Stochastic Analysis and Stochastic Oscillatory Integrals

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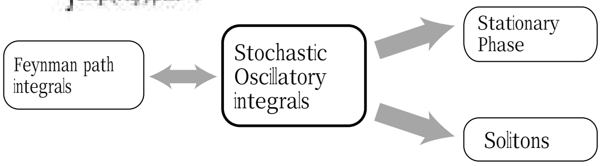
In 1920 's, Nobert Wiener established a probability measure on the path space, the space of continuous functions on the half line of real non-negative numbers. The measure is now called the Wiener measure. A stochastic oscillatory integral is an integral with respect to the Wiener measure, whose integrand is a Wiener functional of exponential type. In other words, it is a Fourier-Laplace type integral on the path space; it is of the form $\int \exp(\lambda q)\psi dP$.



The stochastic oscillatory integrals are studied with the help of two stochastic calculi, the Itô calculus and the Malliavin calculus. They are both based on the Wiener measure. The Itô calculus is a stochastic calculus concerning the time evolution of randam processes, and was established in 1940 's by Professor Kiyosi Itô. The Malliavin calculus is related to the calculus of variation on the path space. The origin of the Malliavin calculus is the paper in 1976 by Professor Paul Malliavin. While the systematic study of stochastic oscillatory integrals developed as the above two stochastic calculi advanced, the study on them with the infinite dimensional change of variables on the path space started just after the establishment of the Wiener measure. The study of the stochastic oscillatory integral has a half-century long history.

The stochastic oscillatory integral is a mathematical counterpart to the Feynman path integral, which plays a key role in the quantum mechanics, but is not justified with any mathematical rigor except some special cases. The stochastic oscillatory integral gives a mathematically firm base to investigate several heuristic arguments made with the Feynman path integrals in the quantum mechanics. For example, the heat kernel associated with the Schrödinger operator with scalar and vector potentials can be expressed as a stochastic oscillatory integral. This expression is used to study the eigenvalues of the Schrödinger operator, to show the trace formulae, and so on. Another important topic in the theory of Feynman path integrals is, so called, the semi-classical limits; if the Planck constant tends to 0, the quantum dynamics converges to the classical one. In the theory of Feynman path integrals, the semi-classical limits are explained by assuming that the Laplace and the stationary phase methods on the finite dimensional Euclidean space continue to hold on the path space, the infinite dimensional space. The Laplace and the stationary phase methods both indicate that the asymptotic main term of the path integral comes from an integral around the stationary points of the phase function giving the exponent of the integrand of the path integral. The Laplace method on the path space was studied mathematically rigorously; the borne fruit is the theory of large deviation. As for the stationary phase method, some partial results have been obtained but no systematic study was made. To give a systematic framework to study the principle of stationary phase on the path space, I introduced a new complexification of the path space attached with the Wiener measure, and achieved complex change of variable formulae (Cauchy formulae) on the complexified path space in the joint work with P. Malliavin. As an application of the complex change of variables formulae, were obtained several concrete expressions of stochastic oscillatory integrals with quadratic phase functions. The expression was applied to investigate the asymptotic behavior of stochastic oscillatory integrals, i.e., the stationary phase method on the path space.

As is well known, the Korteweg-de Vries (KdV) equation is a nonlinear differential equation describing a shallow water wave, and possesses explicit solutions called solitons. I have shown that stochastic oscillatory integrals with quadratic Wiener phase function coming from Ornstein-Uhlenbeck processes give a rise of solitons. Moreover, a concrete correspondence between the processes and the solitons was achieved. The probabilistic study of the KdV equation involves a lot of topics in the stochastic analysis; the Itô formula, the non-linear transformation in the path space, the filtering theory, the infinitely divisible laws, the quadratic Wiener functionals, and so on. Studying the KdV equation from the stochastic analytical point of view is a new and attracting subject. It has been widely known that the stochastic oscillatory integrals are applicable to the study of linear differential equations. The integrals and the equations relate to each other via the celebrated Itô formula. As for the KdV equation, such a correspondence has not been found out yet, and is needed to done.

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