Exceptional zeros of the Selberg zeta function and noncongruence subgroups

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§1. Exceptional zeros of the Selberg zeta function

Γ: a cocompact Fuchsian group

$-1_2 \in \Gamma$, $\overline{\Gamma} = \Gamma/\{\pm 1_2\}$.

χ: a character of Γ such that $\chi(-1_2) = 1$.

We consider χ as a character of $\overline{\Gamma}$.

The Selberg zeta function $Z_\Gamma(s, \chi)$ is defined by

$$Z_\Gamma(s,\chi) = \prod_{\{\gamma\}} \prod_{k=0}^{\infty} (1 - \chi(\gamma)(N(\gamma))^{-s-k})$$

which converges absolutely when $\Re(s) > 1$ and can be continued to an entire function. Here $\{\gamma\}$ extends over all primitive hyperbolic conjugacy classes of $\overline{\Gamma}$ and $N(\gamma)$ denotes the norm of $\{\gamma\}$. When χ is trivial, we denote $Z_\Gamma(s,\chi)$ by $Z_\Gamma(s)$.

I will give a simple proof that there exists Γ for which $Z_\Gamma(s)$ has an exceptional zero.

$L^2(\Gamma\backslash\mathfrak{H},\chi)$: the Hilbert space consisting of all functions φ on $\mathfrak{H}$ which satisfy $\varphi(\gamma z)=\chi(\gamma)\varphi(z)$ for every $\gamma\in\Gamma$ and $|\varphi|\in L^2(\Gamma\backslash\mathfrak{H})$.

Assume that $\overline{\Gamma}$ is torsion free.

Then the trace formula reads as follows. For a test function $F\in C_c^\infty(\mathbf{R})$, put

$$\Phi(s)=\int_{-\infty}^{\infty}F(x)e^{(s-1/2)x}dx.$$

Then, when F is an even function,

$$\sum_\rho \Phi(\rho)=\frac{\mathrm{vol}(\Gamma\backslash\mathfrak{H})}{2\pi}\int_{-\infty}^{\infty}r\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\Phi(\frac{1}{2}+ir)dr$$
$$+2\sum_{\{\gamma\}}\sum_{k=1}^{\infty}\frac{\chi(\gamma)^k\log(N(\gamma))}{N(\gamma)^{k/2}-N(\gamma)^{-k/2}}F(k\log(N(\gamma))).$$

$\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$: non-Euclidean Laplacian

For an eigenvalue λ of Δ occuring in $L^2(\Gamma\backslash\mathfrak{H}, \chi)$ with multiplicity $m(\lambda)$, we let ρ occur with multiplicity $m(\lambda)$ by the relation $\rho = 1/2 \pm s/2$, $\lambda = (1 - s^2)/4$. (When $\lambda = 1/4$, $s = 0$, we let $\rho = 1/2$ occur with the multiplicity $2m(1/4)$. When $\lambda \neq 1/4$, two ρ's occur.) And $\{\gamma\}$ extends over all primitive hyperbolic conjugacy classes of $\overline{\Gamma}$. The ρ's such that $0 < \Re(\rho) < 1$ coincide with the zeros of $Z_\Gamma(\Gamma, \chi)$ with the multiplicities stated above. Both sides of the trace formula converge absolutely.

Interpretation by representation theory.

$G = \mathrm{SL}(2, \mathbf{R})$, $K = \mathrm{SO}(2, \mathbf{R})$.

$L^2(\Gamma\backslash G, \chi)$: the Hilbert space consisting of all functions φ on G which satisfy $\varphi(\gamma g) = \chi(\gamma)\varphi(g)$ for every $\gamma \in \Gamma$ and $|\varphi| \in L^2(\Gamma\backslash G)$.

We put $H_\chi = L^2(\Gamma\backslash G, \chi)$; G acts on H_χ by the right translation. The Hilbert space $L^2(\Gamma\backslash \mathfrak{H}, \chi)$ can be identified with the closed subspace of H_χ consisting of all K-fixed vectors. The unitary representation of G on H_χ decomposes into a discrete direct sum:

$$H_\chi = \oplus_\pi V_\pi$$

where V_π is a closed invariant subspace of H_χ and an irreducible unitary representation π of G is realized on V_π. Then π must satisfy $\pi(-1_2) = \mathrm{id}$; V_π contributes to $L^2(\Gamma\backslash \mathfrak{H}, \chi)$ if and only if π has a (nonzero) K-fixed vector.

The classification of such π

B: the subgroup of G consisting of all upper triangular matrices. For $s \in \mathbf{C}$, we define a quasi-character ω_s of B by

$$\omega_s(\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix}) = |t|^{s+1}.$$

$PS(\omega_s)$: the space of smooth functions f on G which satisfy $f(bg) = \omega_s(b)f(g)$ for $b \in B$.

G acts on $PS(\omega_s)$ by right translation.

When $s \in i\mathbf{R}$, $PS(\omega_s)$ is a pre-Hilbert space with a canonical inner product. Let π_s be the unitary representation of G obtained by completion. It is irreducible and is called a principal series representation. When $-1 < s < 1$, $s \neq 0$, we obtain an irreducible unitary representation π_s by a similar procedure from $PS(\omega_s)$. It is called a complementary series representaion. We have $\pi_s \cong \pi_{-s}$.

The eigenvalue of Δ for a K-fixed vector of π_s (unique up to constant multiple) is $(1-s^2)/4$. This finishes the classification besides the trivial representation.

A principal series representation π_s corresponds to zeros $1/2 \pm s/2$ on the critical line; a complementary series representation π_s corresponds to zeros $\rho = 1/2 \pm s/2$ on the real line, $0 < \rho < 1$, $\rho \neq 1/2$, i.e. exceptional zeros; the trivial representation contributes $\rho = 0$ and 1 for the trace formula.

Now the trivial representation of G occurs in H_χ if and only if $\chi = 1$. Therefore the following observation holds.

(F) The terms $\Phi(0)$ and $\Phi(1)$ appear on the left hand side of the trace formula if and only if $\chi = 1$.

Remark. This fact should not be confused with the existence of trivial zeros of $Z_\Gamma(s,\chi)$; $Z_\Gamma(s,\chi)$ has a trivial zero at $s = 0$ with multiplicity $2g-2$.

The left-hand side of the trace formula defines a distribution $T_{\Gamma,\chi}$:

$$T_{\Gamma,\chi}(F) = \sum_\rho \Phi(\rho).$$

A distribution T is called of positive type if $T(\alpha * \tilde{\alpha}) \geq 0$, $\tilde{\alpha}(x) = \overline{\alpha(-x)}$, for every $\alpha \in C_c^\infty(\mathbf{R})$.

As is well known (due to Weil), $T_{\Gamma,\chi}$ is of positive type if and only if all ρ lie on the critical line.

As a slight refinement of this criterion, I showed that the condition $T_{\Gamma,\chi}(\alpha * \tilde{\alpha}) \geq 0$ for all odd functions α is sufficient to assure this conclusion (Adv. Stud. in pure math. 21 (1992)). Then $F = \alpha * \tilde{\alpha}$ is an even function.

Now from (F), we see that there exists odd $\alpha \in C_c^\infty(\mathbf{R})$ such that $T_{\Gamma,1}(\alpha * \tilde{\alpha}) < 0$. We fix such an α.

$g \geq 2$: the genus of the compact Riemann surface $\Gamma\backslash\mathfrak{H}$.

Since $\overline{\Gamma} \cong \pi_1(\Gamma\backslash\mathfrak{H})$, $\overline{\Gamma}$ has $2g$ generators σ_1, ..., σ_g, τ_1, ..., τ_g whose fundamental relation is

$$(*) \qquad (\sigma_1\tau_1\sigma_1^{-1}\tau_1^{-1})\cdots(\sigma_g\tau_g\sigma_g^{-1}\tau_g^{-1}) = 1.$$

Choose $s_i \in \mathbf{C}$, $|s_i| = 1$, $t_i \in \mathbf{C}$, $|t_i| = 1$, $1 \leq i \leq g$. In view of $(*)$, we can define a character $\overline{\chi}$ of $\overline{\Gamma}$ by

$$\overline{\chi}(\sigma_i) = s_i, \qquad \overline{\chi}(\tau_i) = t_i, \qquad 1 \leq i \leq g.$$

Then we define a character χ of Γ by $\chi = \overline{\chi} \circ p$, where $p : \Gamma \longrightarrow \overline{\Gamma}$ is the canonical homomorphism.

If s_i and t_i are sufficiently close to 1, then we see that $T_{\Gamma,\chi}(\alpha * \tilde{\alpha}) < 0$ from the right-hand side of the trace formula. In view of (F), this implies that $Z_\Gamma(s, \chi)$ has a zero ρ such that $0 < \rho < 1$, $\rho \neq 1/2$ (if $\chi \neq 1$).

In particular, choose $s_i = t_i = e^{2\pi i/N}$, $1 \leq i \leq g$ for a positive integer N. Let Γ_χ be the kernel of χ. Then $\Gamma/\Gamma_\chi \cong \mathbf{Z}/N\mathbf{Z}$ and we have

$$Z_{\Gamma_\chi}(s) = \prod_\eta Z_\Gamma(s, \eta)$$

where η extends over all characters of Γ which are trivial on Γ_χ. Therefore, when N is sufficiently large, $Z_{\Gamma_\chi}(s)$ has a zero ρ such that $0 < \rho < 1$, $\rho \neq 1/2$.

The reason breaking the Riemann hypothesis.

$Z_\Gamma(s, \chi)$ has deformations. The zeros at $s = 0$, 1 move to exceptional zeros by deformation.

Contrary to this, the Hecke L-function $L(s, \psi)$ with Grössenchakter ψ is rigid. ($k_A^1/k^\times$ is compact for a number field k.)

I found this proof about 20 years ago; it seems to be conceptually simpler than

A. Selberg: Proc. Symposia Pure Math. VIII (1965)

B. Randol: Bull. of AMS. 80 (1974)

A conjecture of Selberg states that $Z_{\Gamma}(s)$ has no exceptional zeros if Γ is of arithmetic type. In view of this conjecture, the group Γ_{χ} should be a noncongruence subgroup when Γ is of arithmetic type. In §3, we will examine this problem.

§2. Construction of noncongruence subgroups of $\mathsf{SL}(2, \mathbf{Z})$

I will give a simple construction of noncongruence subgroups of $\mathsf{SL}(2, \mathbf{Z})$.

N: a positive integer

$$\Gamma(N) = \left\{ \gamma \;\middle|\; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2, \mathbf{Z}), a - 1 \equiv b \equiv c \equiv 0 \equiv d - 1 \mod N \right\}$$

(The principal congruence subgroup of level N.)

$$\Gamma_0(N) = \left\{ \gamma \;\middle|\; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2, \mathbf{Z}), c \equiv 0 \mod N \right\}$$

A subgroup of $\mathsf{SL}(2, \mathbf{Z})$ of finite index is called a noncongruence subgroup if it does not contain $\Gamma(N)$ for any positive integer N.

F. Klein: Math. Ann, 17(1880)

asserted the existence of noncongruence subgroups and proofs were given in

R. Fricke: Math. Ann. 28 (1887)

G. Pick: Math. Ann. 28 (1887)

Let

$$\Delta(z) = e^{2\pi iz} \prod_{k=1}^{\infty} (1 - e^{2\pi ikz})^{24}, \qquad z \in \mathfrak{H}$$

be the cusp form of weight 12 with respect to SL$(2, \mathbf{Z})$.

n: a positive integer.

We define a holomorphic function $\Delta(z)^{1/n}$ so that it takes positive values when z is purely imaginary. Then $\Delta(z)^{1/n}$ has the product expansion

$$\Delta(z)^{1/n} = e^{2\pi iz/n} \prod_{k=1}^{\infty} (1 - e^{2\pi ikz})^{24/n}, \qquad z \in \mathfrak{H}.$$

Here the branch of $(1 - e^{2\pi ikz})^{24/n}$ is taken so that it is positive when z is purely imaginary.

$m \geq 2$: an integer

Put

$$f(z) = \Delta(mz)^{1/n}/\Delta(z)^{1/n}.$$

Then $f(z)^n$ is an automorphic function with respect to $\Gamma_0(m)$, since $\Delta(mz) \in S_{12}(\Gamma_0(m))$.

For $\gamma \in \Gamma_0(m)$, put

$$\chi(\gamma) = f(\gamma z)/f(z).$$

Since $\chi(\gamma)^n = 1$, we see that $\chi(\gamma)$ does not depend on z and χ is a character of $\Gamma_0(m)$.

From the product expansion, we see that

$$\Delta(z+1)^{1/n} = e^{2\pi i/n}\Delta(z)^{1/n},$$

$$\Delta(m(z+1))^{1/n} = e^{2m\pi i/n}\Delta(mz)^{1/n}.$$

Hence we obtain

$$\chi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = e^{2\pi i(m-1)/n}. \tag{2.1}$$

Let Γ_χ be the kernel of χ. Write

$$(m-1)/n = p/q$$

with relatively prime positive integers p and q. By (2.1), we see that the order of χ divides n and is divisible by q. Hence $[\Gamma_0(m) : \Gamma_\chi]$ divides n and is divisible by q.

Theorem 2.1. *We assume that q has a prime factor $l \geq 5$ which does not divide m and $t-1$ for every prime factor t of m. Then the group Γ_χ is a noncongruence subgroup.*

Proof. Suppose that Γ_χ contains the principal congruence subgroup $\Gamma(N)$ for a positive interger N. Then m divides N and χ factors through the canonical map $\Gamma_0(m) \longrightarrow \Gamma_0(m)/\Gamma(N)$. Hence $\Gamma_0(m)/\Gamma(N)$ posesses a character whose order is divisible by q.

Let $N = \prod p^{e_p}$ be the prime factorization. We have

$$\Gamma_0(m)/\Gamma(N) \cong \prod_{p|m} G_p \times \prod_{p\nmid m} \mathrm{SL}(2, \mathbf{Z}/p^{e_p}\mathbf{Z}),$$

where, p^{d_p} being the exact power of p dividing m,

$$G_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}/p^{e_2}\mathbf{Z}) \,\middle|\, c \in p^{d_p}\mathbf{Z}/p^{e_2}\mathbf{Z}) \right\}.$$

Let

$$G = \begin{cases} \prod_{p|m} G_p \times \mathsf{SL}(2,\mathbf{Z}/2^{e_2}\mathbf{Z}) \times \mathsf{SL}(2,\mathbf{Z}/3^{e_3}\mathbf{Z}) & \text{if 6 does not divide } m, \\ \prod_{p|m} G_p \times \mathsf{SL}(2,\mathbf{Z}/3^{e_3}\mathbf{Z}) & \text{if 2 divides } m \text{ and 3 does not divide } m, \\ \prod_{p|m} G_p \times \mathsf{SL}(2,\mathbf{Z}/2^{e_2}\mathbf{Z}) & \text{if 3 divides } m \text{ and 2 does not divide } m, \\ \prod_{p|m} G_p & \text{if 6 divides } m. \end{cases}$$

By Lemma 2.2 given below, the commutator subgroup of $\mathsf{SL}(2,\mathbf{Z}/p^{e_p}\mathbf{Z})$ coincides with itself if $p \geq 5$.

(Another simple proof is given as follows. It is well known that the commutator subgroup of $\mathsf{SL}(2,\mathbf{Z})$ contains $\Gamma(6)$. Take $g \in \mathsf{SL}(2,\mathbf{Z}/p^{e_p}\mathbf{Z})$. Write $g = \gamma \mod p^{e_p}$ with $\gamma \in \Gamma(6)$. We can write γ as the product of commutators of elements of $\mathsf{SL}(2,\mathbf{Z})$. Reduce this expression modulo p^{e_p}. Then we obtain an expression of g as the product of commutators of the elements of $\mathsf{SL}(2,\mathbf{Z}/p^{e_p}\mathbf{Z})$.)

Therefore G must have a character whose order is divisible by q. Since the order of G is not divisible by l, this is a contradiction and we complete the proof.

Lemma 2.2. *Let K be a non-archimedean local field, $\mathcal{O}_K$ be the ring of integers, ϖ be a prime element and q be the order of the residue field of K.*

Take a positive integer n and let $G = \mathrm{SL}(2, \mathcal{O}_K/\varpi^n\mathcal{O}_K)$.

If $q > 3$, then the commutator subgroup $[G, G]$ of G coincides with G.

Proof. For a, $b \in G$, we define the commutator by

$$[a, b] = aba^{-1}b^{-1}.$$

First we consider the case $n = 1$.

Let $\mathbf{F}_q = \mathcal{O}_F/\varpi\mathcal{O}_F$ be the finite field with q elements. It is well known that $\mathrm{PSL}(2, \mathbf{F}_q)$ is a simple group when $q > 3$. Therefore we have $[G, G]\{\pm 1_2\} = G$. Since

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

we have $-1_2 \in [G, G]$. Hence the assertion holds in this case.

Now assume $n \geq 2$. We put $R = \mathcal{O}_K / \varpi^n \mathcal{O}_K$.

Define a subgroup H of G by

$$H = \{g \in G \mid g \equiv 1_2 \mod \varpi\}.$$

Then H is a normal subgroup of G such that $G/H \cong \mathrm{SL}(2, \mathbf{F}_q)$. We have $[G, G]H = G$. For $t \in R^\times$ and $u \in R$, we have

$$\left[\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & (t^2-1)u \\ 0 & 1 \end{pmatrix}.$$

Since $q > 3$, we can choose t so that $t^2 - 1 \in R^\times$. Hence we have

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in [G, G]$$

for every $u \in R$. Similarly

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in [G, G]$$

for every $u \in R$.

For $x \in R$, $y \in \varpi\mathcal{O}_K/\varpi^n\mathcal{O}_K$, we have

$$\begin{pmatrix} 1 & 0 \\ -y/(1+xy) & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & -x/(1+xy) \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1+xy & 0 \\ 0 & 1/(1+xy) \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in [G,G]$$

for every $t \in 1 + (\varpi\mathcal{O}_K/\varpi^n\mathcal{O}_K)$. We can check easily that H is generated by such elements together with $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$, x, $y \in \varpi\mathcal{O}_K/\varpi^n\mathcal{O}_K$. Therefore $[G,G] \supset H$. Combined with $[G,G]H = G$, the assertion follows.

Remark 2.3. The condition of the theorem is satisfied if $m = 2$ and $n \geq 5$ is a prime number. In the case $m = 2$, $n = 5$, we obtain a noncongruence subgroup of $\mathrm{SL}(2, \mathbf{Z})$ of index 15.

Remark 2.4. It is well known that the principal congruence subgroup $\Gamma(p)$ is a free group for a prime number p. Using this fact, we can apply the method of the next section to produce noncongruence subgroups.

Remark 2.5. Let D be a hermitian symmetric space. If there exists an everywhere nonvanishing holomorphic automorphic form on D with respect to an arithmetic group Γ, then we can produce noncongruence subgroups of Γ by a similar argument to the above. However the non-existence of such a form is known for a wide class of D.

I found the proof of this section when I wrote the paper

On absolute CM-periods,

Proc. Symposia Pure Math. 66, Part 1, 1999, 221–278.

§3. Construction of noncongruence subgroups for cocompact case

I will give a simple proof for the existence of noncongruence subgroups of a cocompact arithmetic Fuchsian group.

F: a totally real algebraic number field of degree n.

$\mathcal{O}_F$: the ring of integers of F.

B: a division quaternion algebra over F such that

$$B \otimes_{\mathbf{Q}} \mathbf{R} \cong M(2, \mathbf{R}) \times \mathbf{H}^{n-1}.$$

Here $\mathbf{H}$ denotes the Hamilton quaternion algebra.

$*$: the main involution.

$N : B \longrightarrow F$: the reduced norm.

We have $N(x) = xx^*$.

R: a maximal order of B (we fix it).

For a prime ideal $\mathfrak{p}$ of F, we define localizations by

$$B_{\mathfrak{p}} = B \otimes_F F_{\mathfrak{p}}, \qquad R_{\mathfrak{p}} = R \otimes_{\mathcal{O}_F} \mathcal{O}_{F_{\mathfrak{p}}},$$

where $F_{\mathfrak{p}}$ is the completion of F at $\mathfrak{p}$ and $\mathcal{O}_{F_{\mathfrak{p}}}$ is the ring of integers of $F_{\mathfrak{p}}$.

We say that B is ramified at $\mathfrak{p}$ if $B_{\mathfrak{p}}$ is a division algebra and unramified otherwise. In the latter case, $B_{\mathfrak{p}}$ is isomorphic to $M(2, F_{\mathfrak{p}})$ as algebras over $F_{\mathfrak{p}}$.

Put

$$\Gamma = R^1 = \{x \in R \mid N(x) = 1\}.$$

By the projection to the first factor, we can regard Γ as a subgroup of $\mathrm{SL}(2, \mathbf{R})$; Γ is a cocompact Fuchsian group.

$\Gamma\backslash\mathfrak{H}$: (a special case of) the Shimura curve.

For an integral ideal $\mathfrak{n}$ of F, we put

$$\Gamma_{\mathfrak{n}} = \{\gamma \in \Gamma \mid \gamma - 1 \in \mathfrak{n}R\}.$$

We call $\Gamma_{\mathfrak{n}}$ the principal congruence subgroup of level $\mathfrak{n}$. A subgroup of finite index of Γ is called a noncongruence subgroup if it does not contain $\Gamma_{\mathfrak{n}}$ for any $\mathfrak{n}$. We are going to show that Γ contains noncongruence sugbroups.

Lemma 3.1. *There exists an ideal $\mathfrak{n}$ such that $\Gamma_{\mathfrak{n}}$ is torsion free.*

We take an ideal $\mathfrak{n}$ so that $\Gamma_{\mathfrak{n}}$ is torsion free and put $\Delta = \Gamma_{\mathfrak{n}}$.

g: the genus of the compact Riemann surface $\Delta\backslash\mathfrak{H}$.

As in §1, Δ has $2g$ generators $\sigma_1, \ldots, \sigma_g, \tau_1, \ldots, \tau_g$ whose fundamental relation is $(*)$. Let $\mathfrak{p}$ be a prime ideal of F. We put

$$R_{\mathfrak{p}}^1 = \{x \in R_{\mathfrak{p}} \mid N(x) = 1\}.$$

For a nonnegative integer f, we put

$$U_{\mathfrak{p},f} = \{u \in R_{\mathfrak{p}}^1 \mid u - 1 \in \mathfrak{p}^f R_{\mathfrak{p}}\}.$$

S: the finite set of all prime ideals of F at which B is ramified.

Theorem 3.2. *Let m be a positive integer and define a character χ of Δ by $\chi(\sigma_i) = \chi(\tau_i) = e^{2\pi i/m}$, $1 \leq i \leq g$. Let Γ_χ be the kernel of χ. We assume that m has a prime factor $l \geq 5$ which satisfies the following three conditions.*

(i) l does not divide the norm of $\mathfrak{n}$.

(ii) l is relatively prime to every prime ideal $\mathfrak{p} \in S$.

(iii) l does not divide the order of $U_{\mathfrak{p},0}/U_{\mathfrak{p},1}$ for every prime ideal $\mathfrak{p} \in S$.

Then Γ_χ is a noncongruence subgroup of Γ.

Proof. Suppose that Γ_χ contains a principal congruence subgroup of level $\mathfrak{m}$. Then Γ_χ contains $\Gamma_{\mathfrak{nm}}$. We may regard χ as a character of $\Delta/\Gamma_{\mathfrak{nm}}$. Therefore $\Delta/\Gamma_{\mathfrak{nm}}$ has a character of order l.

Let

$$\mathfrak{n} = \prod_{\mathfrak{p}} \mathfrak{p}^{e_\mathfrak{p}}, \qquad \mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{d_\mathfrak{p}}$$

be the prime ideal decompositions. By the strong approximation theorem, we have

$$\Gamma_\mathfrak{n}/\Gamma_{\mathfrak{nm}} \cong \prod_{\mathfrak{p}} (U_{\mathfrak{p},e_\mathfrak{p}}/U_{\mathfrak{p},e_\mathfrak{p}+d_\mathfrak{p}}).$$

Hence there exists $\mathfrak{p}$ such that $U_{\mathfrak{p},e_\mathfrak{p}}/U_{\mathfrak{p},e_\mathfrak{p}+d_\mathfrak{p}}$ has a character ψ of order l. We distinguish two cases.

(I) The case where $\mathfrak{p} \notin S$.

Let $p\mathbf{Z} = \mathfrak{p} \cap \mathbf{Z}$.

If $e_{\mathfrak{p}} > 0$, then $U_{\mathfrak{p},e_{\mathfrak{p}}}/U_{\mathfrak{p},e_{\mathfrak{p}}+d_{\mathfrak{p}}}$ is a p-group. Since $l \neq p$ by (i), this is a contradiction.

Suppose $e_{\mathfrak{p}} = 0$. Since $R_{\mathfrak{p}} \cong M(2, \mathcal{O}_{F_{\mathfrak{p}}})$, we have

$$U_{\mathfrak{p},0}/U_{\mathfrak{p},d_{\mathfrak{p}}} \cong \mathrm{SL}(2, \mathcal{O}_F/\mathfrak{p}^{d_{\mathfrak{p}}}\mathcal{O}_F).$$

If $p \geq 5$, then by Lemma 2.2, the commutator subgroup of $\mathrm{SL}(2, \mathcal{O}_F/\mathfrak{p}^{d_{\mathfrak{p}}}\mathcal{O}_F)$ coincides with itself, which is a contradiction.

Suppose that $p = 2$ or 3. Since $l \geq 5$ and $U_{\mathfrak{p},1}/U_{\mathfrak{p},d_{\mathfrak{p}}}$ is a p-group, ψ is trivial on $U_{\mathfrak{p},1}$. Hence ψ can be identified with a character of $\mathrm{SL}(2, \mathcal{O}_F/\mathfrak{p}\mathcal{O}_F)$. We see that ψ is trivial on the subgroups

$$H = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \,\middle|\, u \in \mathcal{O}_F/\mathfrak{p}\mathcal{O}_F \right\}$$

and ${}^tH$. Since H and ${}^tH$ generate $\mathrm{SL}(2, \mathcal{O}_F/\mathfrak{p}\mathcal{O}_F)$, this is a contradiciton.

(II) The case where $\mathfrak{p} \in S$.

By (ii) and (iii), we see that l does not divide the order of $U_{\mathfrak{p},e_{\mathfrak{p}}}/U_{\mathfrak{p},e_{\mathfrak{p}}+d_{\mathfrak{p}}}$, which is a contradiction.

This completes the proof.

Remark 3.3. If χ is a character of Δ whose order is divisible by a prime number l satisfying the conditions of Theorem 3.2, then Γ_χ is a noncongruence subgroup of Γ.

Problem. The Selberg conjecture states

If Δ is a congruence subgroup of Γ,

then $Z_{\Delta}(s)$ does not have an exceptional zero.

Is the converse true?

§4. An example of modular forms for a noncongruence subgroup

Our construction in §2 has the merit that we can easily obtain examples of modular forms for a noncongruence subgroup. I will present an explicit example.

In Theorem 2.1, we assume that m and n are prime numbers such that $n \geq 5$, $n \neq m$ and n does not divide $m-1$. Then Γ_χ is a noncongruence subgroup such that $[\Gamma_0(m) : \Gamma_\chi] = n$.

Up to equivalence, $\Gamma_0(m)$ has two cusps ∞ and 0. The equivalence classes of the cusps of Γ_χ lying over ∞ are in one to one correspondence with $\Gamma_\chi \backslash \Gamma_0(m) / \Gamma_0(m)_\infty$ where

$$\Gamma_0(m)_\infty = \{\gamma \in \Gamma_0(m) \mid \gamma\infty = \infty\}.$$

By (2.1), we see easily that $\Gamma_0(m) = \Gamma_\chi \Gamma_0(m)_\infty$.

Hence, up to equivalence, there is only one cusp of Γ_χ lying over ∞. Similarly, we see that there is only one cusp of Γ_χ lying over 0, up to equivalence.

The number of equivalence classes of elliptic points of $\Gamma_0(m)$ of order 2 (resp. 3) are ν_2 (resp. ν_3) where

$$\nu_2 = 1 + (\frac{-1}{m}), \qquad \nu_3 = 1 + (\frac{-3}{m}). \tag{4.1}$$

We can show easily that the number of equivalence classes of elliptic points of Γ_χ of order 2 (resp. 3) are $n\nu_2$ (resp. $n\nu_3$).

g_χ: the genus of the compact Riemann surface $\Gamma_\chi \backslash \mathfrak{H} \cup \{\text{cusps}\}$

g_0: the genus of the compact Riemann surface $\Gamma_0(m) \backslash \mathfrak{H} \cup \{\text{cusps}\}$

Calculating using a formula of Shimura's book, we find

$$(4.2) \qquad g_\chi = n\left[\frac{1}{12}(m+1) - \frac{1}{4}\nu_2 - \frac{1}{3}\nu_3\right], \qquad g_\chi = ng_0.$$

Here ν_2 and ν_3 are given by (4.1).

We have $\dim S_2(\Gamma_\chi) = g_\chi$ and for an even integer $k > 2$, we have

$$(4.3) \qquad \dim S_k(\Gamma_\chi) = (k-1)g_\chi - 1 + n\nu_2[\frac{k}{4}] + n\nu_3[\frac{k}{3}].$$

Let

$$f(z) = \Delta(mz)^{1/n}/\Delta(z)^{1/n}$$

be the function used in §2. We see that $f(z)$ is an automorphic function with respect to Γ_χ.

Let $q = e^{2\pi iz/n}$ (resp. q') be the uniformizing parameter at the cusp ∞ (resp. 0) of Γ_χ.

We have

$$\text{(4.4)} \qquad \mathrm{ord}_q(f(z)) = m-1, \qquad \mathrm{ord}_{q'}(f(z)) = -(m-1).$$

For a funtion F on $\mathfrak{H}$, $k \in \mathbf{Z}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{R})$, $\det g > 0$, we define a function $(F|_k\, g)(z)$ on $\mathfrak{H}$ by

$$(F|_k\, g)(z) = (\det g)^{k/2} F(gz)(cz+d)^{-k}, \qquad z \in \mathfrak{H}.$$

For $1 \leq i \leq n-1$, we set

$$S_k(\Gamma_0(m), \chi^i) = \left\{ h \in S_k(\Gamma_\chi) \;\middle|\; h|_k\, \gamma = \chi(\gamma)^i h, \quad \gamma \in \Gamma_0(m) \right\}.$$

Then we have a decomposition:

$$\text{(4.5)} \qquad S_k(\Gamma_\chi) = S_k(\Gamma_0(m)) \oplus (\oplus_{i=1}^{n-1} S_k(\Gamma_0(m), \chi^i)).$$

Let $\omega = \begin{pmatrix} 0 & 1 \\ -m & 0 \end{pmatrix}$. We have

$$\chi(\omega\gamma\omega^{-1}) = \chi(\gamma)^{-1}, \qquad \gamma \in \Gamma_0(m).$$

Hence we see that ω normalizes Γ_χ and that the operator $|_k\, \omega$ gives an isomorphism of $S_k(\Gamma_0(m), \chi^i)$ onto $S_k(\Gamma_0(m), \chi^{-i})$.

Now we take $m = 2$, $n = 5$. We have $\nu_2 = 1$, $\nu_3 = 0$. By (4.2), we have $g_\chi = 0$. By (4.3), we have $\dim S_4(\Gamma_\chi) = 4$. Let

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n z}$$

be the Eisenstein series of weight 4 with respect to $\mathrm{SL}(2, \mathbf{Z})$. Here

$$\sigma_3(n) = \sum_{0<d|n} d^3.$$

Put

$$g(z) = E_4(z) - 2^4 E_4(2z). \tag{4.6}$$

Then $g(z)$ is a modular form of weight 4 with respect to $\Gamma_0(2)$ and we see that

$$\mathrm{ord}_q(g(z)) = 0, \qquad \mathrm{ord}_{q'}(g(z)) = n.$$

In view of (4.4), $f(z)^i g(z) \in S_4(\Gamma_0(2), \chi^i) \subset S_4(\Gamma_\chi)$ for $1 \le i \le 4$. By (4.5), they are linearly independent. Therefore a basis of $S_4(\Gamma_\chi)$ is given by

$$\{f(z)g(z), f(z)^2 g(z), f(z)^3 g(z), f(z)^4 g(z)\}.$$

Remark 4.1. We have

$$f(z)^i g(z)|_k \omega = f(z)^{5-i} g(z), \qquad 1 \le i \le 4.$$

Put $h(z) = E_4(z) - E_4(2z)$. Then we have

$$\mathrm{ord}_q(h(z)) = n, \qquad \mathrm{ord}_{q'}(h(z)) = 0.$$

A basis of $S_4(\Gamma_\chi)$ is also given by

$$\{f(z)^{-1} h(z), f(z)^{-2} h(z), f(z)^{-3} h(z), f(z)^{-4} h(z)\}.$$

Using the fact $\dim S_4(\Gamma_0(2), \chi) = 1$, we can prove the relation

$$h(z) = -16 f(z)^5 g(z).$$

Remark 4.2. We have $\dim S_6(\Gamma_\chi) = 4$ and a basis of this space can be given similarly.

We have $\dim S_8(\Gamma_\chi) = 9$, $\dim S_8(\Gamma_0(2)) = 1$. For $1 \leq i \leq 4$, a basis of $S_8(\Gamma_0(2), \chi^i)$ is given by $\{f(z)^i g(z)^2, f(z)^i g(z) E_4(z)\}$ and $f(z)^5 g(z)^2$ spans $S_8(\Gamma_0(2))$.

Remark 4.3. It would be interesting to examine the example of this section in more detail in view of the Atkin-Swinnerton-Dyer congruences.

Playing with modular forms

A. O. L. Atkin and H. P. F. Swinnerton-Dyer,

Symposia Pure Math. 19 (1971)

A. J. Scholl, Invent. Math. 79 (1985)

Consider $f(z)g(z) \in S_4(\Gamma_\chi)$.

We have

$$f(z) = \Delta(2z)^{1/5}\Delta(z)^{-1/5} = q^{1/5}\prod_{n=1}^{\infty}(1-q^n)^{24/5}.$$

Here $q = e^{2\pi iz}$. Put $x = q^{1/5}$ and

$$f(z)g(z)/(-15) = \sum_{n=1}^{\infty} a(n)x^n.$$

Then $a(1) = 1$, $a(n) \in \mathbf{Z}[1/5]$; $a(n)$ can be nonzero only when $n \equiv 1 \mod 5$.

We can observe A-S type congruence

$$a(pn) - A(p)a(n) + p^3 a(n/p) \equiv 0 \mod p^{3(\alpha+1)}$$

if $\mathrm{ord}_p(n) = \alpha$, $A(p) \in \mathbf{Z}_p$.

Here p is a prime such that $p \equiv 1 \mod 5$.