) < m$ for any hyperplane $H$ in $\mathbb{P}^{m-1}(\mathbb{C})$. We denote by $[p_1, \dots, p_n]$ the corresponding *configuration*, namely, the equivalence class of $(p_1, \dots, p_n)$ under the diagonal $PGL_m(\mathbb{C})$-action. By fixing a system of homogeneous coordinates for $\mathbb{P}^{m-1}(\mathbb{C})$, the configuration space $\mathbb{X}_{m,n}$ may be identified with the double coset space

$$\mathbb{X}_{m,n} = GL_m(\mathbb{C}) \backslash \mathrm{Mat}^*_{m,n}(\mathbb{C}) / T_n, \tag{1}$$

where $\mathrm{Mat}^*_{m,n}(\mathbb{C})$ stands for the space of all $m \times n$ complex matrices whose $m \times m$ minor determinants are all nonzero, and $T_n = (\mathbb{C}^*)^n$ for the diagonal subgroup of $GL_n(\mathbb{C})$. The configuration space $\mathbb{X}_{m,n}$ has the structure of an affine algebraic variety, isomorphic to a Zariski open subset of $\mathbb{C}^{(m-1)(n-m-1)}$ (see [**23**], for instance). Also, it is known [**1**], [**2**] that the Weyl group associated with the tree

(2) $T_{2,m,n-m}$ :




acts birationally on $\mathbb{X}_{m,n}$. This Weyl group $W_{m,n} = W(T_{2,m,n-m})$ is generated by the *simple reflections* $s_0, s_1, \ldots, s_{n-1}$ with the following fundamental relations.

(3) $W_{m,n} = \langle s_0, s_1, \ldots, s_{n-1} \rangle$ :

$$s_i^2 = 1 \qquad \alpha_i \quad \alpha_j$$
$$s_i s_j = s_j s_i \qquad \circ \quad \circ$$
$$s_i s_j s_i = s_j s_i s_j \qquad \circ\!\!-\!\!\circ$$

As we will recall below, $W_{m,n}$ is realized as a group of birational transformations of $\mathbb{X}_{m,n}$ by the standard Cremona transformations with respect to $m$ points among $p_1, \ldots, p_n$.

Given a set of $m$ points $p_1, \ldots, p_m$ in general position, choose a system of homogeneous coordinates $x = (x_1, \ldots, x_m)$ such that

$$p_1 = (1 : 0 : \ldots : 0), \quad p_2 = (0 : 1 : \ldots : 0), \quad \ldots, \quad p_m = (0 : \ldots : 0 : 1). \tag{4}$$

Then the *standard Cremona transformation* with respect to $(p_1, \ldots, p_m)$ is the birational transformation $p \to \widetilde{p}$ of $\mathbb{P}^{m-1}(\mathbb{C})$ defined by $\widetilde{p} = \left(x_1^{-1} : \ldots : x_m^{-1}\right)$ for any $p = (x_1 : \ldots : x_m)$ with $x_i \neq 0$ $(i = 1, \ldots, m)$. Note that this transformation depends on the choice of homogeneous coordinates, and is determined only up to the action of $(\mathbb{C}^*)^m$. The birational (right) action of $W_{m,n}$ on $\mathbb{X}_{m,n}$ is then defined as follows. Firstly, the symmetric group $\mathfrak{S}_n$ acts on $\mathbb{X}_{m,n}$ by the permutation of $n$ points:

$$[p_1, \ldots, p_n].\sigma = [p_{\sigma(1)}, \ldots, p_{\sigma(n)}] \qquad (\sigma \in \mathfrak{S}_n). \tag{5}$$

The adjacent transpositions $s_j = (j, j+1)$ $(j = 1, \ldots, n-1)$ provide the simple reflections attached to the subdiagram of type $A_{n-1}$ in $T_{2,m,n-m}$. The remaining simple reflection $s_0$ is given by the (well-defined) birational transformation

$$[p_1, \ldots, p_n].s_0 = [p_1, \ldots, p_m, \widetilde{p}_{m+1}, \ldots, \widetilde{p}_n], \tag{6}$$

in terms of the standard Cremona transformation $p \to \widetilde{p}$ with respect to the first $m$ points $(p_1, \ldots, p_m)$. These birational transformations $s_0, s_1, \ldots, s_{n-1}$ in fact satisfy the fundamental relations for the simple reflections of $W_{m,n}$. We also remark that, for each subset $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$ of mutually distinct $m$ indices, the standard Cremona transformation with respect to $(p_{j_1}, \ldots, p_{j_m})$ is determined as $\mathrm{cr}_{j_1,\ldots,j_m} = \sigma s_0 \sigma^{-1} \in W_{m,n}$ by a permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma(a) = j_a$ for $a = 1, \ldots, m$.

The right birational action of $W_{m,n}$ on $\mathbb{X}_{m,n}$ induces a left action of $W_{m,n}$ on the field $\mathcal{K}(\mathbb{X}_{m,n})$ of rational functions on $\mathbb{X}_{m,n}$ as a group of automorphisms: For each

$\varphi \in \mathcal{K}(\mathbb{X}_{m,n})$ and $w \in W_{m,n}$, we define $w(\varphi) \in \mathcal{K}(\mathbb{X}_{m,n})$ by

$$w(\varphi)([p_1,\dots,p_n]) = \varphi([p_1,\dots,p_n].w) \tag{7}$$

for any generic $[p_1,\dots,p_n] \in \mathbb{X}_{m,n}$. Let us consider the set $\mathcal{U}_{m,n}$ of all matrices $U \in \mathrm{Mat}_{m,n}(\mathbb{C})^*$ of the form

$$U = \begin{pmatrix} 1 & \dots & 0 & 0 & 1 & u_{1,m+2} & \dots & u_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 0 & 1 & u_{m-1,m+2} & \dots & u_{m-1,n} \\ 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}. \tag{8}$$

It is easily shown that each $(GL_m(\mathbb{C}), T_n)$-orbit in $\mathrm{Mat}^*_{m,n}(\mathbb{C})$ intersects with $\mathcal{U}_{m,n}$ at one point. By using this transversal $\mathcal{U}_{m,n} \xrightarrow{\sim} GL_m(\mathbb{C})\backslash\mathrm{Mat}_{m,n}(\mathbb{C})^*/T_n$, we identify $\mathbb{X}_{m,n}$ with a Zariski open subset of the affine space $\mathbb{C}^{(m-1)(n-m-1)}$ with canonical coordinates $u = (u_{i,j})_{1\le i\le m-1; m+2\le j\le n}$. Through the isomorphism $\mathcal{K}(\mathbb{X}_{m,n}) \xrightarrow{\sim} \mathbb{C}(u)$, the action of $W_{m,n}$ on $\mathcal{K}(\mathbb{X}_{m,n})$ can be described explicitly in terms of the $u$ variables. The following table shows how the simple reflections $s_k$ $(k = 0, 1, \dots, n-1)$ transform the coordinates $u_{i,j}$ $(i = 1,\dots,m-1; j = m+2,\dots,n)$:

$$\begin{aligned}
&k = 0: && s_0(u_{ij}) &= \frac{1}{u_{ij}}, \\
&k = 1,\dots,m-2: && s_k(u_{ij}) &= u_{s_k(i),j}, \\
&k = m-1: && s_{m-1}(u_{ij}) &= \begin{cases} \dfrac{u_{ij}}{u_{m-1,j}} & (i = 1,\dots,m-2), \\ \dfrac{1}{u_{m-1,j}} & (i = m-1), \end{cases} \\
&k = m: && s_m(u_{ij}) &= 1 - u_{ij}, \\
&k = m+1: && s_{m+1}(u_{ij}) &= \begin{cases} \dfrac{1}{u_{i,m+2}} & (j = m+2), \\ \dfrac{u_{ij}}{u_{i,m+2}} & (j = m+3,\dots,n), \end{cases} \\
&k = m+2,\dots,n-1: && s_k(u_{ij}) &= u_{i,s_k(j)},
\end{aligned} \tag{9}$$

where $s_k(i)$ stands for the action of the adjacent transposition $(k, k+1)$ on the index $i \in \{1,\dots,n\}$. ¿ From this representation, for each $w \in W_{m,n}$ we obtain a family of rational functions

$$w(u_{i,j}) = S^w_{i,j}(u) \qquad (i = 1,\dots,m-1; j = m+2,\dots,n) \tag{10}$$

in the $u$ variables; these functions satisfy the consistency relations

$$S^1_{i,j}(u) = u_{ij}, \qquad S^{ww'}_{i,j}(u) = S^{w'}_{i,j}(S^w(u)) \tag{11}$$

for any $i, j$ and $w, w' \in W_{m,n}$.

If we regard the $u$ variables as dependent variables (unknown functions), (10) or equivalently (9) can be regarded as a system of functional equations for $u_{ij}$. Typically,

we take a vector space $V$ with canonical coordinates $t = (t_1, \dots, t_N)$, assuming that $W_{m,n}$ acts linearly on $V$ (from the right). If we regard the coordinate functions $t_j$ of $V$ as the independent variables, then a solution of the system (10) is nothing but a $W_{m,n}$-equivariant mapping $\varphi : V \to \mathbb{X}_{m,n}$. In Section 4, we construct an elliptic solution of the system (10) in this sense, with $V$ being the Cartan subalgebra $\mathfrak{h}_{m,n}$ of the Kac-Moody Lie algebra associated with $T_{2,m,n-m}$.

## 3. Tracing the Cremona transformations

Given a generic configuration $[p_1, \dots, p_n]$ of $n$ points, let us ask how a general point of $\mathbb{P}^{m-1}(\mathbb{C})$, as well as the configuration itself, is transformed by a successive application of standard Cremona transformations. In what follows, by a *Cremona transformation* we mean a birational transformation of $\mathbb{P}^{m-1}(\mathbb{C})$ obtained by a successive application of standard Cremona transformations.

We now consider the relative situation with respect to the projection $\pi : \mathbb{X}_{m,n+1} \to \mathbb{X}_{m,n}$ that maps $[p_1, \dots, p_n, p_{n+1}]$ to $[p_1, \dots, p_n]$. This projection is $W_{m,n}$-equivariant relative to the inclusion $W_{m,n} \subset W_{m,n+1}$ of Weyl groups. We regard $\mathbb{X}_{m,n}$ as the parameter space for Cremona transformations belonging to $W_{m,n}$, and the last point $q = p_{n+1}$ as the general point in $\mathbb{P}^{m-1}(\mathbb{C})$ that should be transformed by such Cremona transformations. (This formulation has been used by [**10**] in the case $(m, n) = (3, 9)$.) Then, our question is how to describe $[p_1, \dots, p_n, q].w$ for each $[p_1, \dots, p_n, q] \in \mathbb{X}_{m,n+1}$ and $w \in W_{m,n}$. By using the coordinates

$$\widetilde{U} = \begin{pmatrix} 1 & \dots & 0 & 0 & 1 & u_{1,m+2} & \dots & u_{1,n} & z_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 1 & u_{m-1,m+2} & \dots & u_{m-1,n} & z_{m-1} \\ 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix} \tag{12}$$

for $\mathcal{U}_{m,n+1}$, we parametrize the configurations $[p_1, \dots, p_n] \in \mathbb{X}_{m,n}$ and the general points $q \in \mathbb{P}^{m-1}(\mathbb{C})$ as

$$\begin{aligned} &p_1 = (1 : 0 : \dots : 0), \ \dots, \ \ p_m = (0 : \dots : 0 : 1), \ \ p_{m+1} = (1 : \dots : 1 : 1), \\ &p_j = (u_{1,j} : \dots : u_{m-1,j} : 1) \ \ (j = m+2, \dots, n), \quad q = (z_1 : \dots : z_{m-1} : 1). \end{aligned} \tag{13}$$

Then for any $w \in W_{m,n}$ the configuration $[p_1, \dots, p_n, q].w = [\widetilde{p}_1, \dots, \widetilde{p}_n, \widetilde{q}]$ is given by

$$\begin{aligned} \widetilde{p}_j &= p_j && (j = 1, \dots, m+1), \\ \widetilde{p}_j &= (w(u_{1,j}) : \dots : w(u_{m-1,j}) : 1) && (j = m+2, \dots, n), \\ \widetilde{q} &= (w(z_1) : \dots : w(z_{m-1}) : 1). \end{aligned} \tag{14}$$

In this sense, for each $w \in W_{m,n}$ the corresponding Cremona transformation of $q = p_{n+1}$ is determined as

$$w(z_i) = R_i^w(u; z) \qquad (i = 1, \dots, m-1), \tag{15}$$

in terms of rational functions $R_i^w(u;z)$ in the variables $u=(u_{ij})_{1\le i\le m-1;m+2\le j\le n}$ and $z=(z_1,\dots,z_{m-1})$. Note also that $R_i^w(u;z)$ satisfy

$$R_i^1(u;z)=z_i,\qquad R_i^{ww'}(u;z)=R_i^{w'}(S^w(u);R^w(u;z)) \tag{16}$$

for any $i$ and $w,w'\in W_{m,n}$. As we will see below, these $R_i^w(u;z)$, regarded as rational functions in the variable $z=(z_1,\dots,z_{m-1})$, have a characteristic property concerning their multiplicities of zero at $p_1,\dots,p_n$.

Consider a free $\mathbb{Z}$-module

$$L_{m,n}=\mathbb{Z}e_0\oplus\mathbb{Z}e_1\oplus\cdots\oplus\mathbb{Z}e_n \tag{17}$$

of rank $n+1$ with basis $\{e_0,e_1,\dots,e_n\}$, and define a symmetric bilinear form $(\,|\,):L_{m,n}\times L_{m,n}\to\mathbb{Z}$ by

$$\begin{gathered}(e_0\,|\,e_0)=-(m-2),\quad (e_j\,|\,e_j)=1\quad (j=1,\dots,n),\\ (e_i\,|\,e_j)=0\qquad (i,j=0,1,\dots,n;\ i\ne j).\end{gathered} \tag{18}$$

This lattice $L_{m,n}$ admits a natural linear action of the Weyl group $W_{m,n}$ defined by

$$s_k.\Lambda=\Lambda-(h_k\,|\,\Lambda)\,h_k\qquad (\Lambda\in L_{m,n}) \tag{19}$$

for each $k=0,1,\dots,n-1$, where

$$h_0=e_0-e_1-\cdots-e_m,\quad h_k=e_k-e_{k+1}\quad (k=1,\dots,n). \tag{20}$$

Note that $(h_j\,|\,h_j)=2$ $(j=0,1,\dots,n-1)$ and that $((h_i\,|\,h_j))_{i,j=0}^{n-1}$ is the (generalized) Cartan matrix associated with the tree $T_{2,m,n-m}$. For each

$$\Lambda=de_0-\nu_1e_1-\cdots-\nu_ne_n\in L_{m,n}\qquad (d,\nu_1,\dots,\nu_n\in\mathbb{Z}) \tag{21}$$

we denote by $L(\Lambda)$ the vector space over $\mathcal{K}(\mathbb{X}_{m,n})$ consisting of all homogeneous polynomials $f(x)\in\mathcal{K}(\mathbb{X}_{m,n})[x]$ of degree $d$ in $m$ variables $x=(x_1,\dots,x_m)$ that have a zero of multiplicity $\ge\nu_j$ at each $p_j$ $(j=1,\dots,n)$:

$$\deg f(x)=d,\qquad \mathrm{ord}_{p_j}f(x)\ge\nu_j\quad (j=1,\dots,n). \tag{22}$$

Here we regard $x=(x_1,\dots,x_m)$ as the homogeneous coordinate system for $\mathbb{P}^{m-1}(\mathcal{K}(\mathbb{X}_{m,n}))$ such that $(z_1:\dots:z_{m-1}:1)=(a_1x_1:\cdots:a_mx_m)$ for some nonzero constants $a_i\in\mathcal{K}(\mathbb{X}_{m,n})$ $(i=1,\dots,m-1)$. Then we have

***Theorem 3.1.*** — (1) *Let $M_{m,n}=W_{m,n}\{e_1,\dots,e_n\}$ be the $W_{m,n}$-orbit of $\{e_1,\dots,e_n\}$ in $L_{m,n}$. Then for any $\Lambda\in M_{m,n}$, one has* $\dim_{\mathcal{K}(\mathbb{X}_{m,n})}L(\Lambda)=1$.
(2) *Given any element $w\in W_{m,n}$, take nonzero polynomials*

$$F_i(x)\in L(w.e_i),\quad G_i(x)\in L(ws_0.e_i)\qquad (i=1,\dots,m). \tag{23}$$

*Then one has*

$$(w(z_1):\dots:w(z_{m-1}):1)=\left(c_1\frac{G_1(x)}{F_1(x)}:\cdots:c_m\frac{G_m(x)}{F_m(x)}\right) \tag{24}$$

*for some nonzero constants $c_i\in\mathcal{K}(\mathbb{X}_{m,n})$ $(i=1,\dots,m)$.*

This theorem can be proved by decomposing each $w \in W_{m,n}$ into a product of simple reflections $w = s_{j_1} \cdots s_{j_p}$, and then by lifting each $s_{j_k}$ to the level of homogeneous coordinates. We remark that there is no canonical way *a priori* to define the action of $s_j$ on homogeneous polynomials. This is the reason why we cannot specify the choice of $F_i$ and $G_i$. We will return to this point later in Section 5 in the context of $\tau$-functions for the elliptic Cremona system. In the case $(m,n) = (3,9)$, for any $\Lambda \in M_{m,n}$ a nontrivial element in $L(\Lambda)$ can be constructed as a certain interpolation determinant(see Section 6).

As we have seen above, the birational action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ can be expressed in the form

$$w(u_{i,j}) = S^w_{i,j}(u), \qquad w(z_i) = R^w_i(u;z). \tag{25}$$

Formula (25) can be thought of as a system of functional equations for the dependent variables $z = (z_1, \dots, z_{m-1})$ including the $u$ variables as parameters. Theorem 3.1 then implies that such a system of equations can be expressed as

$$w(z_i) = R^w_i(z) = \frac{c_i}{c_m} \frac{G^w_i(x) F^w_m(x)}{F^w_i(x) G^w_m(x)} \qquad (i = 1, \dots, m-1) \tag{26}$$

for each $w \in W_{m,n}$, by means of homogeneous polynomials $F^w_i \in L(w.e_i)$ and $G^w_i \in L(ws_0.e_i)$ that are characterized by the degree and the multiplicities of zero at $p_1, \dots, p_n$. (The dependence on the $u$ variables is suppressed in this formula.) Note that any abelian subgroup of the Weyl group $W_{m,n}$ gives rise to a commuting family of birational transformations on $\mathbb{P}^{m-1}(\mathbb{C})$ parameterized by configurations of $n$ points. Such a birational dynamical system (for the $z$-variables) could be called a *discrete Painlevé system* associated with point configurations in $\mathbb{P}^{m-1}(\mathbb{C})$.

When the number $4-(m-2)(n-m-2)$ has the sign $+, 0$, or $-$, the root system associated with the tree $T_{2,m,n-m}$ is of finite type, of affine type, or of indefinite type, respectively, in the sense of [**6**]. In particular there are three cases $(m,n) = (3,9), (4,8)$ and $(6,9)$ of affine type; the corresponding root systems are of type $E^{(1)}_8$, $E^{(1)}_7$ and $E^{(1)}_8$, respectively. In these three cases, the configuration of $n$ points can be parametrized generically by means of elliptic functions. Note also that the Weyl group $W_{m,n} = W(E^{(1)}_l)$ is then expressed as the semidirect product of the root lattice $Q(E_l)$ of rank $l$ and the finite Weyl group $W(E_l)$ acting on it. We call the discrete dynamical system arising from the lattice part of $W_{m,n}$ the *elliptic difference Painlevé equation* of type $(m,n)$.

## 4. Linearization of the $W_{m,n}$-action in terms of elliptic functions

Let $\mathfrak{h}_{m,n} = L_{m,n} \otimes_{\mathbb{Z}} \mathbb{C}$ the complexification of the lattice $L_{m,n}$. In this section we construct a $W_{m,n}$-equivariant meromorphic mapping $\varphi_{m,n} : \mathfrak{h}_{m,n} \dashrightarrow \mathbb{X}_{m,n}$ by means of elliptic functions. This mapping specifies a class of configurations of $n$ points on

an elliptic curve in $\mathbb{P}^{m-1}(\mathbb{C})$, which is preserved by the action of $W_{m,n}$. In order to simplify the presentation, we assume $m \geq 3$. In this case the symmetric bilinear form $(\,|\,) : \mathfrak{h}_{m,n} \times \mathfrak{h}_{m,n} \to \mathbb{C}$ is nondegenerate, and $\mathfrak{h}_{m,n}$ is identified with the Cartan subalgebra of the Kac-Moody Lie algebra associated with the tree $T_{2,m,n-m}$.

We define the linear functions $\varepsilon_j$ $(j = 0, 1, \ldots, n)$ and $\alpha_j$ $(j = 0, 1, \ldots, n-1)$ on $\mathfrak{h}_{m,n}$ by

$$\varepsilon_j(h) = (e_j \,|\, h), \quad \alpha_j(h) = (h_j \,|\, h) \qquad (h \in \mathfrak{h}_{m,n}). \tag{27}$$

We regard $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) : \mathfrak{h}_{m,n} \xrightarrow{\sim} \mathbb{C}^{n+1}$ as the canonical coordinates for $\mathfrak{h}_{m,n}$. The linear functions $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are the *simple roots* of the root system associated with $T_{2,m,n-m}$. Note also that the dual space $\mathfrak{h}^*_{m,n} = \mathbb{C}\varepsilon_0 \oplus \mathbb{C}\varepsilon_1 \oplus \cdots \oplus \mathbb{C}\varepsilon_n$ has the induced symmetric bilinear form: $(\varepsilon_i \,|\, \varepsilon_j) = (e_i \,|\, e_j)$ for any $i, j \in \{0, 1, \ldots, n\}$. The Weyl group $W_{m,n}$ acts on $\mathfrak{h}_{m,n}$ and $\mathfrak{h}^*_{m,n}$ in a standard way: For each $k = 0, 1, \ldots, n-1$,

$$s_k.h = h - \langle h, \alpha_k \rangle h_k \quad (h \in \mathfrak{h}_{m,n}), \quad s_k.\lambda = \lambda - \langle h_k, \lambda \rangle \alpha_k \quad (\lambda \in \mathfrak{h}^*_{m,n}), \tag{28}$$

where $\langle\ ,\ \rangle : \mathfrak{h}_{m,n} \times \mathfrak{h}^*_{m,n} \to \mathbb{C}$ is the canonical pairing. When we consider the right action of $W_{m,n}$ on $\mathfrak{h}_{m,n}$, we use the convention $h.w = w^{-1}.h$ for $h \in \mathfrak{h}_{m,n}$ and $w \in W_{m,n}$.

We now fix a nonzero holomorphic function on $\mathbb{C}$, denoted by $[x]$, assuming that $[x]$ is odd ($[-x] = -[x]$ for any $x \in \mathbb{C}$), and satisfies the *Riemann relation*:

$$[x+y][x-y][u+v][u-v] = [x+u][x-u][y+v][y-v] - [x+v][x-v][y+u][y-u] \tag{29}$$

for any $x, y, u, v \in \mathbb{C}$. If this condition is satisfied, the set $\Omega = \{a \in \mathbb{C} \mid [a] = 0\}$ of zeros of $[x]$ forms a $\mathbb{Z}$-submodule of $\mathbb{C}$, and the function $[x]$ is quasi-periodic with respect to $\Omega$. There are three classes of such functions; elliptic, trigonometric and rational, according to the rank of $\Omega$:

| | | |
|---|---|---|
| Elliptic case: | $[x] = c\, e^{ax^2} \sigma(x; \Omega)$ | $(\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, |
| Trigonometric case: | $[x] = c\, e^{ax^2} \sin(\pi x/\omega_0)$ | $(\Omega = \mathbb{Z}\omega_0)$, |
| Rational case: | $[x] = c\, e^{ax^2} x$ | $(\Omega = \{0\})$. |

Here $\sigma(x; \Omega)$ denotes the Weierstrass sigma function associated with the period lattice $\Omega$. In the context of discrete Painlevé equations, these three cases correspond to the three types of difference equations (elliptic, trigonometric and rational). We will use this symbol $[x]$ whenever the three cases can be treated simultaneously.

Taking constants $\lambda \in \mathbb{C}$ and $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{C}^m$ such that $[\lambda] \neq 0$ and $[\mu_i - \mu_j] \neq 0$ $(1 \leq i < j \leq m)$, we define a holomorphic mapping $p_{\lambda,\mu} : \mathbb{C} \to \mathbb{P}^{m-1}(\mathbb{C})$ by

$$\begin{aligned} p_{\lambda,\mu}(t) &= \left( \frac{[\lambda + \mu_1 - t]}{[\lambda][\mu_1 - t]} : \ldots : \frac{[\lambda + \mu_m - t]}{[\lambda][\mu_m - t]} \right) \\ &= \left( [\lambda + \mu_1 - t] \prod_{k=2}^{m} [\mu_k - t] : \ldots : [\lambda + \mu_m - t] \prod_{k=1}^{m-1} [\mu_k - t] \right) \end{aligned} \tag{30}$$

for any $t \in \mathbb{C}$. Thanks to the quasi-periodicity of $[x]$, this mapping induces a holomorphic mapping $\overline{p}_{\lambda,\mu} : E_\Omega = \mathbb{C}/\Omega \to \mathbb{P}^{m-1}(\mathbb{C})$. We denote by $C_{\lambda,\mu} = \overline{p_{\lambda,\mu}(\mathbb{C})} \subset \mathbb{P}^{m-1}(\mathbb{C})$ the curve obtained as the closure of the image of $p_{\lambda,\mu}$. Note that this curve passes the $m$ coordinate origins in $\mathbb{P}^{m-1}(\mathbb{C})$; in fact we have

$$p_{\lambda,\mu}(\mu_1) = (1:0:\ldots:0), \quad \ldots, \quad p_{\lambda,\mu}(\mu_m) = (0:\ldots:0:1). \tag{31}$$

We also remark that $C_{\lambda,\mu}$ is a smooth elliptic curve when $\operatorname{rank}\Omega = 2$, and a singular elliptic curve with a node (resp. a cusp) when $\operatorname{rank}\Omega = 1$ (resp. $\operatorname{rank}\Omega = 0$) at $(1:\ldots:1)$.

We now consider a configuration of $n$ points on $C_{\lambda,\mu} \subset \mathbb{P}^{m-1}(\mathbb{C})$ defined as

$$[p_1, \ldots, p_n] = [p_{\lambda,\mu}(\varepsilon_1), \ldots, p_{\lambda,\mu}(\varepsilon_n)] \tag{32}$$

by the coordinates $\varepsilon_j \in \mathbb{C}$ $(j = 1, \ldots, n)$. Setting $\varepsilon_0 = \lambda + \mu_1 + \cdots + \mu_m$, we assume that the parameters $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ are generic in the sense

$$\begin{aligned} &[\varepsilon_i - \varepsilon_j] \neq 0, && (1 \le i < j \le n), \\ &[\varepsilon_0 - \varepsilon_{j_1} - \cdots - \varepsilon_{j_m}] \neq 0 && (1 \le j_1 < \ldots < j_m \le n). \end{aligned} \tag{33}$$

Then by the Frobenius formula

$$\det\left(\frac{[\lambda + x_i - y_j]}{[\lambda][x_i - y_j]}\right)_{i,j=1}^{m} = \frac{[\lambda + \sum_{k=1}^{m}(x_k - y_k)] \prod_{1\le i<j\le m} [x_j - x_i][y_i - y_j]}{[\lambda] \prod_{1\le i,j\le m} [x_i - y_j]} \tag{34}$$

for the function $[x]$, we see that the configuration $[p_1, \ldots, p_n]$ defined as above is in general position, and that its $u$ coordinates are given explicitly by

$$\begin{gathered} u_{i,j} = u_{i,j}(\varepsilon) = \frac{[\alpha_0 + \varepsilon_{m,m+1}][\varepsilon_{i,m+1}]}{[\varepsilon_{m,m+1}][\alpha_0 + \varepsilon_{i,m+1}]} \frac{[\alpha_0 + \varepsilon_{i,j}][\varepsilon_{m,j}]}{[\varepsilon_{i,j}][\alpha_0 + \varepsilon_{m,j}]} \\ (i = 1, \ldots, m-1; j = m+2, \ldots, n) \end{gathered} \tag{35}$$

where $\varepsilon_{i,j} = \varepsilon_i - \varepsilon_j$ for $i, j \in \{1, \ldots, n\}$, and $\alpha_0 = \varepsilon_0 - \varepsilon_1 - \cdots - \varepsilon_m$. Note that, by the passage to the double coset space $\mathbb{X}_{m,n}$, the dependence of the configuration on the parameters $\lambda$ and $\mu = (\mu_1, \ldots, \mu_m)$ has been confined in the parameter $\varepsilon_0 = \lambda + \mu_1 + \cdots + \mu_m$. Observe also that these functions $u_{i,j}(\varepsilon)$ are $\Omega$-periodic in all the variables $\varepsilon_j$ $(j = 0, 1, \ldots, n)$.

Under the identification of the parameters $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ with the canonical coordinates for $\mathfrak{h}_{m,n}$, the construction described above implies two meromorphic mappings

$$\begin{aligned} \varphi_{m,n} : &\quad \mathfrak{h}_{m,n} = L_{m,n} \otimes_{\mathbb{Z}} \mathbb{C} &&\dashrightarrow \mathbb{X}_{m,n}, \quad \text{and} \\ \overline{\varphi}_{m,n} : &\quad E_{m,n} = L_{m,n} \otimes_{\mathbb{Z}} (\mathbb{C}/\Omega) &&\dashrightarrow \mathbb{X}_{m,n}. \end{aligned} \tag{36}$$

Note that, when $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) \in \mathfrak{h}_{m,n}$ is generic, the corresponding configuration $\varphi_{m,n}(\varepsilon) = [p_1, \ldots, p_n] \in \mathbb{X}_{m,n}$ is realized by an $n$-tuple of points on the elliptic curve

$C_{\lambda,\mu} \subset \mathbb{P}^{m-1}(\mathbb{C})$ for any choice of $\lambda, \mu = (\mu_1, \dots, \mu_m)$ such that $\varepsilon_0 = \lambda + \mu_1 + \cdots + \mu_m$. The meromorphic mapping $\varphi_{m,n}$ is equivariant by construction under the action of the symmetric group $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \rangle$. Also, the equivariance with respect to $s_0$ is clearly seen by the explicit formula (35) for the $u$ coordinates. Hence we obtain

***Theorem 4.1.*** — *The meromorphic mapping $\varphi_{m,n} : \mathfrak{h}_{m,n} \dashrightarrow \mathbb{X}_{m,n}$ (resp. $\overline{\varphi}_{m,n} : E_{m,n} \dashrightarrow \mathbb{X}_{m,n}$ ) defined as above is $W_{m,n}$-equivariant with respect to the canonical linear action of the Weyl group $W_{m,n}$ on $\mathfrak{h}_{m,n}$ and its nonlinear birational action on $\mathbb{X}_{m,n}$ by Cremona transformations.*

This theorem means that the $\Omega$-periodic functions (35) satisfy the equations

$$u_{i,j}(w(\varepsilon)) = S^w_{i,j}(u(\varepsilon)) \qquad (i = 1, \dots, m-1; j = m+2, \dots, n) \tag{37}$$

for any $w \in W_{m,n}$, where $w(\varepsilon) = (w(\varepsilon_0), w(\varepsilon_1), \dots, w(\varepsilon_n))$. Namely, (35) give a solution of the system of functional equations (10) for the $u$ variables.

As we discussed in the previous section, in the relative situation $\mathbb{X}_{m,n+1} \to \mathbb{X}_{m,n}$, the birational action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ is expressed as

$$w(u_{i,j}) = S^w_{i,j}(u), \qquad w(z_i) = R^w_i(u; z). \tag{38}$$

By substituting $u_{ij} = u_{ij}(\varepsilon)$ as in (35), we obtain a realization of $W_{m,n}$ as a group of automorphisms of the field $\mathcal{M}(E_{m,n})(z)$ of rational functions in $z$ with coefficients in the field of meromorphic functions on $E_{m,n}$:

$$w(z_i) = R^w_i(\varepsilon; z) \qquad (i = 1, \dots, m-1), \tag{39}$$

where we have used the notation $R^w_i(\varepsilon; z)$ again instead of $R^w_i(u(\varepsilon); z)$. We will refer to this system (39) of functional equations for the $z$ variables as the *elliptic Cremona system* of type $(m, n)$. The action of the simple reflection $s_k$ $(k = 0, 1, \dots, n-1)$ on the $z$ variables is now given as follows:

$$\begin{aligned}
&k = 0: && s_0(z_i) &&= \frac{1}{z_i}, \\
&k = 1, \dots, m-2: && s_k(z_i) &&= z_{s_k(i)}, \\
&k = m-1: && s_{m-1}(z_i) &&= \begin{cases} \dfrac{z_i}{z_{m-1}} & (i = 1, \dots, m-2), \\ \dfrac{1}{z_{m-1}} & (i = m), \end{cases} \\
&k = m: && s_m(z_i) &&= 1 - z_i, \\
&k = m+1: && s_{m+1}(z_i) &&= \frac{z_i}{u_{i,m+2}(\varepsilon)} \\
&k = m+2, \dots, n-1: && s_k(z_i) &&= z_i.
\end{aligned} \tag{40}$$

Note that the dependence on the $\varepsilon$ variables enters in the action of $s_{m+1}$. Also, from Theorem 4.1 for the case of $\mathbb{X}_{m,n+1}$ with $\varepsilon_{n+1} = t$, we see that the functions

$$z_i(\varepsilon;t) = \frac{[\alpha_0+\varepsilon_{m,m+1}][\varepsilon_{i,m+1}]}{[\varepsilon_{m,m+1}][\alpha_0+\varepsilon_{i,m+1}]}\frac{[\alpha_0+\varepsilon_i-t][\varepsilon_m-t]}{[\varepsilon_i-t][\alpha_0+\varepsilon_m-t]} \qquad (i=1,\dots,m-1) \tag{41}$$

satisfy the functional equations

$$z_i(w(\varepsilon);t) = R_i^w(\varepsilon;z(\varepsilon;t)) \qquad (i=1,\dots,m-1), \tag{42}$$

for any $w \in W_{m,n}$. Namely, (41) provides a one-parameter family of solutions to the elliptic Cremona system (39) of type $(m,n)$. This solution will be called the *canonical solution* of the elliptic Cremona system, which corresponds to the *vertical solutions* in the context of differential Painlevé equations.

## 5. Tau functions for the elliptic Cremona system

In the case of the elliptic Cremona systems as we introduced above, the homogeneous polynomials $F_i^w(x)$ and $G_i^w(x)$ in (26) can be determined without ambiguity by means of the action of the Weyl group on the *$\tau$-functions*. In this section, we introduce a framework of $\tau$-functions for the elliptic Cremona systems, and show that the Weyl group action is translated into bilinear equations of Hirota-Miwa type for $\tau$-functions on the lattice.

By using the natural linear action of $W_{m,n}$ on $\mathfrak{h}_{m,n}^*$, we define the set of *real roots* by $\Delta_{m,n}^{\mathrm{Re}} = W_{m,n}\{\alpha_0,\alpha_1,\dots,\alpha_{n-1}\}$, and denote by $\mathbb{K} = \mathbb{C}([\alpha];\alpha\in\Delta_{m,n}^{\mathrm{Re}})$ the field of meromorphic functions on $\mathfrak{h}_{m,n}$ generated by $[\alpha]$ for all real roots $\alpha \in \Delta_{m,n}^{\mathrm{Re}}$.

In order to formulate $\tau$-functions for the elliptic Cremona system of type $(m,n)$, we use two kinds of variables (indeterminates) $f_1,\dots,f_m$, which will be identified with a system of homogeneous coordinates for $\mathbb{P}^{m-1}(\mathbb{C})$, and $\tau_1,\dots,\tau_n$ corresponding to the $n$ points $p_1,\dots,p_n$ in the configuration $[p_1,\dots,p_n]$. We denote by

$$\mathcal{L} = \mathbb{K}(f_1,\dots,f_m;\tau_1,\dots,\tau_n) \tag{43}$$

the field of rational functions in the $f$ variables and the $\tau$ variables with coefficients in $\mathbb{K}$. On this field $\mathcal{L}$, we define the automorphisms $s_0,s_1,\dots s_{n-1}$ as follows. These elements act on the coefficient field $\mathbb{K}$ through the canonical $W_{m,n}$-action on the real roots: For each $k=0,1,\dots,n-1$,

$$s_k([\alpha]) = [s_k(\alpha)] = [\alpha - \langle h_k,\alpha\rangle\alpha_k] \qquad (\alpha\in\Delta_{m,n}^{\mathrm{Re}}). \tag{44}$$

The action of $s_k$ on $\tau_j$ $(j=1,\dots,n)$ is defined by

$$\begin{aligned} s_0(\tau_j) &= \begin{cases} \tau_j\, f_j & (j=1,\dots,m),\\ \tau_j & (j=m+1,\dots,n),\end{cases}\\ s_k(\tau_j) &= \tau_{s_k(j)} \qquad (k=1,\dots,n-1;\, j=1,\dots,n). \end{aligned} \tag{45}$$

The action of $s_k$ on $f_i$ $(i = 1, \dots, m)$ is defined by

$$\begin{aligned} s_0(f_i) &= \frac{1}{f_i} \\ s_k(f_i) &= f_{s_k(i)} \qquad (k = 1, \dots, m-1), \\ s_k(f_i) &= f_i \qquad (k = m+1, \dots, n), \end{aligned} \tag{46}$$

and

$$\begin{aligned} s_m(f_i) &= \frac{\tau_m}{\tau_{m+1}} \frac{[\alpha_0 + \varepsilon_{m,m+1}][\varepsilon_{i,m+1}] f_i - [\alpha_0 + \varepsilon_{i,m+1}][\varepsilon_{m,m+1}] f_m}{[\alpha_0][\varepsilon_{i,m}]} \\ & \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \dots, m-1), \\ s_m(f_m) &= \frac{\tau_m}{\tau_{m+1}} f_m. \end{aligned} \tag{47}$$

***Theorem 5.1***. — *The automorphisms $s_0, s_1, \dots, s_{n-1}$ of $\mathcal{L}$ defined as above satisfy the fundamental relations for the simple reflections of the Weyl group $W_{m,n}$.*

This theorem can be proved directly by using the Riemann relation for $[x]$.

We remark that the action of the Weyl group $W_{m,n}$ preserves the subalgebra

$$\mathcal{R} = S[\tau_1^{\pm 1}, \dots, \tau_n^{\pm 1}] \subset \mathcal{L}, \tag{48}$$

where

$$S = \bigoplus_{d \in \mathbb{Z}} S_d, \qquad S_d = f_m^d \, \mathbb{K}(f_1/f_m, \dots, f_{m-1}/f_m). \tag{49}$$

Consider the following elements in $S_0$:

$$z_i = \frac{a_i}{a_m} \frac{f_i}{f_m}, \qquad a_i = \frac{[\varepsilon_{i,m+1}]}{[\alpha_0 + \varepsilon_{i,m+1}]} \quad (i = 1, \dots, m), \tag{50}$$

so that $S_0 = \mathbb{K}(z_1, \dots, z_{m-1})$ and that $(z_1 : \dots : z_{m-1} : 1) = (a_1 f_1 : \dots : a_m f_m)$. Then the action of $s_k$ $(k = 0, 1, \dots, n-1)$ on these $z$ variables coincides with the one given earlier in (40). In this sense the $f$ variables are thought of as *normalized* homogeneous coordinates. We also remark that the $f$ variables and the $\tau$ variables for the canonical solution are given by

$$f_i^C = \frac{[\alpha_0 + \varepsilon_i - t]}{[\varepsilon_i - t]} \quad (i = 1, \dots, m), \qquad \tau_j^C = [\varepsilon_j - t] \quad (j = 1, \dots, n). \tag{51}$$

(We have used the superscript $C$ to indicate that they are "canonical", and also associated with the elliptic curve $C_{\lambda,\mu}$.)

It is also convenient to introduce another $\tau$ variable $\tau_0$ and define $x_i$ $(i = 1, \dots, m)$ by the formula

$$f_i = \frac{\tau_0 \, x_i}{\tau_1 \cdots \tau_m} \qquad (i = 1, \dots, m). \tag{52}$$

To be more precise, consider the field of rational functions

$$\widetilde{\mathcal{L}} = \mathbb{K}(x_1, \dots, x_m; \tau_0, \tau_1, \dots, \tau_m) \tag{53}$$

and identify $\mathcal{L}$ with its subfield

$$\mathcal{L} = \mathbb{K}(\tau_0 x_1, \ldots, \tau_0 x_m; \tau_1, \ldots, \tau_m) \tag{54}$$

by the formula (52). The action of the symmetric group $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ can be extended to $\widetilde{\mathcal{L}}$ by setting

$$s_k(x_i) = x_{s_k(i)} \quad (k = 1, \ldots, m-1), \quad s_k(x_i) = x_i \quad (k = m+1, \ldots, n-1) \tag{55}$$

for $i = 1, \ldots, m$, and

$$s_m(x_i) = \frac{[\alpha_0 + \varepsilon_{m,m+1}][\varepsilon_{i,m+1}]x_i - [\alpha_0 + \varepsilon_{i,m+1}][\varepsilon_{m,m+1}]x_m}{[\alpha_0][\varepsilon_{i,m}]} \tag{56}$$

for $i = 1, \ldots, m-1$ and $s_m(x_m) = x_m$. The action of $\mathfrak{S}_n$ on the $\tau$ variables are defined as

$$s_k(\tau_0) = \tau_0, \qquad s_k(\tau_j) = \tau_{s_k(j)} \quad (j = 1, \ldots, n) \tag{57}$$

for any $k = 1, \ldots, n-1$. Note that the action of $s_0 \in W_{m,n}$ is defined only on the subfield $\mathcal{L} \subset \widetilde{\mathcal{L}}$. The products $\tau_0\, x_i \in \mathcal{L}$ are transformed by $s_0$ as follows:

$$s_0(\tau_0\, x_i) = \frac{\tau_0^{m-1} x_1 \cdots \widehat{x_i} \cdots x_m}{(\tau_1 \cdots \tau_m)^{m-2}} \qquad (i = 1, \ldots, m). \tag{58}$$

We regard $x = (x_1, \ldots, x_m)$ as a normalized homogeneous coordinate system such that $(z_1 : \ldots : z_{m-1} : 1) = (a_1 x_1 : \ldots : a_m x_m)$. Note that the canonical solution is given by

$$\tau_0\, x_i = x_i^C(t) = [\alpha_0 + \varepsilon_i - t] \prod_{1 \le k \le m; k \ne i} [\varepsilon_k - t] \qquad (i = 1, \ldots, m). \tag{59}$$

Accordingly, we define the $x$ coordinates of the reference points $p_1, \ldots, p_n$ by

$$x(p_j) = (x_1^C(\varepsilon_j) : \ldots : x_m^C(\varepsilon_j)), \qquad x_i^C(\varepsilon_j) = [\alpha_0 + \varepsilon_{i,j}] \prod_{1 \le k \le m; k \ne i} [\varepsilon_{k,j}], \tag{60}$$

for $j = 1, \ldots, n$. For each element

$$\Lambda = d e_0 - \nu_1 e_1 - \cdots - \nu_n e_n \in L_{m,n} \qquad (d, \nu_1, \ldots, \nu_n \in \mathbb{Z}), \tag{61}$$

we define the *formal exponential* $\tau^\Lambda$ by

$$\tau^\Lambda = \tau_0^d\, \tau_1^{-\nu_1} \cdots \tau_n^{-\nu_n}. \tag{62}$$

Since $f_i = \tau^{h_0} x_i$ $(i = 1, \ldots, m)$, the algebra $\mathcal{R}$ can be expressed alternatively as

$$\mathcal{R} = \bigoplus_{\Lambda \in L_{m,n}} \mathbb{K}(x)_{\deg(\Lambda)}\, \tau^\Lambda, \qquad \mathbb{K}(x)_d = x_m^d\, \mathbb{K}(x_1/x_m, \ldots, x_{m-1}/x_m), \tag{63}$$

where $\deg(\Lambda)$ stands for the coefficient $d$ of $e_0$ in (61). In the $x$ and $\tau$ variables, the action of the Weyl group on $\mathcal{R}$ is described as follows:

$$s_0(\tau^\Lambda \varphi(x)) = \tau^{s_0 \cdot \Lambda}\, x_1^{d-\nu_1} \cdots x_m^{d-\nu_m}\, {}^{s_0}\varphi(x^{-1}) \tag{64}$$

and

$$s_k(\tau^{\Lambda}\varphi(x)) = \tau^{s_k.\Lambda}\, {}^{s_k}\varphi(s_k(x)) \qquad (k = 1, \dots, n-1) \tag{65}$$

for any $\varphi(x) = \varphi(x_1, \dots, x_m) \in \mathbb{K}(x)_d$, where ${}^{w}\varphi(x)$ stands for the polynomial obtained from $\varphi(x)$ by applying $w$ to its coefficients. The expression $s_k(x) = (s_k(x_1), \dots, s_k(x_m))$ $(k = 1, \dots, n-1)$ should be understood in the sense of the action of $\mathfrak{S}_n$ on $\widetilde{\mathcal{L}}$ described in (55) and (56).

We are now ready to introduce the lattice $\tau$-functions for the elliptic Cremona system of type $(m, n)$. We consider the $W_{m,n}$-orbit of the lattice point $e_n$ in $L_{m,n}$:

$$M_{m,n} = W_{m,n}e_n = W_{m,n}\{e_1, \dots, e_n\} \subset L_{m,n}. \tag{66}$$

Notice that the stabilizer of $e_n$ is $W_{m,n-1}$, and that $\tau_n$ is $W_{m,n-1}$-invariant. Hence, for each $\Lambda \in M_{m,n}$ we can define an element $\tau(\Lambda) = w(\tau_n) \in \mathcal{R}$ by taking any element $w \in W_{m,n}$ such that $w.e_n = \Lambda$. This family of $(\tau(\Lambda))_{\Lambda \in M_{m,n}}$ of elements of $\mathcal{R}$, indexed by the lattice points $M_{m,n} \subset L_{m,n}$ will be called the *lattice $\tau$-functions* for the elliptic Cremona system of type $(m, n)$. These $\tau$-functions are determined by the condition $\tau(e_j) = \tau_j$ $(j = 1, \dots, n)$ together with the compatibility condition

$$w(\tau(\Lambda)) = \tau(w.\Lambda) \qquad (\Lambda \in M_{m,n}; w \in W_{m,n}) \tag{67}$$

with respect to the action of $W_{m,n}$ on $M_{m,n}$. By using

$$f_i = \frac{s_0(\tau_i)}{\tau_i} = \frac{\tau(e_i + h_0)}{\tau(e_i)} \qquad (i = 1, \dots, m), \tag{68}$$

from the action of $s_m$ (47) on the $f$ variables we obtain the following bilinear relations for the lattice $\tau$-functions:

$$\begin{aligned} &[\alpha_0][\varepsilon_{i,m}]\tau(h_0 + e_i + e_m - e_{m+1})\tau(e_{m+1}) \\ &= [\alpha_0 + \varepsilon_{m,m+1}][\varepsilon_{i,m+1}]\tau(h_0 + e_i)\tau(e_m) \\ &\quad - [\alpha_0 + \varepsilon_{i,m+1}][\varepsilon_{m,m+1}]\tau(h_0 + e_m)\tau(e_i). \end{aligned} \tag{69}$$

¿ From this we obtain

***Theorem 5.2***. — *The lattice $\tau$-functions $(\tau(\Lambda))_{\Lambda \in M_{m,n}}$ defined as above satisfy the following bilinear equations of Hirota-Miwa type: For any choice of mutually distinct indices $i, j, k, l_1, \dots, l_{m-2} \in \{1, \dots, n\}$,*

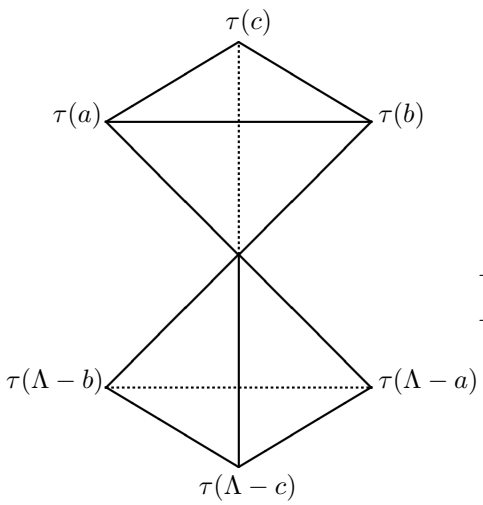
$$\begin{aligned} &[\varepsilon_{j,k}][\lambda - \varepsilon_j - \varepsilon_k]\tau(e_i)\tau(\Lambda - e_i) + [\varepsilon_{k,i}][\lambda - \varepsilon_k - \varepsilon_i]\tau(e_j)\tau(\Lambda - e_j) \\ &\qquad + [\varepsilon_{i,j}][\lambda - \varepsilon_i - \varepsilon_j]\tau(e_k)\tau(\Lambda - e_k) = 0, \end{aligned} \tag{70}$$

*where $\Lambda = e_0 - e_{l_1} - \cdots - e_{l_{m-2}}$ and $\lambda = (\Lambda \,|\, \cdot) = \varepsilon_0 - \varepsilon_{l_1} - \cdots - \varepsilon_{l_{m-2}}$.*

The lattice $\tau$-functions for the canonical solution are given simply by

$$\tau^C(\Lambda) = [\lambda - t] \qquad \text{with} \quad \lambda = (\Lambda \,|\, \cdot) \in \mathfrak{h}^*_{m,n} \tag{71}$$

for any $\Lambda \in M_{m,n}$. In this case the bilinear equations in Theorem 5.2 recover the Riemann relation for the function $[x]$.




$$\begin{aligned}&[\beta-\gamma][\lambda-\beta-\gamma]\ \tau(a)\tau(\Lambda-a)\\&+\ [\gamma-\alpha][\lambda-\gamma-\alpha]\ \tau(b)\tau(\Lambda-b)\\&+\ [\alpha-\beta][\lambda-\alpha-\beta]\ \tau(c)\tau(\Lambda-c)=0\end{aligned}$$

FIGURE 1. Bilinear equations for the lattice $\tau$-functions

We also remark that these bilinear equations of Hirota-Miwa type characterize the lattice $\tau$-functions for the elliptic Cremona system. To be more precise, suppose that the natural action of $W_{m,n}$ on $\mathbb{K}$ is extended to a field $\mathcal{K}$ containing $\mathbb{K}$ (as that of a group of automorphisms). If a family of nonzero elements $(\tau(\Lambda))_{\Lambda\in M_{m,n}}$ of $\mathcal{K}$, indexed by $M_{m,n}$, satisfies the two conditions (67) and (70), then the elements

$$f_i=\frac{\tau(e_i+h_0)}{\tau(e_i)}\quad(i=1,\dots,m),\qquad \tau_j=\tau(e_j)\quad(j=1,\dots,n) \tag{72}$$

of $\mathcal{K}$ recover the relations (45)–(47) for the action of $W_{m,n}$ on $\mathcal{L}$.

Recall that the algebra $\mathcal{R}$ is expressed as

$$\mathcal{R}=\bigoplus_{\Lambda\in L_{m,n}}\mathbb{K}(x)_{\deg(\Lambda)}\,\tau^{\Lambda}. \tag{73}$$

As in Section 3, for each $\Lambda\in L_{m,n}$ of the form (61) we define the $\mathbb{K}$-vector subspace $L(\Lambda)\subset\mathbb{K}[x]_d$ by

$$L(\Lambda)=\{f(x)\in\mathbb{K}[x]_d\mid \mathrm{ord}_{p_j}f(x)\geq\nu_j\quad(j=1,\dots,n)\}, \tag{74}$$

with the reference points $p_j$ specified by the $x$ coordinates as in (60). Then one can show that the subalgebra

$$\mathcal{S}=\bigoplus_{\Lambda\in L_{m,n}}L(\Lambda)\,\tau^{\Lambda}\subset\mathcal{R} \tag{75}$$

is preserved under the action of $W_{m,n}$. In fact each $w \in W_{m,n}$ induces a $\mathbb{C}$-isomorphism

$$w. : L(\Lambda)\,\tau^{\Lambda} \xrightarrow{\sim} L(w.\Lambda)\,\tau^{w.\Lambda} \tag{76}$$

for any $\Lambda \in L_{m,n}$. (This fact can be established by examining the cases of simple reflections $s_0, s_1, \ldots, s_{n-1}$.) Since $\tau_j \in L(e_j)\tau^{e_j}$ $(j = 1, \ldots, n)$, the lattice $\tau$-function $\tau(\Lambda)$ for $\Lambda \in M_{m,n}$ can be expressed in the form

$$\tau(\Lambda) = \tau^{\Lambda}\,\phi(\Lambda; x), \qquad \phi(\Lambda; x) \in L(\Lambda) \tag{77}$$

in terms of the original $\tau$ variables $\tau_0, \tau_1, \ldots, \tau_n$ and the homogeneous coordinates $x_1, \ldots, x_m$. We remark that this family of homogeneous polynomials $\phi(\Lambda; x)$ $(\Lambda \in M_{m,n})$ is determined uniquely by the following properties:

$$\begin{aligned} &\phi(e_j; x) = 1 && (j = 1, \ldots, n), \\ &\phi(s_0.\Lambda; x) = x_1^{d-\nu_1} \cdots x_m^{d-\nu_m}\, {}^{s_0}\phi(\Lambda; x^{-1}), && \\ &\phi(s_k.\Lambda; x) = {}^{s_k}\phi(\Lambda; s_k(x)) && (k = 1, \ldots, n-1), \end{aligned} \tag{78}$$

where $\Lambda = de_0 - \nu_1 e_1 - \ldots - \nu_n e_n$. We sometimes refer to this family of polynomials $\phi(\Lambda; x)$ as the *$\tau$-cocycle*. They correspond to what are called *$\phi$-factors* in [**13**].

Since $f_i = \tau^{h_0} x_i$ $(i = 1, \ldots, m)$, the formula (77) is rewritten also into

$$\tau(\Lambda) = \tau^{\Lambda - \deg(\Lambda)e_0}\,\phi(\Lambda; \tau_0\, x) = \tau^{\Lambda - \deg(\Lambda)h_0}\,\phi(\Lambda; f) \tag{79}$$

in terms of the variables $\tau_0 x_i$ or $f_i$. For $\Lambda$ as in (61), we have

$$\tau(\Lambda) = \frac{\phi(\Lambda; \tau_0 x)}{\tau_1^{\nu_1} \cdots \tau_n^{\nu_n}} = \frac{\tau_1^{d-\nu_1} \cdots \tau_m^{d-\nu_m}}{\tau_{m+1}^{\nu_{m+1}} \cdots \tau_n^{\nu_n}}\,\phi(\Lambda; f). \tag{80}$$

Note here that $\phi(\Lambda; x)$ is a homogeneous polynomial of degree $d$ having a zero of multiplicity $\geq \nu_j$ at each $p_j$ $(j = 1, \ldots, n)$. In the case of the canonical solution, the above formula gives rise to

$$\phi(\Lambda; x^C(t)) = [\lambda - t][\varepsilon_1 - t]^{\nu_1} \cdots [\varepsilon_n - t]^{\nu_n}, \qquad \lambda = (\Lambda \,|\, \cdot). \tag{81}$$

In this form, it is clearly seen that $\phi(\Lambda; x^C(t))$ has zeros at $p_1, \ldots, p_n$ with expected multiplicities.

Note that

$$f_i = \frac{\tau(e_i + h_0)}{\tau(e_i)} = \frac{\tau(s_0.e_i)}{\tau(e_i)} \qquad (i = 1, \ldots, m) \tag{82}$$

implies

$$w(f_i) = \frac{\tau(ws_0.e_i)}{\tau(w.e_i)} = \frac{\tau^{ws_0.e_i}}{\tau^{w.e_i}}\,\frac{\phi(ws_0.e_i; x)}{\phi(w.e_i; x)} = \tau^{w.h_0}\,\frac{\phi(ws_0.e_i; x)}{\phi(w.e_i; x)} \tag{83}$$

for any $w \in W_{m,n}$. Hence, the action of $w \in W_{m,n}$ on the variables $f_1, \dots, f_m$ and $\tau_1, \dots, \tau_n$ can be written in a closed form as follows:

$$\begin{aligned} w(f_i) &= \tau^{w.h_0 - \deg(w.h_0)h_0} \frac{\phi(ws_0.e_i; f)}{\phi(w.e_i; f)} && (i = 1, \dots, m), \\ w(\tau_j) &= \tau^{w.e_j - \deg(w.e_j)h_0} \phi(w.e_j; f) && (j = 1, \dots, n). \end{aligned} \tag{84}$$

Also, from

$$(z_1 : \dots : z_{m-1} : 1) = (a_1 f_1 : \dots : a_m f_m), \qquad a_i = \frac{[\varepsilon_{i,m+1}]}{[\alpha_0 + \varepsilon_{i,m+1}]}, \tag{85}$$

we obtain

$$(w(z_1) : \dots : w(z_{m-1}) : 1) = \left( w(a_1) \frac{\phi(ws_0.e_1; x)}{\phi(w.e_1; x)} : \dots : w(a_m) \frac{\phi(ws_0.e_m; x)}{\phi(w.e_m; x)} \right) \tag{86}$$

for any $w \in W_{m,n}$. This formula provides a refinement of Theorem 3.1 for the elliptic Cremona system of type $(m, n)$.

## 6. Elliptic difference Painlevé equation

In the rest of this article, we confine ourselves to the elliptic Cremona system of type $(3, 9)$ which has the affine Weyl group symmetry of type $E_8^{(1)}$. The discrete dynamical system arising from the lattice part of $W(E_8^{(1)})$ is the elliptic difference Painlevé equation.

The parameter space for the elliptic Cremona system of type $(3, 9)$ is the 10-dimensional vector space

$$\mathfrak{h}_{3,9} = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_9 \tag{87}$$

with the nondegenerate symmetric bilinear form $(\cdot \,|\, \cdot) : \mathfrak{h}_{3,9} \times \mathfrak{h}_{3,9} \to \mathbb{C}$ such that

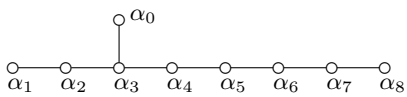
$$\begin{aligned} &(e_0 \,|\, e_0) = -1, && (e_j \,|\, e_j) = 1 \quad (j = 1, \dots, 9), \\ &(e_i \,|\, e_j) = 0 && (i, j \in \{1, \dots, 9\}; i \neq j), \end{aligned} \tag{88}$$

which we regard as the complexification of the lattice $L_{3,9} = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_9$. We take the linear functions $\varepsilon_j = (e_j \,|\, \cdot)$ $(j = 0, 1, \dots, 9)$ as the canonical coordinates for $\mathfrak{h}_{3,9}$, so that $\mathfrak{h}_{3,9}^* = \mathbb{C}\varepsilon_0 \oplus \mathbb{C}\varepsilon_1 \oplus \cdots \oplus \mathbb{C}\varepsilon_9$. The *simple coroots* $h_j \in \mathfrak{h}_{3,9}$ and the *simple roots* $\alpha_j = (h_j \,|\, \cdot) \in \mathfrak{h}_{3,9}^*$ $(j = 0, 1, \dots, 8)$ for this case are

$$\begin{aligned} &h_0 = e_0 - e_1 - e_2 - e_3, && h_j = e_j - e_{j+1} \quad (j = 1, \dots, 8), \\ &\alpha_0 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3, && \alpha_j = \varepsilon_j - \varepsilon_{j+1} \quad (j = 1, \dots, 8). \end{aligned} \tag{89}$$

The $9 \times 9$ matrix $(\langle h_i, \alpha_j \rangle)_{i,j=0}^8$ is the Cartan matrix of type $E_8^{(1)}$ with the following Dynkin diagram.

(90) $E_8^{(1)}$ :



We denote by

$$Q_{3,9} = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_8 \subset \mathfrak{h}_{3,9}^*, \quad Q_{3,9} \xrightarrow{\sim} Q(E_8^{(1)}), \tag{91}$$

the corresponding root lattice. The affine Weyl group $W_{3,9} = \langle s_0, s_1, \ldots, s_8 \rangle$ of type $E_8^{(1)}$ acts in a standard way on $\mathfrak{h}_{3,9}$ and $\mathfrak{h}_{3,9}^*$, so that the symmetric bilinear forms of $\mathfrak{h}_{3,9}$ and $\mathfrak{h}_{3,9}^*$ are both $W_{3,9}$-invariant. The *canonical central element*

$$c = 3e_0 - e_1 - \cdots - e_9 \in \mathfrak{h}_{3,9} \tag{92}$$

is orthogonal to all $h_j$ $(j = 0, 1, \ldots, 8)$, and hence $W_{3,9}$-invariant. The corresponding $W_{3,9}$-invariant element in $\mathfrak{h}_{3,9}^*$

$$\delta = (c \,|\, \cdot) = 3\varepsilon_0 - \varepsilon_1 - \cdots - \varepsilon_9 \in \mathfrak{h}_{3,9}^* \tag{93}$$

is called the *null root*; it plays the role of the scaling constant for difference equations in the context of discrete Painlevé equation. The set $\Delta_{3,9}^{\mathrm{Re}} = W_{3,9}\{\alpha_0, \alpha_1, \ldots, \alpha_8\}$ of *real roots* is now given by

$$\begin{aligned} \Delta_{3,9}^{\mathrm{Re}} = \{\pm\varepsilon_{ij} + n\delta \mid 1 \le i < j \le 9,\ n \in \mathbb{Z}\} \\ \cup \{\pm\varepsilon_{ijk} + n\delta \mid 1 \le i < j < k \le 9,\ n \in \mathbb{Z}\}, \end{aligned} \tag{94}$$

where $\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$ and $\varepsilon_{ijk} = \varepsilon_0 - \varepsilon_i - \varepsilon_j - \varepsilon_k$ for $i, j, k \in \{1, \ldots, 9\}$. For each real root $\alpha \in \Delta_{3,9}^{\mathrm{Re}}$, the reflection $s_\alpha : \mathfrak{h}_{3,9}^* \to \mathfrak{h}_{3,9}^*$ with respect to $\alpha$ is defined by

$$s_\alpha(\lambda) = \lambda - (\alpha \,|\, \lambda)\alpha \qquad (\lambda \in \mathfrak{h}_{3,9}^*). \tag{95}$$

Note also $ws_\alpha w^{-1} = s_{w.\alpha}$ for any $\alpha \in \Delta_{3,9}^{\mathrm{Re}}$ and $w \in W_{3,9}$. When $\alpha = \varepsilon_{ij}$ or $\varepsilon_{ijk}$, we denote the reflection $s_\alpha$ simply by $s_{ij}$ or $s_{ijk}$, respectively.

Following [**6**], for each $\alpha \in Q_{3,9}$ we define the translation $T_\alpha : \mathfrak{h}_{3,9}^* \to \mathfrak{h}_{3,9}^*$ by

$$T_\alpha(\lambda) = \lambda + (\delta \,|\, \lambda)\alpha - \left(\frac{1}{2}(\alpha \,|\, \alpha)(\delta \,|\, \lambda) + (\alpha \,|\, \lambda)\right)\delta \qquad (\lambda \in \mathfrak{h}_{3,9}^*). \tag{96}$$

Note that

$$\begin{aligned} &T_\alpha(\beta) = \beta - (\alpha \,|\, \beta)\,\delta && (\alpha, \beta \in Q_{3,9}), \\ &T_\alpha T_\beta = T_\beta T_\alpha = T_{\alpha+\beta} && (\alpha, \beta \in Q_{3,9}), \\ &wT_\alpha w^{-1} = T_{w.\alpha} && (\alpha \in Q_{3,9} \text{ and } w \in W_{3,9}). \end{aligned} \tag{97}$$

For any real root $\alpha \in \Delta_{3,9}^{\mathrm{Re}}$, the translation $T_\alpha$ can be expressed as the composition of two reflections $T_\alpha = s_{\delta-\alpha}s_\alpha$. ¿ From this fact it follows that $T_\alpha \in W_{3,9}$ for any $\alpha \in Q_{3,9}$. Furthermore, it is known that the affine Weyl group $W_{3,9}$ is decomposed into the semidirect product of the root lattice

$$Q_{3,8} = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_7 \subset Q_{3,9}, \qquad Q_{3,8} \xrightarrow{\sim} Q(E_8), \tag{98}$$

and the finite Weyl group $W_{3,8} = \langle s_0, s_1, \ldots, s_7 \rangle$ of type $E_8$ acting on $Q_{3,8}$. In fact the correspondence $(\alpha, w) \mapsto T_\alpha w$ induces the isomorphism $Q_{3,8} \rtimes W_{3,8} \xrightarrow{\sim} W_{3,9}$ (see

[**6**]). We remark that the action of $T_\alpha$ on $\mathfrak{h}_{3,9}$ is expressed in the form

$$T_\alpha(\Lambda) = \Lambda + (c\,|\,\Lambda)h - \left(\frac{1}{2}(h\,|\,h)(c\,|\,\Lambda) + (h\,|\,\Lambda)\right)c \qquad (\Lambda \in \mathfrak{h}_{3,9}) \tag{99}$$

by using the element $h \in \mathfrak{h}_{3,9}$ such that $\alpha = (h\,|\,\cdot)$. When $\alpha = \varepsilon_{ij}$ or $\varepsilon_{ijk}$, we write the translation $T_\alpha$ simply as $T_{ij}$ or $T_{ijk}$, respectively. In this case of $(m,n) = (3,9)$, the $W_{3,9}$-orbit $M_{3,9} = W_{3,9}\{e_1,\dots,e_9\} \subset L_{3,9}$ can be characterized as follows:

$$M_{3,9} = \{\Lambda \in L_{3,9} \mid (c\,|\,\Lambda) = -1, \quad (\Lambda\,|\,\Lambda) = 1\}. \tag{100}$$

Also, the correspondence $\alpha \mapsto T_\alpha.e_9$ induces a bijection $Q_{3,8} \xrightarrow{\sim} M_{3,9}$. Any element $\Lambda \in M_{3,9}$ can be expressed in the form

$$\Lambda = de_0 - \nu_1 e_1 - \ldots - \nu_9 e_9, \qquad d \geq 0, \quad \nu_j \geq -1 \quad (j = 1,\dots,9). \tag{101}$$

In fact, except for the cases $\Lambda = e_k$ $(k = 1,\dots,9)$, the coefficients $\nu_j$ are all nonnegative.

As in the previous section, we consider the $\mathbb{K}$-algebra

$$\mathcal{R} = S[\tau_1^{\pm 1},\dots,\tau_9^{\pm 1}], \qquad S = \bigoplus_{d\in\mathbb{Z}} f_3^d\,\mathbb{K}(f_1/f_3, f_2/f_3) \tag{102}$$

of $f$ variables and $\tau$ variables. The standard Cremona transformation $s_0$ with respect to $(p_1,p_2,p_3)$ acts on the $f$ variables and $\tau$ variables as

$$s_0(f_i) = \frac{1}{f_i} \quad (i = 1,2,3), \qquad s_0(\tau_j) = \begin{cases} \tau_j f_j & (j = 1,2,3), \\ \tau_j & (j = 4,5,6,7,8,9). \end{cases} \tag{103}$$

Among the 8 adjacent transpositions $s_k$ $(k = 1,\dots,8)$, $s_3$ acts nontrivially on the $f$ variables:

$$\begin{aligned} s_3(f_1) &= \frac{\tau_3}{\tau_4}\frac{[\varepsilon_{124}][\varepsilon_{14}]f_1 - [\varepsilon_{234}][\varepsilon_{34}]f_3}{[\varepsilon_{123}][\varepsilon_{13}]}, \\ s_3(f_2) &= \frac{\tau_3}{\tau_4}\frac{[\varepsilon_{124}][\varepsilon_{24}]f_1 - [\varepsilon_{134}][\varepsilon_{34}]f_3}{[\varepsilon_{123}][\varepsilon_{23}]}, \\ s_3(f_3) &= \frac{\tau_3}{\tau_4} f_3. \end{aligned} \tag{104}$$

(Note that $\alpha_0 = \varepsilon_{123}$.) Recall that the lattice $\tau$-functions $(\tau(\Lambda))_{\Lambda\in M_{3,9}}$ are defined as a family of dependent variables indexed by the $W_{3,9}$-orbit

$$M_{3,9} = W_{3,9}\{e_1,\dots,e_9\} = \{\Lambda \in L_{3,9} \mid (c\,|\,\Lambda) = -1, \quad (\Lambda\,|\,\Lambda) = 1\}. \tag{105}$$

These $\tau$-functions $\tau(\Lambda)$ are characterized by the consistency condition

$$\tau(e_j) = \tau_j \quad (j = 1,\dots,9), \qquad w(\tau(\Lambda)) = \tau(w.\Lambda) \quad (\Lambda \in M_{3,9}; w \in W_{3,9}) \tag{106}$$

with respect to the action of $W_{3,9}$ on $M_{3,9}$, and the bilinear equations

$$\begin{aligned} &[\varepsilon_{jk}][\varepsilon_{jkl}]\tau(e_i)\tau(e_0 - e_i - e_l) + [\varepsilon_{ki}][\varepsilon_{kil}]\tau(e_j)\tau(e_0 - e_j - e_l) \\ &\qquad + [\varepsilon_{ij}][\varepsilon_{ijl}]\tau(e_k)\tau(e_0 - e_k - e_l) = 0 \end{aligned} \tag{107}$$

for any mutually distinct $i,j,k,l \in \{1,\dots,9\}$. These bilinear equations guarantee the equivalence of our formulation of the elliptic difference Painlevé equation of type $E_8^{(1)}$ to that of Ohta-Ramani-Grammaticos [**15**] on the $E_8$ lattice.

We already know that the lattice $\tau$-functions can be expressed in the form

$$\tau(\Lambda) = \tau^{\Lambda}\,\phi(\Lambda;x) = \tau^{\Lambda-\deg(\Lambda)h_0}\phi(\Lambda;f), \quad \phi(\Lambda;x)\in L(\Lambda). \tag{108}$$

where the $x$ coordinates are defined by $f_i = \tau^{h_0}x_i$ $(i=1,2,3)$. When $\Lambda = de_0 - \nu_1 e_1 - \cdots - \nu_9 e_9$, $\phi(\Lambda;x)$ is a homogeneous polynomial of degree $d=\deg(\Lambda)$, and for each $j=1,\dots,9$ it has a zero of multiplicity $\geq \nu_j$ at $p_j$; the $x$ coordinates of $p_j$ are now given by

$$x(p_j) = ([\varepsilon_{23j}][\varepsilon_{2j}][\varepsilon_{3j}] : [\varepsilon_{13j}][\varepsilon_{1j}][\varepsilon_{3j}] : [\varepsilon_{12j}][\varepsilon_{1j}][\varepsilon_{2j}]) \qquad (j=1,\dots,9). \tag{109}$$

By the homogeneous polynomials $\phi(\Lambda;x)$, the action of $W_{3,9}$ on the algebra $\mathcal{R}$ is described as

$$\begin{aligned} w(f_i) &= \tau^{w.h_0-\deg(w.h_0)h_0}\,\frac{\phi(ws_0.e_i;f)}{\phi(w.e_i;f)} && (i=1,2,3),\\ w(\tau_j) &= \tau^{w.e_j-\deg(w.e_j)h_0}\,\phi(w.e_j;f) && (j=1,\dots,9). \end{aligned} \tag{110}$$

Recall that the $z$ variables $z=(z_1,z_2)$ are recovered from the $f$ variables by the formula

$$(z_1:z_2:1) = \left(\frac{[\varepsilon_{14}]}{[\varepsilon_{234}]}f_1 : \frac{[\varepsilon_{24}]}{[\varepsilon_{134}]}f_2 : \frac{[\varepsilon_{34}]}{[\varepsilon_{124}]}f_3\right), \tag{111}$$

and hence

$$(w(z_1):w(z_2):1) = \left(\frac{[w(\varepsilon_{14})]}{[w(\varepsilon_{234})]}w(f_1) : \frac{[w(\varepsilon_{24})]}{[w(\varepsilon_{134})]}w(f_2) : \frac{[w(\varepsilon_{34})]}{[w(\varepsilon_{124})]}w(f_3)\right) \tag{112}$$

for any $w\in W_{3,9}$. This implies

$$\begin{aligned} w(z_1) &= \frac{[w(\varepsilon_{14})][w(\varepsilon_{124})]}{[w(\varepsilon_{234})][w(\varepsilon_{34})]}\,\frac{G_1^w(z)F_3^w(z)}{F_1^w(z)G_3^w(z)},\\ w(z_2) &= \frac{[w(\varepsilon_{24})][w(\varepsilon_{124})]}{[w(\varepsilon_{134})][w(\varepsilon_{34})]}\,\frac{G_2^w(z)F_3^w(z)}{F_2^w(z)G_3^w(z)}, \end{aligned} \tag{113}$$

where $F_i^w(z)$ and $G_i^w(z)$ are the inhomogenizations of $\phi(w.e_i;f)$ and $\phi(ws_0.e_i;f)$, respectively, by (111). The formulas (110) (resp. (113)) for the translations $w=T_\alpha$ $(\alpha\in Q_{3,9})$ give the elliptic difference Painlevé equation of type $(3,9)$ in the homogeneous form of $f$ and $\tau$ variables (resp. in the inhomogeneous form in $z$ variables). In the following, we will mainly work with the homogeneous form (110).

Note that, for each $\Lambda\in M_{3,9}$, the homogeneous polynomial $\phi(\Lambda;x)$ is characterized by its degree and the multiplicity of zeros at $p_1,\dots,p_9$. Thanks to this fact, we are able to express $\phi(\Lambda;x)$ by means of an interpolation determinant. For each $d=0,1,2,\dots$, we denote by

$$\boldsymbol{m}_d(x) = (x^\mu)_{|\mu|=d} = (x_1^{\mu_1}x_2^{\mu_2}x_3^{\mu_3})_{\mu_1+\mu_2+\mu_3=d} \tag{114}$$

the column vector of monomials of degree $d$ in $x=(x_1,x_2,x_3)$. Note that the number of such monomials is given by

$$\dim_{\mathbb{K}} \mathbb{K}[x]_d = \binom{d+2}{2} = \frac{1}{2}(d+1)(d+2). \tag{115}$$

For each $k=0,1,\dots,d$ we denote by

$$\boldsymbol{m}_d^{(k)}(x) = (\partial_x^{\kappa}(x^{\mu}))_{|\mu|=d,|\kappa|=k} = \big(\partial_{x_1}^{\kappa_1}\partial_{x_2}^{\kappa_2}\partial_{x_3}^{\kappa_3}(x_1^{\mu_1}x_2^{\mu_2}x_3^{\mu_3})\big)_{|\mu|=d,|\kappa|=k} \tag{116}$$

the $\binom{d+2}{2}\times\binom{k+2}{2}$ matrix defined by the partial derivatives $\boldsymbol{m}_d(x)$ of order $k$. (For $k<0$, we consider $\boldsymbol{m}_d^{(k)}(x)$ as an empty matrix.) Given an element

$$\Lambda = de_0 - \nu_1 e_1 - \dots - \nu_9 e_9 \in M_{3,9}, \tag{117}$$

we consider the homogeneous polynomial

$$F(\Lambda;x) = \det\Big(\boldsymbol{m}_d^{(\nu_1-1)}(p_1),\dots,\boldsymbol{m}_d^{(\nu_9-1)}(p_9),\boldsymbol{m}_d(x)\Big) \tag{118}$$

of degree $d$ in $x=(x_1,x_2,x_3)$, where $\boldsymbol{m}_d^{(k)}(p_j)$ stands for the matrix obtained from $\boldsymbol{m}_d^{(k)}(x)$ by the substitution $x=x(p_j)$ as in (109). Note here that

$$\binom{d+2}{2} - \sum_{j=1}^{9}\binom{\nu_j+1}{2} - 1 = -\frac{1}{2}\big((c\,|\,\Lambda)+(\Lambda\,|\,\Lambda)\big) = 0 \tag{119}$$

for any $\Lambda\in M_{3,9}$; this means that the number of column vectors in (118) is equal to $\binom{d+2}{2}$. Also, from $\dim_{\mathbb{K}} L(\Lambda)=1$ it follows that $F(\Lambda;x)$ is a nonzero polynomial. Combining this fact with the normalization condition (71), we obtain the following theorem.

***Theorem 6.1***. — *For each $\Lambda\in M_{3,9}$, define the homogeneous polynomial $F(\Lambda;x)$ in $x=(x_1,x_2,x_3)$ by the determinant* (118). *Then, the specialization of $F(\Lambda;x)$ to the canonical solution $x^C(t)$ is expressed in the form*

$$F(\Lambda;x^C(t)) = C_\Lambda\,[\lambda-t]\prod_{j=1}^{9}[\varepsilon_j-t]^{\nu_j}, \qquad \lambda=(\Lambda\,|\,\cdot)\in\mathfrak{h}_{3,9}^*, \tag{120}$$

*with a nonzero constant $C_\Lambda\in\mathbb{K}$. With this normalization constant $C_\Lambda$, the homogeneous polynomial $\phi(\Lambda;x)\in L(\Lambda)$ is expressed as the determinant*

$$\phi(\Lambda;x) = C_\Lambda^{-1}F(\Lambda;x). \tag{121}$$

Let us consider the action of translations $w=T_\alpha$ ($\alpha\in\Delta_{3,9}^{\mathrm{Re}}$) on $\mathcal{R}$:

$$\begin{aligned} T_\alpha(f_i) &= \tau^{T_\alpha.h_0-\deg(T_\alpha.h_0)h_0}\,\frac{\phi(T_\alpha.(e_i+h_0);f)}{\phi(T_\alpha.e_i;f)} && (i=1,2,3),\\ T_\alpha(\tau_j) &= \tau^{T_\alpha.e_j-\deg(T_\alpha.e_j)h_0}\,\phi(T_\alpha.e_j;f) && (j=1,\dots,9). \end{aligned} \tag{122}$$

These formulas are the elliptic difference Painlevé equations of type $(3,9)$ for the $f$ and $\tau$ variables. The polynomials $\phi(\Lambda;x)$ can be determined either recursively by

(78), or by using the determinants of Theorem 6.1. In this sense, these polynomials are computable in principle. We will show later some explicit formulas for small $\phi(\Lambda;x)$.

For $\alpha \in \Delta_{3,9}^{\mathrm{Re}}$, the translation $T_\alpha$ is defined by

$$T_\alpha(\Lambda) = \Lambda - h + (1-(h\,|\,\Lambda))c \tag{123}$$

for any $\Lambda \in M_{3,9}$, where $h \in L_{3,9}$, $\alpha = (h\,|\,\cdot)$. (Note that $(\alpha\,|\,\alpha) = 2$.) As an example, we consider the case $T_{78} = T_{\alpha_7}$. (The argument below applies to any $T_{ij} = T_{\varepsilon_{ij}}$ for mutually distinct $i,j \in \{4,5,6,7,8,9\}$). In this case, the discrete time evolution of the $f$ variables by $T_{78}$ can be written in the form

$$T_{78}(f_i) = \frac{G_i(f_1,f_2,f_3)}{F_i(f_1,f_2,f_3)} \quad (i=1,2,3) \tag{124}$$

where $F_i(x) = \phi(T_{78}(e_i);x)$ and $G_i(x) = \phi(T_{78}(e_i+h_0);x)$. For $i=1$ we have

$$T_{78}(e_1) = 3e_0 - e_2 - e_3 - e_4 - e_5 - e_6 - 2e_7 - e_9. \tag{125}$$

This means that $F_1(x) = \phi(T_{78}(e_1);x)$ is a homogeneous polynomial of degree 3 with multiplicities of zeros $(0,1,1)$ at $(p_1,p_2,p_3)$, and $(1,1,1,2,0,1)$ at $(p_4,p_5,p_6,p_7,p_8,p_9)$. In particular $F_1(x)$ can be written in the form

$$F_1(x) = a_0x_1^3+a_1x_1^2x_2+a_2x_1^2x_3+a_3x_1x_2^2+a_4x_1x_2x_3+a_5x_1x_3^2+a_6x_2^2x_3+a_7x_2x_3^2. \tag{126}$$

Observe that the monomials $x_2^3$ and $x_3^3$ are missing in this formula; this is because $F_1(x)$ should have zeros at $p_2 = (0:1:0)$ and $p_3 = (0:0:1)$. (The coefficients $a_k$ could be determined in principle from the pattern of multiplicities of zeros at $(p_4,\dots,p_9)$.) Similarly from

$$T_{78}(e_1+h_0) = 4e_0 - e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6 - 2e_7 - e_9, \tag{127}$$

we see that $G_i(x)$ is of degree 4. Since $s_0T_{78}(e_1) = T_{78}(e_1+h_0)$, we know that $G_1(x)$ is in fact determined from $F_1(x)$ as $G_1(x) = x_1^3x_2^2x_3^2\,{}^{s_0}F(x^{-1})$. These polynomials $F_i(x)$ and $G_i(x)$ are in fact too big to write down explicitly. A way to *see* this time evolution of the $f$ variables is to decompose $T_{78}$ into two steps by using the expression

$$T_{78} = w^2, \qquad w = s_{11'7}s_{22'7}s_{33'7}s_{89}s_{78}, \tag{128}$$

where $\{1',2',3'\} = \{4,5,6\}$. If we set $g_i = w(f_i)$, we obtain

$$g_i = \frac{Q_i(f_1,f_2,f_3)}{P_i(f_1,f_2,f_3)}, \quad T_{78}(f_i) = \frac{S_i(g_1,g_2,g_3)}{R_i(g_1,g_2,g_3)} \quad (i=1,2,3), \tag{129}$$

where $P_i(x) = \phi(w(e_i);x)$ and $Q_i(x) = \phi(w(e_i+h_0);x)$; $R_i(x)$ and $S_i(x)$ are determined as $R_i(x) = {}^wP_i(x)$ and $S_i(x) = {}^wQ_i(x)$, by applying $w$ to the coefficients. Since

$$w(e_i) = e_0 - e_{j'} - e_{k'}, \quad w(e_i+h_0) = 2e_0 - e_1 - e_2 - e_3 - e_{j'} - e_{k'} \tag{130}$$

for $\{i,j,k\}=\{1,2,3\}$, we see that the corresponding $\phi(\Lambda;x)$ have degree 1 and 2, respectively. In fact we have

$$\phi(e_0-e_a-e_b;x)=\frac{[\varepsilon_{1a}][\varepsilon_{1b}][\varepsilon_{1ab}]}{[\varepsilon_{12}][\varepsilon_{13}][\varepsilon_{123}]}x_1-\frac{[\varepsilon_{2a}][\varepsilon_{2b}][\varepsilon_{2ab}]}{[\varepsilon_{12}][\varepsilon_{23}][\varepsilon_{123}]}x_2+\frac{[\varepsilon_{3a}][\varepsilon_{3b}][\varepsilon_{3ab}]}{[\varepsilon_{13}][\varepsilon_{23}][\varepsilon_{123}]}x_3 \tag{131}$$

for $1\le a<b\le 9$, and

$$\begin{aligned}&\phi(2e_0-e_1-e_2-e_3-e_a-e_b;x)\\&=-\frac{[\varepsilon_{23a}][\varepsilon_{23b}][\varepsilon_{1ab}]}{[\varepsilon_{12}][\varepsilon_{13}][\varepsilon_{123}]}x_2x_3+\frac{[\varepsilon_{13a}][\varepsilon_{13b}][\varepsilon_{2ab}]}{[\varepsilon_{12}][\varepsilon_{23}][\varepsilon_{123}]}x_1x_3-\frac{[\varepsilon_{12a}][\varepsilon_{12b}][\varepsilon_{3ab}]}{[\varepsilon_{13}][\varepsilon_{23}][\varepsilon_{123}]}x_1x_2\end{aligned} \tag{132}$$

for $4\le a<b\le 9$. Hence we obtain the explicit formulas for $P_i.Q_i,R_i,S_i$:

(133)

$$\begin{aligned}P_i(x)&=\frac{[\varepsilon_{1i'}][\varepsilon_{17}][\varepsilon_{1i'7}]}{[\varepsilon_{12}][\varepsilon_{13}][\varepsilon_{123}]}x_1-\frac{[\varepsilon_{2i'}][\varepsilon_{27}][\varepsilon_{2i'7}]}{[\varepsilon_{12}][\varepsilon_{23}][\varepsilon_{123}]}x_2+\frac{[\varepsilon_{3i'}][\varepsilon_{37}][\varepsilon_{3i'7}]}{[\varepsilon_{13}][\varepsilon_{23}][\varepsilon_{123}]}x_3,\\
Q_i(x)&=-\frac{[\varepsilon_{23i'}][\varepsilon_{237}][\varepsilon_{1i'7}]}{[\varepsilon_{12}][\varepsilon_{13}][\varepsilon_{123}]}x_2x_3+\frac{[\varepsilon_{13i'}][\varepsilon_{137}][\varepsilon_{2i'7}]}{[\varepsilon_{12}][\varepsilon_{23}][\varepsilon_{123}]}x_1x_3-\frac{[\varepsilon_{12i'}][\varepsilon_{127}][\varepsilon_{3i'7}]}{[\varepsilon_{13}][\varepsilon_{23}][\varepsilon_{123}]}x_1x_2,\\
R_i(x)&=-\frac{[\varepsilon_{i1'}][\varepsilon_{1'79}][\varepsilon^-_{i1'8}]}{[\varepsilon_{1'2'}][\varepsilon_{1'3'}][\varepsilon_{123}]}x_1+\frac{[\varepsilon_{i2'}][\varepsilon_{2'79}][\varepsilon^-_{i2'8}]}{[\varepsilon_{1'2'}][\varepsilon_{2'3'}][\varepsilon_{123}]}x_2-\frac{[\varepsilon_{i3'}][\varepsilon_{3'79}][\varepsilon^-_{i3'8}]}{[\varepsilon_{1'3'}][\varepsilon_{2'3'}][\varepsilon_{123}]}x_3,\\
S_i(x)&=-\frac{[\varepsilon_{jk1'}][\varepsilon^-_{2'3'8}][\varepsilon^-_{i1'8}]}{[\varepsilon_{1'2'}][\varepsilon_{1'3'}][\varepsilon_{123}]}x_2x_3+\frac{[\varepsilon_{jk2'}][\varepsilon^-_{1'3'8}][\varepsilon^-_{i2'8}]}{[\varepsilon_{1'2'}][\varepsilon_{2'3'}][\varepsilon_{123}]}x_1x_3-\frac{[\varepsilon_{jk3'}][\varepsilon^-_{1'2'8}][\varepsilon^-_{i3'8}]}{[\varepsilon_{1'3'}][\varepsilon_{2'3'}][\varepsilon_{123}]}x_1x_2\end{aligned}$$

for $\{i,j,k\}=\{1,2,3\}$, where $\varepsilon^-_{ij8}=\varepsilon_{ij8}-\delta$.

We remark that the translation $T_\alpha$ for any $\alpha\in\Delta^{\mathrm{Re}}_{3,9}$ can be expressed as $T_\alpha=v\,T_{78}\,v^{-1}$ for some $v\in W_{3,9}$ such that $v(\alpha_7)=\alpha$. If we introduce the dependent variables $\varphi_i=v(f_i)$, $\psi_i=v(g_i)$ $(i=1,2,3)$, the discrete time evolution of these variables by $T_\alpha$ can be expressed in the same form as in the case of $T_{78}$.

Explicit description for the time evolutions of the elliptic difference Painlevé equation is discussed also in [**16**] and [**11**] from different viewpoints.

## 7. In terms of geometry of plane curves

The discrete time evolution of type $T_{ij}$ $(i,j\in\{1,\dots,9\};i\ne j)$ for the elliptic difference Painlevé equation can be described by means of geometry of plane cubic curves. In this final section we give an explanation of this fact in the scope of this paper.

Take three constants $c_1,c_2,c_3\in\mathbb{C}$ such that

$$[c_1+c_2+c_3]\ne 0,\qquad [c_i-c_j]\ne 0\quad(1\le i<j\le 3), \tag{134}$$

and set $c_0=-c_1-c_2-c_3$. With these constants fixed, let us consider the holomorphic mapping $p:\mathbb{C}\to\mathbb{P}^2(\mathbb{C})$ defined by

$$\begin{aligned}p(u)&=(x_1(u):x_2(u):x_3(u))\quad(u\in\mathbb{C}),\\ x_i(u)&=[c_0+c_i-u][c_j-u][c_k-u]\qquad(\{i,j,k\}=\{1,2,3\}).\end{aligned} \tag{135}$$

(By the quasi-periodicity of the function $[u]$, this mapping induces a holomorphic mapping $\overline{p} : E = \mathbb{C}/\Omega \to \mathbb{P}^2(\mathbb{C})$ as well.) We denote by $C_0 = \overline{p(\mathbb{C})}$ the plane curve obtained by the parametrization (135). The defining equation for this curve $C_0$ is given explicitly by

$$\begin{aligned} &-\frac{[c_0+c_3-c_1]}{[c_3-c_1]}x_1^2x_2+\frac{[c_0+c_2-c_1]}{[c_2-c_1]}x_1^2x_3+\frac{[c_0+c_3-c_2]}{[c_3-c_2]}x_1x_2^2 \\ &+2\frac{[c_0]}{[0]'}\left(\frac{[c_0+c_2-c_3]'}{[c_2-c_3]}+\frac{[c_0+c_3-c_1]'}{[c_3-c_1]}+\frac{[c_0+c_1-c_2]'}{[c_1-c_2]}\right)x_1x_2x_3 \\ &-\frac{[c_0+c_2-c_3]}{[c_2-c_3]}x_1x_3^2-\frac{[c_0+c_1-c_2]}{[c_1-c_2]}x_2^2x_3+\frac{[c_0+c_1-c_3]}{[c_1-c_3]}x_2x_3^2=0, \end{aligned} \tag{136}$$

where $[u]'$ stands for the derivative of $[u]$. (The coefficient of $x_1x_2x_3$ can be written in various ways.) We remark that this definition of $p$ and $C_0$ is related to that of $p_{\lambda,\mu}$ and $C_{\lambda,\mu}$ for the case $m=3$ in Section 4 by the change of variables

$$\lambda = c_0, \quad \mu_i = c_i + \frac{\varepsilon_0}{3} \quad (i=1,2,3), \quad t = u + \frac{\varepsilon_0}{3}. \tag{137}$$

In particular, the $W_{3,n}$-equivariant meromorphic mapping $\varphi_{3,n} : \mathfrak{h}_{3,n} \dashrightarrow \mathbb{X}_{3,n}$ can be realized by means of point configurations on one single curve $C_0 \subset \mathbb{P}^2(\mathbb{C})$:

$$\varphi_{3,n}(\varepsilon) = [p(u_1), \cdots, p(u_n)], \qquad u_j = \varepsilon_j - \frac{\varepsilon_0}{3} \quad (j=1,\dots,n) \tag{138}$$

for each generic $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \in \mathfrak{h}_{3,n}$. Thanks to the condition $c_0+c_1+c_2+c_3=0$, we see that a set of $3d$ points $p(a_1), \dots p(a_{3d})$ on $C_0$ is realized as the intersection of $C_0$ and a curve of degree $d$ if and only if $[a_1+\cdots+a_{3d}]=0$. In particular, three points $p(a_1), p(a_2), p(a_3)$ on $C_0$ are colinear if and only if $[a_1+a_2+a_3]=0$. The affine Weyl group $W_{3,n}$ acts on these variables $u_1, \dots, u_n$ as follows:

$$\begin{aligned} s_0(u_j) &= \begin{cases} u_j - \dfrac{2}{3}(u_1+u_2+u_3) & (j=1,2,3), \\ u_j + \dfrac{1}{3}(u_1+u_2+u_3) & (j=4,\dots,n). \end{cases} \\ s_k(u_j) &= u_{s_k(j)} \qquad (k=1,\dots,n-1; j=1,\dots,n). \end{aligned} \tag{139}$$

Before going further, we remark that any irreducible cubic curve in $\mathbb{P}^2(\mathbb{C})$ can be obtained by a projective linear transformation from a curve of the form $C_0$ with $[u]$ and $c_1, c_2, c_3$ appropriately chosen. In fact the curve $C_0$ is related with the Weierstrass canonical form of a cubic curve in the following way. Consider the case when $\operatorname{rank}\Omega = 2$, and for $[u]$ take the Weierstrass sigma function $\sigma(u) = \sigma(u;\Omega)$ associated with the period lattice $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. If we set

$$c_0 = -\frac{\omega_1+\omega_2}{2}, \quad c_1 = \frac{\omega_1}{2}, \quad c_2 = \frac{\omega_2}{2}, \quad c_3 = 0, \tag{140}$$

then the parametrization of $C_0$ is given by

$$\begin{aligned} x_1 &= \sigma(u)\sigma(c_2+u)\sigma(c_2-u) = \sigma(c_2)^2\sigma(u)^3(\wp(u)-\wp(c_2)), \\ x_2 &= \sigma(u)\sigma(c_1+u)\sigma(c_1-u) = \sigma(c_1)^2\sigma(u)^3(\wp(u)-\wp(c_1)), \\ x_3 &= \sigma(c_0-u)\sigma(c_1-u)\sigma(c_2-u) = -\tfrac{1}{2}\sigma(c_0)\sigma(c_1)\sigma(c_2)\sigma(u)^3\wp'(u), \end{aligned} \tag{141}$$

where $\wp(u) = \wp(u;\Omega)$ is the Weierstrass $\wp$ function associated with $\Omega$. Hence the curve $C_0$ is transformed into the canonical form

$$y_1y_3^2 = 4(y_2-\wp(c_0)y_1)(y_2-\wp(c_1)y_1)(y_2-\wp(c_2)y_1) \tag{142}$$

by the projective linear transformation

$$\begin{aligned} &x_1 = \sigma(c_2)^2(y_2-\wp(c_2)y_1), \qquad x_2 = \sigma(c_1)^2(y_2-\wp(c_1)y_1), \\ &x_3 = -\tfrac{1}{2}\sigma(c_0)\sigma(c_1)\sigma(c_2)y_3. \end{aligned} \tag{143}$$

This implies that any smooth cubic curve can be expressed in the form of $C_0$ up to a projective linear transformation. Note that through this transformation to the Weierstrass canonical form, formula (139) recovers the same Weyl group action in the parametrization by the $\wp$ function as in [**20**].

In what follows, we consider the translation $T_{89} \in W_{3,9}$ as an example, and describe the corresponding Cremona transformation in the language of geometry of plane curves. Namely, given an generic configuration $[p_1,\dots,p_9,q] \in \mathbb{X}_{3,10}$, we explain in geometric terms how to specify $\overline{p}_1,\dots,\overline{p}_9$ and $\overline{q}$ in $\mathbb{P}^2(\mathbb{C})$ such that

$$[p_1,\dots,p_9,q].T_{89} = [\overline{p}_1,\dots,\overline{p}_9,\overline{q}]. \tag{144}$$

We first consider the case where all the 10 points $p_1,\dots,p_9,q=p_{10}$ are on a smooth cubic curve $C$. Given four points $p,q,p',q' \in C$, we say that $p+q=p'+q'$ *under the addition of* $C$ if the third intersection point of the line $L_{p,q}$, passing through $p,q$, with $C$ coincides with that of the line $L_{p',q'}$.

***Lemma 7.1***. — *Let $[p_1,\dots,p_9,p_{10}] \in \mathbb{X}_{3,10}$ be generic and assume that the 10 points $p_1,\dots,p_9,p_{10}$ are on a smooth cubic curve $C$. Then the action of $T_{89}$ on $[p_1,\dots,p_9,p_{10}]$ is expressed in the form*

$$[p_1,\dots,p_7,p_8,p_9,p_{10}].T_{89} = [p_1,\dots,p_7,\overline{p}_8,\overline{p}_9,\overline{p}_{10}] \tag{145}$$

*by using the three points $\overline{p}_8,\overline{p}_9,\overline{p}_{10} \in C$ that are determined by the following three conditions.*

(1) *The 9 points $p_1,\dots,p_8,\overline{p}_9$ form the base points of a pencil of cubic curves.*
(2) $\overline{p}_8+\overline{p}_9 = p_8+p_9$ *under the addition of $C$.*
(3) $\overline{p}_9+\overline{p}_{10} = p_8+p_{10}$ *under the addition of $C$.*

In order to prove Lemma 7.1, by a projective linear transformation, we may assume that this curve $C$ is of the form $C_0$, and that the 10 points are parametrized as

$$[p_1,\dots,p_9,p_{10}] = [p(u_1),\dots,p(u_9),p(u_{10})]. \tag{146}$$

As we already know, such a configuration is transformed by $T_{89}$ into

$$\begin{aligned} &[p_1,\dots,p_9,p_{10}].T_{89} = [\overline{p}_1,\dots,\overline{p}_9,\overline{p}_{10}], \\ &\overline{p}_j = p(\overline{u}_j), \quad \overline{u}_j = T_{89}(u_j) \qquad (j=1,\dots,10). \end{aligned} \tag{147}$$

Since

$$\begin{aligned} &T_{89}(u_j) = u_j \qquad (j=1,\dots,7), \\ &T_{89}(u_8) = u_8 - \delta, \quad T_{89}(u_9) = u_9 + \delta, \\ &T_{89}(u_{10}) = u_{10} + u_8 - u_9 - \delta, \\ &u_1 + \cdots + u_9 = -\delta, \end{aligned} \tag{148}$$

the new coordinates $\overline{u}_j$ $(j=1,\dots,10)$ are determined by the conditions

$$\begin{aligned} &(0) \quad \overline{u}_j = u_j \qquad (j=1,\dots,7), \\ &(1) \quad u_1 + \cdots + u_8 + \overline{u}_9 = 0, \\ &(2) \quad \overline{u}_8 + \overline{u}_9 = u_8 + u_9, \\ &(3) \quad \overline{u}_9 + \overline{u}_{10} = u_8 + u_{10}. \end{aligned} \tag{149}$$

Lemma 7.1 is a paraphrase of this characterization of $\overline{u}_j$ $(j=1,\dots,10)$ in geometric terms. We remark that the point $\overline{p}_9$ is determined only from $p_1,\dots,p_8$, and does not depend on the position of $p_9$, while $\overline{p}_8$ depends essentially on $p_9$.

Lemma 7.1 can be extended to the general case as follows.

***Theorem 7.2.*** — *Let $[p_1,\dots,p_9,q]$ be a configuration of 10 points in $\mathbb{P}^2(\mathbb{C})$ in general position. Suppose that this configuration is generic, and take two smooth cubic curves $C_0$ and $C$ such that*

$$p_1,\dots,p_8,p_9 \in C_0, \quad \text{and} \quad p_1,\dots,p_8,q \in C, \tag{150}$$

*respectively. Then the action of the translation $T_{89}$ on the configuration $[p_1,\dots,p_9,q] \in \mathbb{X}_{3,10}$ is expressed as*

$$[p_1,\dots,p_9,q].T_{89} = [p_1,\dots,p_7,\overline{p}_8,\overline{p}_9,\overline{q}], \tag{151}$$

*in terms of the points $\overline{p}_8,\overline{p}_9$ on $C_0$ and $\overline{q} \in C$ that are determined by the following conditions:*

(1) *The 9 points $p_1,\dots,p_8,\overline{p}_9 \in C_0$ form the base points of a pencil of cubic curves.*
(2) *Under the addition of $C_0$, $\overline{p}_8 + \overline{p}_9 = p_8 + p_9$.*
(3) *Under the addition of $C$, $\overline{p}_9 + \overline{q} = p_8 + q$.*

*In particular $\overline{p}_9$ is determined as the ninth point in the intersection of $C_0$ and $C$.*

¿ From Lemma 7.1 applied to $C_0$, we have

$$[p_1,\dots,p_7,p_8,p_9].T_{89} = [p_1,\dots,p_7,\overline{p}_8,\overline{p}_9] \tag{152}$$

with $\overline{p}_8,\overline{p}_9 \in C_0$ determined by the conditions (1), (2) of Theorem 7.2. (This part does not depend on the tenth point.) Hence, the action of $T_{89}$ on $[p_1,\dots,p_8,p_9,q]$ can be written as

$$[p_1,\dots,p_7,p_8,p_9,q].T_{89} = [p_1,\dots,p_7,\overline{p}_8,\overline{p}_9,\overline{q}] \tag{153}$$

for some $\overline{q} \in \mathbb{P}^2(\mathbb{C})$. We remark here that $T_{89}$ can be expressed in the form

$$T_{89} = w\, s_{89}, \qquad w = s_{128}s_{348}s_{567}s_{348}s_{128} \in W_{3,8}, \tag{154}$$

where $W_{3,8} = \langle s_0, s_1, \ldots, s_7 \rangle$. Hence, by applying $s_{89}\, s_{9,10} \in W_{3,10}$ to (153) from the right, we obtain

$$[p_1, \ldots, p_7, p_8, q, p_9].w = [p_1, \ldots, p_7, \overline{p}_9, \overline{q}, \overline{p}_8] \qquad \text{in} \quad \mathbb{X}_{3,10}. \tag{155}$$

(Note that $s_{9,10}$ commute with $w \in W_{3,8}$.) Since $w \in W_{3,8}$, this formula projects to

$$[p_1, \ldots, p_7, p_8, q].w = [p_1, \ldots, p_7, \overline{p}_9, \overline{q}] \qquad \text{in} \quad \mathbb{X}_{3,9}. \tag{156}$$

This implies that $\overline{q}$ does not depend on $p_9$. (This fact can be seen clearly in formula (133) for $T_{78}$. In fact, none of the polynomials $P_i, Q_i, R_i, S_i$ depends on the parameter $\varepsilon_8$.) Hence, by considering the configuration $[p_1, \ldots, p_8, \overline{p}_9, q]$ on $C$ with $\overline{p}_9$ replaced for $p_9$, we have

$$[p_1, \ldots, p_7, p_8, \overline{p}_9, q].T_{89} = [p_1, \ldots, p_7, p_8, \overline{p}_9, \overline{q}]. \tag{157}$$

(The 8th and 9th components remain invariant since $p_1, \ldots, p_8, \overline{p}_9$ are already the base points of a pencil of cubic curves containing $C$.) Then by applying Lemma 7.1 to $C$, we conclude that $\overline{q}$ is determined by condition (3).

Geometric description of discrete time evolutions of type $T_{ij}$ as described above is proved in [**7**] by a more geometric argument based on the results of [**9**] and [**12**]. We remark that this geometric approach has been employed in the study of hypergeometric solutions to elliptic and multiplicative discrete Painlevé equations in [**7**], [**8**]. It is also used by [**22**] in order to clarify the relationship between the elliptic difference Painlevé equation and the integrable mapping of Quispel-Roberts-Thompson [**17**].

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