# ON $L$-FUNCTIONS FOR THE SPACE OF BINARY QUADRATIC FORMS

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This note is a summary of my talk in "Workshop on L-FUNCTIONS" at Fukuoka in 22 April 2011. I thank Professor Weng for his invitation to the conference. In this note, we give explicit forms of $L$-functions with nontrivial quadratic characters for the space of binary quadratic forms.

## 1. Main results

Let $F$ be an algebraic number field. We set

$$X = \{x \in M(2) \mid x = {}^t x\} \quad \text{and} \quad G = \mathrm{GL}(1) \times \mathrm{PGL}(2)$$

over $F$. The group $G$ acts on $X$ by

$$g \cdot x = \frac{a}{\det h} h x \, {}^t h \,, \quad g = (a,h) \in G\,,\ a \in \mathrm{GL}(1)\,,\ h \in \mathrm{PGL}(2)\,,\ x \in X\,.$$

This action is faithful. Let $\mathbb{A}_F$ be the adele ring of $F$, $|\ |_{\mathbb{A}_F}$ the idele norm of $\mathbb{A}_F^\times$, $\mathbb{A}_F^1 = \{a \in \mathbb{A}_F^\times \mid |a|_{\mathbb{A}_F} = 1\}$, $\omega$ a character on $\mathbb{A}_F^1/F^\times$, and $\mathcal{S}(X(\mathbb{A}_F))$ the Schwartz space on $X(\mathbb{A}_F)$. We assume that $\omega$ is a quadratic character, that is, $\omega^2 = 1$. We put

$$\omega(g) = \omega\big(\frac{a}{\det h}\big) \text{ and } \chi(g) = a^2 \text{ for } g = (a,h) \in G.$$

We set

$$X^*(F) = \{x \in X(F) \mid \det x \neq 0 \ \text{ and } \ -\det x \notin (F^\times)^2\}.$$

Let $\mathrm{d}g$ be the Tamagawa measure on $G(\mathbb{A})$. We define the zeta integral $Z(\Phi, s, \omega)$ by

$$Z(\Phi, s, \omega) = \int_{G(\mathbb{A}_F)/G(F)} |\chi(g)|^s \, \omega(g) \sum_{x \in X^*(F)} \Phi(g \cdot x) \, \mathrm{d}g$$

for $s \in \mathbb{C}$ and $\Phi \in \mathcal{S}(X(\mathbb{A}_F))$. $Z(\Phi, s, \omega)$ is absolutely convergent for $\mathrm{Re}(s) > 3/2$. We will give an explicit form of $Z(\Phi, s, \omega)$ for any non-trivial quadratic character $\omega$. Let $\Sigma$ denote the set of places of $F$. For any $v \in \Sigma$, we denote by $F_v$ the completion of $F$ at $v$ and $|\ |_v$ the normal valuation of $F_v$. Let $\mathcal{S}(X(F_v))$ be the Schwartz space on $X(F_v)$. For each $v < \infty$, we denote by $\mathfrak{O}_v$ the ring of integers of $F_v$, $\mathfrak{p}_v$ the maximal ideal of $\mathfrak{O}_v$, and $\pi_v$ a prime element. We set $q_v = |\mathfrak{O}_v/\mathfrak{p}_v|$. Let $d$ be an element of $F^\times$ such that the quadratic extension $F(\sqrt{d})$ corresponds to $\omega$ via the class field theory. We set $\omega = \prod_{v \in \Sigma} \omega_v$ where $\omega_v$ is a character on $F_v^\times$. Let $\zeta_F(s)$ denote the Dedekind zeta function defined by $\zeta_F(s) = \prod_{v<\infty}(1 - q_v^{-s})^{-1}$ and $L(s,\omega)$ the Hecke $L$-function defined by $L(s,\omega) = \prod_{v \in \Sigma} L_v(s, \omega_v)$ where

$$L_v(s,\omega_v) = \begin{cases} (1 - \omega_v(\pi_v) q_v^{-s})^{-1} & \text{if } v < \infty \text{ and } \omega_v \text{ is unramified} \\ 1 & \text{otherwise} \end{cases}.$$

Let $\Delta_F$ be the discriminant of $F$, $e_v$ the ramification index of $F_v$ for $v|2$, and $\Phi_{0,v}$ the characteristic function of

$$\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in X(F_v) \,|\, a,\, c \in \mathfrak{O}_v \ \text{ and } \ b \in \frac{1}{2}\mathfrak{O}_v \right\}$$

for $v < \infty$. For each $v \in \Sigma$, we define the local zeta function $Z(\Phi_v, s, \omega_v, d)$ by

$$Z(\Phi_v, s, \omega_v, d) = \frac{2\, c_v}{L_v(1, \omega_v)} \times \int_{G(F_v)\cdot x_d} |\det x_v|_v^{s-\frac{3}{2}} \omega_v(x_v)\, \Phi_v(x_v)\, \mathrm{d}x_v$$

where $x_d = \begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix}$, $c_v = \begin{cases} (1-q_v^{-1})^{-1} & \text{if } v < \infty \\ 1 & \text{if } v|\infty \end{cases}$, $\mathrm{d}x_v$ is the Haar measure on $X(F_v)$ normalized by $\int_{X(\mathfrak{O}_v)} \mathrm{d}x_v = 1$, and $\omega_v(x_v) = \omega_v(\frac{a_v}{\det h_v})$ for $x_v = g_v \cdot x_d$, $g_v = (a_v, h_v) \in G(F_v)$. The following formula is deduced from Saito's works [Saito1, Saito2].

**Theorem 1.** *We assume that $\Phi = \prod_{v\in\Sigma} \Phi_v$ where $\Phi_v \in \mathcal{S}(X(F_v))$. Let $S$ be any finite subset of $\Sigma$, which contains $\{v \in \Sigma \,|\, v|\infty$ or $\omega_v$ is ramified or $\Phi_v \neq \Phi_{0,v}\}$. Then, we have*

$$Z(\Phi, s, \omega) = \frac{L(1,\omega)\, |\Delta_F|^{-3/2}}{\mathrm{Residue}_{s=1}\zeta_F(s)} \times \prod_{v|2\,,\, v\notin S} \frac{2q_v^{(2s-1)e_v}}{|\mathfrak{O}_v^\times : (\mathfrak{O}_v^\times)^2|} \times \frac{\zeta_F^S(2s-1)}{\zeta_F^S(2)} \times \prod_{v\in S} Z(\Phi_v, s, \omega_v, d)$$

*where we set $\zeta_F^S(s) = \prod_{v\notin S}(1-q_v^{-s})^{-1}$.*

From Theorem 1 and some results for local zeta functions (cf. [Igusa1, Igusa2, SS]) we find that $Z(\Phi, s, \omega)$ is meromorphically continued to the whole complex $s$-plane. It is proved by [Yukie] in general. We define the function $\Phi_{1,v}$ as

$$\Phi_{1,v}(x_v) = \begin{cases} \omega_v(a) & \text{if } {}^{\exists}h \text{ s.t. } x_v = h \begin{pmatrix} a & b \\ b & c \end{pmatrix},\ a \in \mathfrak{O}_v^\times,\ b, c \in \mathfrak{p}_v \\ 0 & \text{otherwise} \end{cases}$$

for each $v < \infty$. By Theorem 1 and local computations we have the following.

**Theorem 2.** *We set*

$$\begin{aligned} S_{\mathrm{fin}} =& \{v \in \Sigma \,|\, v < \infty \ \textit{and} \ \omega_v \ \textit{is ramified}\}, \\ S_{\mathrm{fin},\not{|}2} =& \{v \in S_{\mathrm{fin}} \,|\, v \not{|} 2\}, \\ S_{\mathrm{fin},2,1} =& \{v \in S_{\mathrm{fin}} \,|\, v|2 \ \textit{and} \ d \in \mathfrak{O}_v^\times (F_v^\times)^2\}, \\ S_{\mathrm{fin},2,\pi_v} =& \{v \in S_{\mathrm{fin}} \,|\, v|2 \ \textit{and} \ d \in \pi_v \mathfrak{O}_v^\times (F_v^\times)^2\}, \\ \Sigma_\infty =& \{v \in \Sigma \,|\, v|\infty\}. \end{aligned}$$

*We have the disjoint union $S_{\mathrm{fin}} = S_{\mathrm{fin},v\not{|}2} \cup S_{\mathrm{fin},2,1} \cup S_{\mathrm{fin},2,\pi_v}$. We assume $\Phi = \prod_{v\in\Sigma}\Phi_v$. If we set $\Phi_v = \Phi_{0,v}$ for $v \notin S_{\mathrm{fin}} \cup \Sigma_\infty$ and $\Phi_v = \Phi_{1,v}$ for $v \in S_{\mathrm{fin}}$, then we obtain*

$$\begin{aligned} Z(\Phi, s, \omega) =& \frac{L(1,\omega)\, |\Delta_F|^{-3/2}\, 2^{-[F:\mathbb{Q}]}}{\mathrm{Residue}_{s=1}\zeta_F(s)} \times \frac{\zeta_F(2s-1)}{\zeta_F(2)} \times \prod_{v\in\Sigma_\infty} Z(\Phi_v, s, \omega_v, d) \\ & \times \prod_{v|2\,,\, v\notin S_{\mathrm{fin}}} q_v^{(2s-1)e_v} \times \prod_{v\in S_{\mathrm{fin},v\not{|}2}} q_v^{-s+\frac{1}{2}} \times \prod_{v\in S_{\mathrm{fin},2,1}} q_v^{-2s+1} \times \prod_{v\in S_{\mathrm{fin},2,\pi_v}} q_v^{-s+\frac{1}{2}}. \end{aligned}$$

We will show that the formula for $L(s, L_2^*, \psi)$ in [IS, Theorem 1] is derived from Theorem 2. Hence, Theorem 2 is a generalization of the formula. We can similarly deduce the other formulas of [IS, Theorem 1] from Theorem 1.

We assume $F = \mathbb{Q}$. Let $m$ be a square-free integer. We assume that $m$ is not 0 and 1. Let $\omega_m$ be the character on $\mathbb{A}_{\mathbb{Q}}^1/\mathbb{Q}^\times$ which corresponds to the quadratic field $\mathbb{Q}(\sqrt{m})$, $D$ the discriminant of $\mathbb{Q}(\sqrt{m})$, and $\psi_m$ the Dirichlet character on $\mathbb{Z}/D\mathbb{Z}$ which corresponds to the quadratic field $\mathbb{Q}(\sqrt{m})$. We set

$$L = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in X(\mathbb{Q}) \,|\, a,\, c \in \mathbb{Z} \text{ and } b \in \frac{1}{2}\mathbb{Z} \right\},$$

$$L_1 = \{\, x \in L \,|\, x \text{ is positive definite} \,\},$$

$$L_2 = \{\, x \in L \,|\, \det x < 0\,,\, -\det(x) \notin (\mathbb{Q}^\times)^2 \}.$$

For $x \in L$, we set

$$\psi_m(x) = \begin{cases} \psi_m(a) & \text{if there exists an element } g \in \mathrm{SL}(2,\mathbb{Z}) \text{ such that } gx\,{}^t g = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \\ & b \in \frac{1}{2}m\mathbb{Z} \text{ and } c \in m\mathbb{Z} \text{ if } m \equiv 1 \mod 4, \\ & b, c \in m\mathbb{Z} \text{ if } m \equiv 2 \mod 4, \\ & \text{and } b, c \in 2m\mathbb{Z} \text{ if } m \equiv 3 \mod 4, \\ 0 & \text{otherwise.} \end{cases}$$

For each $x \in L$, we denote by $\mathfrak{O}_x$ the maximal order of $\mathbb{Q}(\sqrt{-\det x})$ and $\varepsilon_x > 1$ the fundamental unit of $\mathbb{Q}(\sqrt{-\det x})$. We set

$$\mu(x) = \begin{cases} \pi\, |\mathfrak{O}_x^\times|^{-1} & \text{if } x \in L_1 \\ \log \varepsilon_x & \text{if } x \in L_2 \end{cases}.$$

Let $G_x$ denote the stabilizer of $x \in X(F)$ in $G$ and let $G_x^0$ denote the connected component of 1 in $G_x$. We define the $L$-function $L(s, m, i)$ by

$$L(s, m, i) = \sum_{x \in G(\mathbb{Z}) \backslash L_i} \frac{\mu(x)\, \psi_m(x)}{[G_x(\mathbb{Z}) : G_x^0(\mathbb{Z})]\, |\det x|^s}$$

except for the case $m < 0$ and $i = 2$. We put $\zeta(s) = \zeta_{\mathbb{Q}}(s)$. From Theorem 2 we deduce the following.

**Theorem 3.** *We have*

$$L(s, m, \frac{3 + \mathrm{sgn}(m)}{2}) = L(1, \omega_m) \times \zeta(2s-1) \times |m|^{-s+\frac{1}{2}} \times \begin{cases} 2^{-2s} & \textit{if } m \equiv 3 \mod 4 \\ 2^{-1} & \textit{if } m \equiv 2 \mod 4 \\ 2^{2s-2} & \textit{if } m \equiv 1 \mod 4 \end{cases}.$$

*We also have* $L(s, m, 1) = 0$ *for* $m > 0$.

If we substitute $m = -p$ ($p \equiv 3 \mod 4$) into Theorem 3, then the above formula is the same as the formula for $L(s, L_2^*, \psi)$ in [IS, Theorem 1].

## 2. Endoscopy

In this section, we explain why we are interested in the zeta integrals with quadratic characters.

First, we review a formula given by Labesse-Langlands [LL, (5.11)]. We consider the parabolic subgroup

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \mid a \in F^\times, \, v \in F \right\}$$

of $\mathrm{SL}(2, F)$. By the adjoint action of the Levi subgroup $a \in \mathbb{G}_m$ on the unipotent radical $v \in \mathbb{G}_a$, we can define the prehomogeneous zeta function $\xi(\phi, s)$ as

$$\xi(\phi, s) = \int_{\mathbb{A}_F^\times / F^\times} |a^2|^s \sum_{v \in F^\times} \phi(a^2 v) \, \mathrm{d}^\times a$$

where $\phi \in \mathcal{S}(\mathbb{A}_F)$ and $\mathrm{Re}(s) > 1$. They proved the following formula in [LL].

**Theorem 4** (Labesse-Langlands)**.** *For* $\mathrm{Re}(s) > 1$*, we have*

$$\xi(\phi, s) = \frac{1}{2} \sum_{\omega} \zeta(\phi, s, \omega)$$

*where* $\omega$ *runs over all quadratic characters on* $\mathbb{A}_F^1 / F^\times$ *and* $\zeta(\phi, s, \omega)$ *is the Tate integral, that is,*

$$\zeta(\phi, s, \omega) = \int_{\mathbb{A}_F^\times} |a|^s \, \omega(a) \, \phi(a) \, \mathrm{d}^\times a.$$

This theorem is proved by the Poisson summation formula for $\mathbb{A}_F^1/(F^\times)^2$ and $F^\times/(F^\times)^2$. They used this formula to stabilize the trace formula for $\mathrm{SL}(2)$. We can understand the meaning of the formula from point of view of trace formula.

Let $f \in C_c^\infty(\mathrm{SL}(2, \mathbb{A}_F))$ and $\phi(v) = \int_K f(k^{-1} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} k) \, \mathrm{d}k$ where $K$ is the standard maximal compact subgroup of $\mathrm{SL}(2, \mathbb{A}_F)$. Then, the unipotent term in the geometric side of the trace formula for $f$ is

$$\lim_{s \to 1} \frac{\mathrm{d}}{\mathrm{d}s} (s-1) \xi(\phi, s) = \frac{1}{2} \lim_{s \to 1} (s-1) \zeta(\phi, s, 1) + \frac{1}{2} \sum_{\omega \neq 1} \zeta(\phi, 1, \omega).$$

The first term corresponds to an unipotent term of trace formula for $\mathrm{GL}(2)$. Hence, it is stable. If $L$ is the quadratic extension of $F$ which corresponds to $\omega \neq 1$, then $H = R^{(1)}_{L/F} \mathbb{G}_m$ is an elliptic endoscopic group of $\mathrm{SL}(2)$. Furthermore, if $f^H$ is the transfer of $f$ to $H$, then we have

$$\zeta(\phi, 1, \omega) = L(1, \omega) \, f^H(1) \, .$$

From this we have obtained a stabilization of the unipotent term of $\mathrm{SL}(2)$. The stabilization directly followed from Theorem 4. Hence, Theorem 4 looks like a stabilization of the zeta function $\xi(\phi, s)$. We are interested in stabilizations of prehomogeneous zeta functions as a generalization of Theorem 4. In addition, we also want to know relations between such stabilizations and explicit forms of prehomogeneous zeta functions, which were studied by Ibukiyama and Saito.

If we see the Siegel parabolic subgroup

$$\left\{ \begin{pmatrix} h & O_2 \\ O_2 & {}^th^{-1} \end{pmatrix} \begin{pmatrix} I_2 & S \\ O_2 & I_2 \end{pmatrix} \in \mathrm{Sp}(2,F) \,\middle|\, h \in \mathrm{GL}(2,F) \text{ and } S \in X(F) \right\}$$

of $\mathrm{Sp}(2,F)$, we can define the prehomogeneous zeta function $\Xi(\Phi,s)$ by

$$\Xi(\Phi,s) = \int_{\mathrm{GL}(2,\mathbb{A}_F)/\mathrm{GL}(2,F)} |\det h|_{\mathbb{A}_F}^{2s} \sum_{x \in X^*(F)} \Phi(hx\,{}^th)\mathrm{d}h$$

where $\Phi \in \mathcal{S}(X(\mathbb{A})_F)$, $\mathrm{Re}(s) > 3/2$, and $\mathrm{d}h$ is the Tamagawa measure on $\mathrm{GL}(2)$. If we apply the above-mentioned argument of [LL] to $\Xi(\Phi,s)$ and the center of $\mathrm{GL}(2)$, then we have

$$\Xi(\Phi,s) = \sum_\omega Z(\Phi,s,\omega)$$

where $\omega$ runs over all quadratic characters on $\mathbb{A}_F^1/F^\times$. Let $f \in C_c^\infty(\mathrm{Sp}(2,\mathbb{A}_F))$ and $\Phi(x) = \int_K f(k^{-1}\begin{pmatrix} I_2 & x \\ O_2 & I_2 \end{pmatrix})\mathrm{d}k$, where $K$ is a suitable maximal compact subgroup of $\mathrm{Sp}(2,\mathbb{A}_F)$. We denote by $\{\gamma\}_{\mathrm{Sp}(2,F)}$ the $Sp(2,F)$-conjugacy class of $\gamma \in \mathrm{Sp}(2,F)$. Hoffmann and I proved that the unipotent term for $\cup_{x\in X^*(F)}\{\begin{pmatrix} I_2 & x \\ O_2 & I_2 \end{pmatrix}\}_{\mathrm{Sp}(2,F)}$ in the geometric side of the trace formula for $f$ is equal to

$$\lim_{s\to 3/2} \frac{\mathrm{d}}{\mathrm{d}s}(s-\frac{3}{2})\,\Xi(\Phi,s).$$

Furthermore, it follows from the above-mentioned equality that

$$\lim_{s\to 3/2} \frac{\mathrm{d}}{\mathrm{d}s}(s-\frac{3}{2})\,\Xi(\Phi,s) = \lim_{s\to 3/2} \frac{\mathrm{d}}{\mathrm{d}s}(s-\frac{3}{2})\,Z(\Phi,s,1) + \sum_{\omega\neq 1} Z(\Phi,\frac{3}{2},\omega)\,.$$

The first term should be unstable. However, I do not know how to stabilize it. If we substitute $s = 3/2$ into Theorem 1, then we have

$$Z(\Phi,\frac{3}{2},\omega) = \frac{L(1,\omega)\,|\Delta_F|^{-3/2}}{\mathrm{Residue}_{s=1}\zeta_F(s)} \times \prod_{v|2\,,\,v\notin S} \frac{2q_v^{2e_v}}{|\mathfrak{O}_v^\times : (\mathfrak{O}_v^\times)^2|} \times \prod_{v\in S} Z(\Phi_v,\frac{3}{2},\omega_v,d)\,.$$

If we see the result of [Assem], then it seems that $Z(\Phi,\frac{3}{2},\omega)$ is related to the elliptic endoscopic group $H = (R^{(1)}_{L/F}\mathbb{G}_m) \times \mathrm{SL}(2)$ of $\mathrm{Sp}(2)$, where $L$ is the quadratic extension of $F$ corresponding to $\omega \neq 1$. Spallone and I are studying this stabilization now.

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