# Double Euler products

Hirotaka Akatsuka

Kyushu University

April, 2011

# Contents

# Zeta regularized products (I)

For a support discrete map $m : \mathbb{C} \to \mathbb{Z}$

$$\zeta_m(w,s) := \sum_{\substack{\rho \in \mathbb{C} \\ m(\rho) \neq 0}} \frac{m(\rho)}{(s-\rho)^w} \qquad (|\arg(s-\rho)| < \pi).$$

Assume that, for each $\mathrm{Re}(s) \gg 1$,

- the series converges absolutely in $\mathrm{Re}(w) \gg 1$,
- $\zeta_m(w,s)$ has a meromorphic continuation w.r.t. $w$ to a region including $w = 0$.

Then the zeta regularized product is defined by

$$\prod_{\rho \in \mathbb{C}} (s-\rho)^{m(\rho)} := \exp\left( -\mathrm{Res}_{w=0} \frac{\zeta_m(w,s)}{w^2} \right).$$

# Zeta regularized products (II)

**Property.** (cf. Illies, 1999) Under the previous assumptions,

$$\coprod_{\rho\in\mathbb{C}} (s-\rho)^{m(\rho)}$$

has a meromorphic continuation to the whole plane $s \in \mathbb{C}$. It has zeros at $s = \rho$ with multiplicity $m(\rho)$.

**Example.** (1) For $M > 1$

$$\coprod_{n\in\mathbb{Z}} \left( s - \frac{2\pi i n}{\log M} \right)^{-1} = (1 - M^{-s})^{-1}.$$

# Zeta regularized products (III)

(2) (Deninger, 1992)

$\zeta(s)$: the Riemann zeta function,

$R := \{\rho \in \mathbb{C} : \zeta(\rho) = 0, 0 < \mathrm{Re}(\rho) < 1\}$,

$m(\rho)$: the order of zeros for $\zeta(s)$ at $s = \rho$.

Then,

$$\zeta(s) \overset{\text{distinct}}{=} \frac{\displaystyle\prod_{\rho \in R} (s-\rho)^{m(\rho)} \prod_{n=1}^{\infty} (s+2n)}{s-1}.$$

**Note**: Zeros and a pole are located at

- zeros: $s = \rho$(nontrivial zeros), $s = -2n(n = 1, 2, \ldots)$
- pole: $s = 1$

# Absolute tensor products (I)

**Definition.** (Kurokawa, 1992)

Assume $Z_j(s)$ are expressed by

$$Z_j(s) = \prod_{\rho \in \mathbb{C}} (s-\rho)^{m_j(\rho)} \qquad (j = 1, \ldots, r).$$

Then their absolute tensor product is defined by

$$(Z_1 \otimes \cdots \otimes Z_r)(s) := \prod_{\rho_1, \ldots, \rho_r \in \mathbb{C}} (s - \rho_1 - \cdots - \rho_r)^{m(\rho_1, \ldots, \rho_r)},$$

where

$$m(\rho_1, \ldots, \rho_r) := m_1(\rho_1) \cdots m_r(\rho_r) \times \begin{cases} 1 & \mathrm{Im}(\rho_j) \geq 0 (\forall j), \\ (-1)^{r-1} & \mathrm{Im}(\rho_j) < 0 (\forall j), \\ 0 & \text{otherwise.} \end{cases}$$

# Absolute tensor products (II)

$(Z_1 \otimes \cdots \otimes Z_r)(s)$ has the following additive structure on zeros and poles:

$$Z_j(\rho_j) = 0 \text{ or } \infty \text{ for } \forall j;\ \ \mathrm{Im}(\rho_j) \text{ have the same sign}$$
$$\Longrightarrow (Z_1 \otimes \cdots \otimes Z_r)(\rho_1 + \cdots + \rho_r) = 0 \text{ or } \infty.$$

A similar structure is important for congruence zeta functions.

# Congruence zeta functions

**Definition.**

For a nonsingular projective variety $X$ over $\mathbb{F}_q$, its congrucence zeta function is defined by

$$Z(q^{-s}; X) = \zeta(s; X) := \exp\left(\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} q^{-ns}\right) (\in \mathbb{Q}(q^{-s})).$$

**(A part of) Weil's conjecture.** (proved by Deligne)

All zeros and poles of $\zeta(s; X)$ are located at

$$\begin{cases} \text{zeros} : \mathrm{Re}(s) = \frac{j}{2}, & j = 1, 3, \ldots, 2\dim X - 1, \\ \text{poles} : \mathrm{Re}(s) = \frac{j}{2}, & j = 0, 2, \ldots, 2\dim X. \end{cases}$$

# Deligne's proof of Weil's conjecture

1. (Grothendieck) Zeros and poles of $Z(t; X)$ are identified with the inverse of eigenvalues of the Frobenius operator $\mathrm{Frob}$ acting on $H^i(X)$ with $i = 0, 1, \ldots, 2\dim X$.
2. $q^{(i-1)/2} \leq |\alpha| \leq q^{(i+1)/2}$ holds for any eigenvalues $\alpha$ of $\mathrm{Frob}$ on $H^i(X)$.
3. For eigenvalues $\alpha_j$ of $\mathrm{Frob}$ acting on $H^{i_j}(X_j)$, $\alpha_1 \times \cdots \times \alpha_m$ is an eigenvalue of $\mathrm{Frob}$ on a $(i_1 + \cdots + i_m)$-th cohomology.

$$
\begin{aligned}
& \alpha : \text{eigenvalue of } \mathrm{Frob} \text{ on } H^i(X) \\
\stackrel{3}{\Longrightarrow} \quad & \alpha^m : \text{eigenvalue of } \mathrm{Frob} \text{ on } im\text{-th cohomology for } \forall m \\
\stackrel{2}{\Longrightarrow} \quad & q^{(im-1)/2} \leq |\alpha^m| \leq q^{(im+1)/2} \text{ for } \forall m \\
\Longrightarrow \quad & q^{\frac{i}{2}-\frac{1}{2m}} \leq |\alpha| \leq q^{\frac{i}{2}+\frac{1}{2m}} \text{ for } \forall m \\
\Longrightarrow \quad & |\alpha| = q^{\frac{i}{2}}.
\end{aligned}
$$

# Motivation I

For many other zeta functions

- spaces (like $H^i(X)$)
- operators (like $\mathrm{Frob}$)

are unknown.

The absolute tensor product is a trial to replace spaces and operators by a family of our multiple zeta functions $Z^{\otimes r}(s)$.

**Note:**

$$Z(\rho) = 0 \text{ or } \infty \Longrightarrow Z^{\otimes r}(r\rho) = 0 \text{ or } \infty.$$

How do we analyze $Z^{\otimes r}(s)$?

# Motivation II

An Euler product expression and a functional equation are fundamental tools when we investigate zeros of zeta functions.

By construction, we see that there exists a functional equation for $(Z_1 \otimes \cdots \otimes Z_r)(s)$ (up to the exponential of polynomials) if each $Z_j(s)$ has a functional equation.

On the other hand, an existance of Euler products for $(Z_1 \otimes \cdots \otimes Z_r)(s)$ is nontrivial even if each $Z_j(s)$ has an Euler product expression.

In this talk we mainly deal with an Euler product expression for $(\zeta \otimes \zeta)(s)$, where $\zeta(s)$ is the Riemann zeta function.

# $(\zeta_p \otimes \zeta_q)(s)$ (I)

Recall that

$$\zeta_p(s) := (1 - p^{-s})^{-1} = \coprod_{n \in \mathbb{Z}} \left( s - \frac{2\pi i n}{\log p} \right)^{-1}.$$

Thus,

$$(\zeta_p \otimes \zeta_q)(s) = \frac{\displaystyle\coprod_{m,n \geq 0} \left( s - \frac{2\pi i m}{\log p} - \frac{2\pi i n}{\log q} \right)}{\displaystyle\coprod_{m,n \geq 1} \left( s + \frac{2\pi i m}{\log p} + \frac{2\pi i n}{\log q} \right)}.$$

Critical line: $\mathrm{Re}(s) = 0$.

**Note:** This function is essentially a double sine function, which is a quotient of Barnes' double gamma functions.

# $(\zeta_p \otimes \zeta_q)(s)$ (II)

**Theorem.** (Koyama-Kurokawa, 2004) Let $p, q$ be prime numbers.

1. When $p \neq q$, for $\mathrm{Re}(s) > 0$

$$\begin{aligned}&(\zeta_p \otimes \zeta_q)(s)\\ =\quad &\exp\left(-\sum_{n=1}^{\infty} \frac{p^{-ns}}{n(1-e(n\frac{\log p}{\log q}))} - \sum_{n=1}^{\infty} \frac{q^{-ns}}{n(1-e(n\frac{\log q}{\log p}))}\right),\end{aligned}$$

   where $e(x) := \exp(2\pi i x)$.

2. When $p = q$, for $\mathrm{Re}(s) > 0$

$$\begin{aligned}&(\zeta_p \otimes \zeta_p)(s)\\ =\quad &\exp\left(-\frac{1}{2\pi i}\sum_{n=1}^{\infty} \frac{p^{-ns}}{n^2} - \left(1 + \frac{s\log p}{2\pi i}\right)\sum_{n=1}^{\infty} \frac{p^{-ns}}{n}\right).\end{aligned}$$

# $(\zeta_p \otimes \zeta_q)(s)$ (III)

**Remark.**

1. To check the convergence in the case $p \neq q$, we have to estimate the denominators. We need the following Baker's results on linear forms in logarithms:

$$\exists c = c(p,q) > 0 \text{ s.t. } \min_{m \in \mathbb{Z}} \left| n \frac{\log p}{\log q} - m \right| \geq n^{-c} \text{ for } n \geq 2.$$

2. Similar formulas for $(\zeta_{p_1} \otimes \cdots \otimes \zeta_{p_r})(s)$ were obtained by Kurokawa-Wakayama(2004), A(2006).

# $(\zeta \otimes \zeta)(s)$ (I)

Recall that

$$\zeta(s) = \frac{\prod_{\rho}(s-\rho)\prod_{n=1}^{\infty}(s+2n)}{s-1}.$$

Thus,

$$(\zeta \otimes \zeta)(s) = \frac{(s-2)\prod_{\mathrm{Im}(\rho_j)>0}(s-\rho_1-\rho_2)}{\prod_{\mathrm{Im}(\rho_j)<0}(s-\rho_1-\rho_2)\left(\prod_{\mathrm{Im}(\rho)>0}(s-1-\rho)\right)^2} \times R_1(s),$$

where $\rho$, $\rho_1$, $\rho_2$ run over the nontrivial zeros counted with multiplicity and $R_1(s)$ is nonzero holomorphic in $\mathrm{Re}(s) > -1$.

# $(\zeta \otimes \zeta)(s)$ (II)

Critical strip: $0 \leq \mathrm{Re}(s) \leq 2$.

Symmetry: $(\zeta \otimes \zeta)(s) \longleftrightarrow (\zeta \otimes \zeta)(2-s)^{-1} \longrightarrow$ afterward,
$(\zeta \otimes \zeta)(s) \longleftrightarrow \overline{(\zeta \otimes \zeta)(\overline{s})}^{-1}$.

$$(\zeta \otimes \zeta)(s) \times \overline{(\zeta \otimes \zeta)(\overline{s})} = \frac{\left(\prod_{n=1}^{\infty} \zeta(s+2n)\right)^2}{\zeta(s+1)^2}.$$

# $(\zeta \otimes \zeta)(s)$ (III)

**Theorem 1.**(A) For $\mathrm{Re}(s) > 2$,

$$
\begin{aligned}
&(\zeta \otimes \zeta)(s) \\
= \quad & \exp\Bigg( \frac{1}{\pi i} \sum_{p} \sum_{m=1}^{\infty} \sum_{q} \sum_{\substack{n=1 \\ q^n \neq p^m}}^{\infty} \frac{p^{-m(s-1)} q^{-n} \log p}{n(m \log p - n \log q)} \\
& \quad - \frac{1}{\pi i} \sum_{p} \sum_{m=1}^{\infty} \sum_{q} \sum_{n=1}^{\infty} \frac{p^{-ms} q^{-n} \log p}{n(m \log p + n \log q)} \\
& \quad + \frac{1}{\pi i} \int_0^1 \frac{\zeta'}{\zeta}(s-u) \log |\zeta(u)| du \Bigg) \zeta(s-1)^{-1} \times R_2(s),
\end{aligned}
$$

where $p$, $q$ run over the prime numbers and $R_2(s)$ is nonzero holomorphic in $\mathrm{Re}(s) > 1$.

# $(\zeta \otimes \zeta)(s)$ (IV)

**Remark.**

1. The second sum converges absolutely in $\mathrm{Re}(s) > 1$.
2. The first sum and the third integral converge absolutely in $\mathrm{Re}(s) > 2$.
3. Furthermore, as $s \to 2$,

$$\frac{1}{\pi i} \sum_{p} \sum_{m=1}^{\infty} \sum_{q} \sum_{\substack{n=1 \\ q^n \neq p^m}}^{\infty} \frac{p^{-m(s-1)} q^{-n} \log p}{n(m \log p - n \log q)} \sim \frac{1}{2\pi i} (\log(s-2))^2,$$

$$\frac{1}{\pi i} \int_0^1 \frac{\zeta'}{\zeta}(s-u) \log |\zeta(u)| du \sim -\frac{1}{2\pi i} (\log(s-2))^2.$$

4. Koyama-Kurokawa (2005) also obtained an Euler product for $(\zeta \otimes \zeta)(s)$. However, their Euler product is more complicated than ours.

# Proof of Theorem 1 (I)

$\rho_j = \frac{1}{2} + i\tau_j$: nontrivial zeros of $\zeta(s)$.

We start with

$$\begin{aligned}
&\sum_{\mathrm{Re}(\tau_j)>0} \frac{1}{(z+\tau_1+\tau_2)^w} \\
= \;& \frac{1}{\Gamma(w)} \sum_{\mathrm{Re}(\tau_j)>0} \int_0^\infty e^{-(z+\tau_1+\tau_2)t} t^w \frac{dt}{t} \\
= \;& \frac{1}{\Gamma(w)} \int_0^\infty e^{-zt} \theta(t)^2 t^w \frac{dt}{t},
\end{aligned}$$

where

$$\theta(t) = \sum_{\mathrm{Re}(\tau)>0} e^{-\tau t}.$$

Want to connect this with prime numbers.

# Property of $\theta(t)$ (I)

$$U(t) := \theta(t) + \frac{\log t}{4\pi \sin(t/2)}.$$

**Proposition.** (Cramér(1919) and Guinand(1949))

- (explicit formula)

$$\begin{aligned} & \theta(t) \\ = \quad & \frac{t}{2\pi i} e^{\frac{it}{2}} \sum_{p} \sum_{n=1}^{\infty} \frac{p^{-m}}{m(t + im \log p)} \\ - \quad & \frac{t}{2\pi i} e^{-\frac{it}{2}} \sum_{p} \sum_{n=1}^{\infty} \frac{p^{-m}}{m(t - im \log p)} + \cdots . \end{aligned}$$

# Property of $\theta(t)$ (II)

(continued)

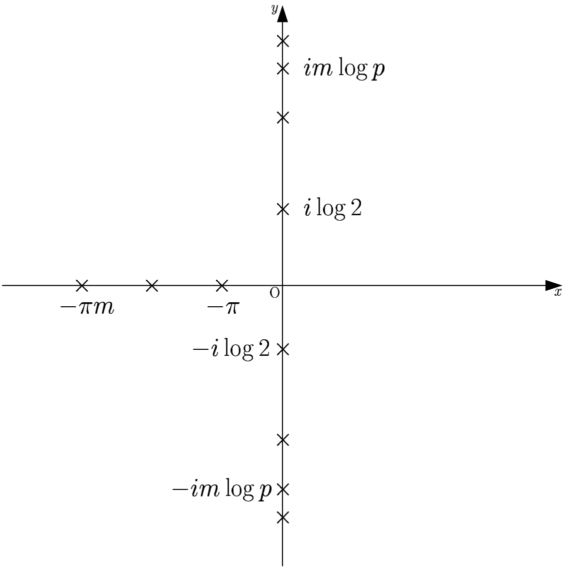
- (analytic continuation and functional equation) $U(t)$ has a (single valued) meromorphic continuation to $\mathbb{C}$ and

$$U(t) + U(-t) = 2\cos\left(\frac{t}{2}\right) - \frac{1}{4\cos(t/2)}.$$

- (Poles) Poles of $U(t)$ are located at
  - $t = \pm im \log p$ ($p$ : prime numbers; $m = 1, 2, \ldots$),
  - $t = -\pi m$ ($m = 1, 2, \ldots$),
  - $t = 2\pi m$ ($m = 0, 1, \ldots$)

  and nowhere else.

**Poles of $\theta(t)$:**



# Proof of Theorem 1 (II)

Want to express

$$\sum_{\mathrm{Re}(\tau_j)>0} \frac{1}{(z+\tau_1+\tau_2)^w} = \frac{1}{\Gamma(w)} \int_0^\infty e^{-zt}\theta(t)^2 t^w \frac{dt}{t},$$

as coutour integral expression and apply the residue theorem. But

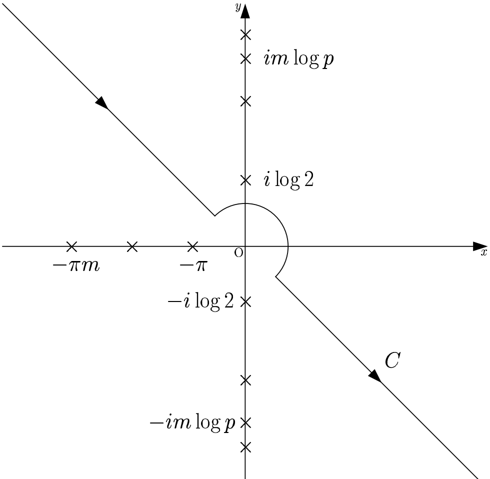
$$\sum_{\alpha:\text{all poles of } \theta(t)} \mathrm{Res}_{t=\alpha}\, e^{-zt}\theta(t)^2 t^{w-1}$$

does not converge absolutely for $\forall(w,z) \in \mathbb{C}^2$.

To halve the summation range, consider the following integral:

# Proof of Theorem 1 (III)

$$F(w,z) = \frac{1}{\Gamma(w)} \int_C e^{-zt}\theta(t)^2 t^{w-1} dt,$$

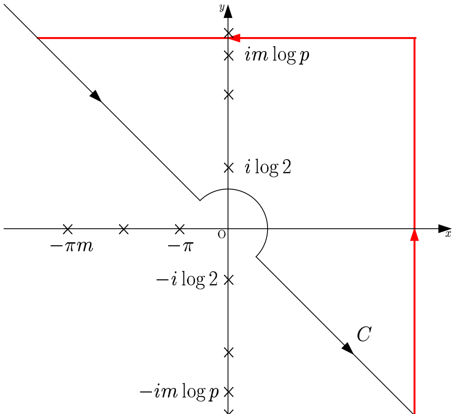


# Proof of Theorem 1 (IV)

- The residue theorem gives

$$F(w,z) = \frac{2\pi i}{\Gamma(w)} \sum_{p,m} \mathrm{Res}_{t=im\log p}\, e^{-zt}\theta(t)^2 t^{w-1},$$

which converges absolutely for certain domain.



# Proof of Theorem 1 (V)

- From functional equation $\theta(t) \longleftrightarrow \theta(-t)$, $F(w,z)$ can be expressed in terms of sums over zeros of $\zeta(s)$:

$$\begin{aligned}
F(w,z) &= \frac{1}{\Gamma(w)}\Big(\int_{\infty e^{3\pi i/4}}^{0} + \int_{0}^{\infty e^{-\pi i/4}}\Big) e^{-zt}\theta(t)^2 t^{w-1} dt \\
&= \frac{1}{\Gamma(w)}\int_{0}^{\infty e^{-\pi i/4}} e^{zt}\theta(-t)^2 (te^{\pi i})^{w-1} dt \\
&+ \frac{1}{\Gamma(w)}\int_{0}^{\infty e^{-\pi i/4}} e^{-zt}\theta(t)^2 t^{w-1} dt \\
&\overset{ft.eq}{=} \frac{1}{\Gamma(w)}\int_{0}^{\infty e^{-\pi i/4}} e^{zt}(-\theta(t) + \cdots)^2 (te^{\pi i})^{w-1} dt + \cdots
\end{aligned}$$

Combining these, we obtain Theorem 1.

# Some comments (I)

Do we expect a relation between $(\zeta_p \otimes \zeta_q)(s)$ and $(\zeta \otimes \zeta)(s)$?

It may be helpful to investigate $(\zeta \otimes \zeta_p)(s)$. By the same manner as the proof of Theorem 1, for $\mathrm{Re}(s) > 2$ we have

$$\begin{aligned}(\zeta \otimes \zeta_p)(s) &= \exp\Bigg( -\sum_{\substack{q \\ q \neq p}} \sum_{n=1}^{\infty} \frac{q^{-ns}}{n(1 - e(n\frac{\log q}{\log p}))} \\ &\qquad + \frac{1}{2\pi i} \sum_{m=1}^{\infty} \sum_{q} \sum_{\substack{n=1 \\ q^n \neq p^m}}^{\infty} \frac{p^{-m(s-1)} q^{-n} \log p}{n(m \log p - n \log q)} \\ &\qquad - \frac{\log p}{2\pi i} \int_0^1 \frac{\log|\zeta(u)|}{p^{s-u} - 1} du \Bigg) \\ &\times \quad (1 - p^{-(s-1)})^{-1/2} R_p(s),\end{aligned}$$

where $R_p(s)$ is nonzero holomorphic in $\mathrm{Re}(s) > 0$.

# Some comments (II) (functional equation)

Put

$$\xi(s) := s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \cong \prod_{\rho}(s-\rho).$$

By definition,

$$(\xi \otimes \xi)(s) = \frac{\prod_{\mathrm{Im}(\rho_j)>0}(s-\rho_1-\rho_2)}{\prod_{\mathrm{Im}(\rho_j)<0}(s-\rho_1-\rho_2)}.$$

Then we have

$$(\xi \otimes \xi)(s) \times (\xi \otimes \xi)(2-s) = e^{Q(s)},$$

where $Q(s)$ is a polynomial of order 2.

# Some comments (III) (functional equation)

More precisely, $Q(s)$ is explicitly written in terms of $a_n (n = -1, 0, 1)$, $\Gamma'(1)$ and $\Gamma''(1)$, where

$$U(t) = \sum_{n=-1}^{\infty} a_n t^n.$$

**Note**: $a_{-1}$ and $a_0$, $a_2, \ldots$ are computable numbers. In fact,

$$a_{-1} = -\frac{\gamma + \log(2\pi)}{2\pi}, \qquad a_{2n} = \frac{1}{(2n)!}\left(1 - \frac{E_{2n}}{2^{2n+4}}\right),$$

where $\gamma$ is the Euler constant and $E_n$ is the $n$-th Euler number defined by

$$\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n.$$

# Some comments (IV) (functional equations)

$$
\begin{aligned}
Q(s) \;=\; & -2\pi i(a_1 a_{-1} + a_0^2) + 4\pi i\gamma(a_1 b_{-1} + a_{-1} b_1) \\
& - 8i\left(\pi\gamma_3 - \frac{\pi^3}{24}\right) b_1 b_{-1} \\
& + (s-1)^2 \left\{ \pi i a_{-1}^2 - 2\pi i\gamma a_{-1} b_{-1} + 2i\left(\pi\gamma_3 - \frac{\pi^3}{24}\right) b_{-1}^2 \right\},
\end{aligned}
$$

where

$$
-\frac{1}{4\pi \sin\frac{t}{2}} = \sum_{n=-1}^{\infty} b_n t^n,
$$

$$
\frac{1}{\Gamma(t)} = \sum_{n=1}^{\infty} \gamma_n t^n.
$$

Can we evaluate $a_1$?