On a relation between p-adic gamma functions and p-adic periods

Tomokazu Kashio, Tokyo univ. of Science

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1 Motivation.

We recall **Stark's conjecture**. Let K/F bf an abelian extension of number fields. For $\tau \in G := \mathrm{Gal}(K/F)$, we define the partial zeta function as a partial sum of the Dedekind zeta function, as follows.

$$\zeta(\tau, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_F,\ \left(\frac{K/F}{\mathfrak{a}}\right) = \tau} N\mathfrak{a}^{-s}.$$

Here $\left(\frac{K/F}{\mathfrak{a}}\right)$ is the Artin symbol.

Assume F is a totally real field, there exists an embedding $K \hookrightarrow \mathbb{R}$, and fix this embedding. In this case Stark's conjecture states that

$$u(\tau) := \exp(2\zeta'(0,\tau)) \in K,$$
$$u(\tau)^{\sigma} = u(\sigma\tau)$$

for $\tau, \sigma \in G$. We call $u(\tau)$ Stark's units because these are units in K in almost cases.

Today we consider the case of $F = \mathbb{Q}$, $K = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ ($\zeta_m := \exp(2\pi i/m)$). Then we have the isomorphism

$$\begin{aligned}(\mathbb{Z}/m\mathbb{Z})^{\times}/\{\pm 1\} &\cong \mathrm{Gal}(\mathbb{Q}(\zeta_m + \zeta_m^{-1})/\mathbb{Q}),\\ \pm a \quad &\mapsto [\overline{\sigma}_a \colon \zeta_m + \zeta_m^{-1} \mapsto \zeta_m^a + \zeta_m^{-a}]\end{aligned}$$

and we can calculate

$$u(\overline{\sigma}_a) = \exp(2\zeta'(0, \overline{\sigma}_a)) = \left(2\sin(\frac{a}{m}\pi)\right)^{-2}$$

by using the Hurwitz-Lerch formula and Euler's reflection formula. Therefore Stark's conjecture in this case, which is proved, states **a reciprocity law on sin-values**

$$\sin^2\left(\frac{a}{m}\pi\right)^{\overline{\sigma}_b} = \sin^2\left(\frac{ab}{m}\pi\right).$$

The **aim** of this talk is

- to drive this reciprocity law from a "**reciprocity law on beta functions**",
- by studying **periods of Fermat curves** and **its p-adic analogues**.

2 Periods of Fermat curves.

Let F_m be the mth Fermat curve $x^m + y^m = 1$. We consider the differentials of second kind $\eta_{r/m,s/m} := x^{r-1}y^{s-m}dx$ $(0 < r, s < m,\ r + s \neq m)$. Then
• Rohrlich showed that for any $\gamma \in H_1(F_m(\mathbb{C}), \mathbf{Q})$

$$\frac{\int_\gamma \eta_{r/m,s/m}}{B(r/m, s/m)} \in \mathbb{Q}(\zeta_m),$$

where we define the beta function by $B(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.
• Since $H_1(F_m(\mathbb{C}), \mathbf{Q})$ is a $\mathbb{Q}(\zeta_m)$-vector space, there exists γ_0 s.t.

$$\frac{\int_{\gamma_0} \eta_{r/m,s/m}}{B(r/m, s/m)} = 1.$$

Note. The reason why we will study Fermat curves is because **Euler's reflection formula**

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

relates such periods to Stark's units over $\mathbb{Q}$ which are sin-values.

3 p-adic gamma functions.

For simplicity, assume p is an odd prime.
• Morita constructed the p-adic gamma function $\Gamma_p \colon \mathbb{Z}_p \to \mathbb{Z}_p^\times$, which is continuous and characterized by

$$\Gamma_p(n) = (-1)^n \prod_{i=1,\ p \nmid i}^{n-1} i \quad (n \in \mathbb{N}).$$

• We can generalize it to $\mathbb{Q}_p$. Namely there exists a function $\Gamma_p \colon \mathbb{Q}_p \to \overline{\mathbb{Q}_p}^\times$, unique up to $\mathrm{Ker}(\log_p)$, satisfying

- it is continuous, and is a generalization of Morita's Γ_p,
- $\Gamma_p(z+1) \equiv z\Gamma_p(z) \bmod \mathrm{Ker}(\log_p)$ for $z \in \mathbb{Q}_p - \mathbb{Z}_p$,
- $\Gamma_p(2z) \equiv 2^{2z-1/2}\Gamma_p(z)\Gamma_p(z+1/2) \bmod \mathrm{Ker}(\log_p)$ for $z \in \mathbb{Q}_p - \mathbb{Z}_p$.

Note.

1. $\mathrm{Ker}(\log_p)$ is the subgroup of $\overline{\mathbb{Q}_p}^\times$ generated by rational powers of p and roots of unity.
2. The functional equations which characterize our Γ_p are p-adic analogues of classical formulas
$$\Gamma(z+1) = z\Gamma(z),\ \Gamma(2z) = \frac{2^{2z-1/2}}{\sqrt{2\pi}}\Gamma(z)\Gamma(z+1/2).$$

4 p-adic periods of Fermat curves.

Facts.

- By comparison theorems of cohomologies ($H_1 = \widehat{H}^1_B$, $H_B \otimes \mathbb{Q}_p = H_{p,et}$, $H_{p,et} \otimes B_{dR} = H_{dR} \otimes B_{dR}$), we define a pairing

$$H_1(F_m(\mathbb{C}), \mathbb{Q}) \times H^1_{dR}(F_m/\mathbb{Q}) \to B_{dR}$$

  and denote by $\int_{\gamma,p} \eta$ the image of (γ, η) under this map. This is a p-adic counterpart of the usual integral $\int_\gamma \eta \in \mathbb{C}$.

- Actually, images $\int_{\gamma,p} \eta$ are in a smaller ring $B_{cris}\overline{\mathbb{Q}_p} \cap B^+_{dR}$.

- On the ring $B_{cris}\overline{\mathbb{Q}_p}$, there exist (more than one) absolute Frobenius actions

$$\Phi_\sigma := \Phi_{cris}^{\deg \sigma} \otimes \sigma \ (\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)).$$

  Here Φ_{cris} is the absolute Frobenius action on B_{cris} and $\deg\sigma$ is defined by

$$\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p), \ \sigma \mapsto \mathrm{Frob}_p^{\deg\sigma}.$$

Note.

1. B_{cris}, B_{dR} are Fontaine's rings of p-adic periods.
2. The definition of our pairing depends on the choices of embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, etc.
3. The Jacobian of F_m has CM, therefore is potentially good at p.

Theorem 1. *(K-, a corollary of Coleman's Theorem.)*

- *Define the p-adic beta function by* $B_p(\alpha, \beta) := \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}$.
- *Define the action of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ *on rational numbers* $\mathbb{Q} \cap [0,1)$ *by identifying* $\mathbb{Q} \cap [0,1) \cong \{$ *roots of unity* $\in \overline{\mathbb{Q}_p}\}$, $a/m \mapsto \zeta_m^a$.
- *For simplicity, assume that* $p > 3$, $p | m$, $p \nmid r, s, r+s$.

Then we have

$$\Phi_\sigma \left(\frac{\int_{\gamma,p} \eta_{\frac{r}{m},\frac{s}{m}}}{B_p(\frac{r}{m}, \frac{s}{m})} \right) \equiv \frac{\int_{\gamma,p} \eta_{\sigma(\frac{r}{m}),\sigma(\frac{s}{m})}}{B_p(\sigma(\frac{r}{m}), \sigma(\frac{s}{m}))} \bmod \mathbb{Q}^\times \mathrm{Ker}(\log_p),$$

for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, $0 < r, s < m$, $r + s \neq m$.

5 A reciprocity law.

- At first we saw $\frac{B(\frac{r}{m},\frac{s}{m})}{\int_\gamma \eta_{\frac{r}{m},\frac{s}{m}}} \in \mathbb{Q}(\zeta_m)$.
- Next we considered its p-adic analogue $\frac{\int_{\gamma,p} \eta_{\frac{r}{m},\frac{s}{m}}}{B_p(\frac{r}{m},\frac{s}{m})} \in B_{dR}$.
- Now let us product them

$$\frac{B(\frac{r}{m},\frac{s}{m})}{\int_\gamma \eta_{\frac{r}{m},\frac{s}{m}}} \frac{\int_{\gamma,p} \eta_{\frac{r}{m},\frac{s}{m}}}{B_p(\frac{r}{m},\frac{s}{m})} \in B_{dR}.$$

  Then it does not depend on the choice of $\gamma \in H_1(F_m(\mathbb{C}),\mathbb{Z})$ because the dependences of $\int_\gamma$, $\int_{\gamma,p}$ on γ are canceled each other.

- For $\tau_1,\tau_2 \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, we take r, s so that τ_1,τ_2 correspond to r,s under the isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$. We put

$$B(\tau_1,\tau_2) := \frac{B(\frac{r}{m},\frac{s}{m})}{\int_\gamma \eta_{\frac{r}{m},\frac{s}{m}}} \frac{\int_{\gamma,p} \eta_{\frac{r}{m},\frac{s}{m}}}{B_p(\frac{r}{m},\frac{s}{m})}.$$

Since there exists γ_0 s.t. $\frac{B(\frac{r}{m},\frac{s}{m})}{\int_{\gamma_0} \eta_{\frac{r}{m},\frac{s}{m}}} = 1$, i.e., $B(\tau_1,\tau_2) = \frac{\int_{\gamma_0,p} \eta_{\frac{r}{m},\frac{s}{m}}}{B_p(\frac{r}{m},\frac{s}{m})}$, we may apply our theorem to our $B(\tau_1,\tau_2)$. Then we get a **reciprocity law on beta-functions**

$$\Phi_\sigma(B(\tau_1,\tau_2)) \equiv B(\sigma\tau_1,\sigma\tau_2) \bmod \mathbb{Q}^\times \mathrm{Ker}(\log_p).$$

Here we identify $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ as the image of σ under the map

$$\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p) \subset \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}).$$

6 The relation to Stark's units/$\mathbb{Q}$.

Let ρ be the complex conjugation map. We consider the product

$$B(\tau_1,\tau_2)B(\rho\tau_1,\rho\tau_2) = \frac{B(\frac{r}{m},\frac{s}{m})}{\int_\gamma \eta_{\frac{r}{m},\frac{s}{m}}} \frac{\int_{\gamma,p} \eta_{\frac{r}{m},\frac{s}{m}}}{B_p(\frac{r}{m},\frac{s}{m})} \frac{B(\frac{m-r}{m},\frac{m-s}{m})}{\int_\gamma \eta_{\frac{m-r}{m},\frac{m-s}{m}}} \frac{\int_{\gamma,p} \eta_{\frac{m-r}{m},\frac{m-s}{m}}}{B_p(\frac{m-r}{m},\frac{m-s}{m})}.$$

We can calculate some pairs.

- $B(\frac{r}{m},\frac{s}{m})B(\frac{m-r}{m},\frac{m-s}{m}) = \frac{\sin(\frac{r+s}{m}\pi)\pi}{\sin(\frac{r}{m}\pi)\sin(\frac{s}{m}\pi)} \frac{m}{m-r-s}$ (by Euler's reflection formula).
- $B_p(\frac{r}{m},\frac{s}{m})B_p(\frac{m-r}{m},\frac{m-s}{m}) \equiv \frac{m}{m-r-s} \bmod \mathrm{Ker}(\log_p)$ (by a direct calculation).
- $\frac{\int_{\gamma,p} \eta_{\frac{r}{m},\frac{s}{m}} \int_{\gamma,p} \eta_{\frac{m-r}{m},\frac{m-s}{m}}}{\int_\gamma \eta_{\frac{r}{m},\frac{s}{m}} \int_\gamma \eta_{\frac{m-r}{m},\frac{m-s}{m}}}$ should be $\frac{(2\pi i)_p}{2\pi i}$, where $(2\pi i)_p \in B_{dR}$ is the p-adic counterpart of $2\pi i$ (a consequence of Shimura's monomial relations on CM-periods and its p-adic analogues by de Shalit etc).

Moreover by $B(\tau_1, \tau_2)$-symbol, we can write a power of each Stark's unit $/\mathbb{Q}$. For example, assume that m is odd and take f s.t. $2^f \equiv 1 \bmod m$. Then

$$\begin{aligned}
&\frac{B(\frac{r}{m},\frac{r}{m})B(\frac{-r}{m},\frac{-r}{m})}{2\pi B_p(\frac{r}{m},\frac{r}{m})B(\frac{-r}{m},\frac{-r}{m})} \\
&\equiv \frac{B(\frac{r}{m},\frac{r}{m})B(\frac{m-r}{m},\frac{m-r}{m})}{2\pi B_p(\frac{r}{m},\frac{r}{m})B(\frac{m-r}{m},\frac{m-r}{m})} \\
&\equiv \frac{\sin(\frac{2r}{m}\pi)}{2\sin(\frac{r}{m}\pi)\sin(\frac{r}{m}\pi)} \bmod \mathrm{Ker}(\log_p),
\end{aligned}$$

$$\begin{aligned}
&\prod_{i=0}^{f-1}\left[\frac{B(\frac{2^ia}{m},\frac{2^ia}{m})B(\frac{-2^ia}{m},\frac{-2^ia}{m})}{2\pi B_p(\frac{2^ia}{m},\frac{2^ia}{m})B(\frac{-2^ia}{m},\frac{-2^ia}{m})}\right]^{2^{f-i}} \\
&\equiv \frac{\sin(\frac{2a}{m}\pi)^{2^f}}{2^{2^f}\sin(\frac{a}{m}\pi)^{2^{f+1}}}\frac{\sin(\frac{4a}{m}\pi)^{2^{f-1}}}{2^{2^{f-1}}\sin(\frac{2a}{m}\pi)^{2^f}}\cdots\frac{\sin(\frac{2^fa}{m}\pi)}{2\sin(\frac{2^{f-1}a}{m}\pi)^2} \\
&\equiv \frac{1}{2}\left(2\sin(\frac{a}{m}\pi)\right)^{2-2^{f+1}} \\
&\equiv \frac{1}{2}u(\overline{\sigma}_a)^{2^f-1} \bmod \mathrm{Ker}(\log_p).
\end{aligned}$$

Since Φ_σ is σ-semi linear, we can drive

$$u(\tau)^\sigma \equiv u(\sigma\tau) \bmod \mathbb{Q}^\times\mathrm{Ker}(\log_p)$$

for $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p) \subset \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, from the reciprocity law on **beta-functions**. We may vary $p \mid m$. Therefore we get the reciprocity law on **Stark's units** $/\mathbb{Q}$, mod $\mathbb{Q}^\times\mathrm{Ker}(\log_p)$.

Remark.

- The proof of our main Theorem involves Coleman's calculation of absolute Frobenius automorphism on F_m.
- "mod $\mathrm{Ker}(\log_p)$" comes from the problem of the normalization of Γ_p, "mod $\mathbb{Q}^\times$" comes up when we rewrite Coleman's results in terms of our Γ_p.
- Our beta symbol $B(\tau_1, \tau_2)$ should be generalized to any totally real field. The Gamma function (the Beta function) and the period of Fermat curves should become multiple gamma functions studied by Barnes, Shintani, Yoshida and Shimura's CM-period symbol. Yoshida and K- also studied their p-adic analogues.
- We saw the product $\prod_{\sigma\in<\rho>} B(\sigma\tau_1, \sigma\tau_2)$ relates to cyclotomic units. In fact, the Gross-Koblitz formula relates the product $\prod_{\sigma\in D_p} B(\sigma\tau_1, \sigma\tau_2)$ to Gauss sums, where D_p is the decomposition group.

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