On the zeros of Weng zeta functions for Chevalley groups

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The Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\text{: prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\mathrm{Re}\, s > 1).$$

The functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

The number of zeros in $0 < \mathrm{Im}\, s < T$

$$\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

where

$$S(T) = \frac{1}{\pi}\arg\zeta(1/2 + iT).$$

Why difficult?

$$S(T) = O(\log T) \qquad (\textit{unconditionally});$$

$$S(T) = O\left(\frac{\log T}{\log\log T}\right) \qquad (RH);$$

$$S(T) = \Omega(\sqrt{\log T}/\sqrt{\log\log T}) \text{ (RH, Montgomery)};$$

$$S(T) = o\left(\frac{\log T}{\log\log T}\right) \qquad (RH, ?);$$

$$\limsup_{t\to\infty} \frac{S(t)}{\sqrt{\log t \cdot \log\log t}} = \frac{1}{\pi\sqrt{2}} \qquad (RMT, ?).$$

Montgomery's pair correlation conjecture (RH)

$$\sum_{\substack{0<\gamma,\tilde{\gamma}\leqslant T \\ 0<\gamma-\tilde{\gamma}\leqslant 2\pi\beta/\log T}} 1 \sim \frac{T}{2\pi}\log T \int_0^\beta \left[1-\left(\frac{\sin \pi u}{\pi u}\right)^2\right] du.$$

Here $1/2 + i\gamma$, $1/2 + i\tilde{\gamma}$ are zeros of $\zeta(s)$.

The Riemann-Siegel formula I

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = F(s) + \overline{F(1-\overline{s})},$$

where

$$F(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\int_{0\swarrow 1}\frac{e^{i\pi x^2}x^{-s}}{e^{i\pi x} - e^{-i\pi x}}dx.$$

The Riemann-Siegel formula II

$$\zeta(s) \sim \sum_{n \leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{n^s} + \chi(s) \sum_{n \leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{1-s}},$$

where

$$\chi(s) = \frac{\pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right)}{\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)}.$$

Why the Riemann-Siegel formula is good?

$$E(s) + \overline{E(1 - \overline{s})},$$

or

$$W(z) + \overline{W(\overline{z})}.$$

RH will follow if we show that $E(s)$ has no zeros in $\mathrm{Re}(s) > 1/2$ or $\mathrm{Re}(s) < 1/2$!

Hermite-Biehler's theorem

Suppose a polynomial $W(z)$ has exactly n zeros in the lower half-plane.
Then, $W(z) + \overline{W(\overline{z})}$ can have at most n pairs of conjugate complex zeros.

Theorem

Let $W(z)$ be a function in $\mathbb{C}$. Suppose $W(z)$ satisfies

$$W(z) = H(z)e^{\alpha z} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\rho_n}\right) \left(1 + \frac{z}{\overline{\rho_n}}\right) \right],$$

where $H(z)$ is a nonzero polynomial having N many (counted with multiplicity) in the lower half-plane, $\alpha \in \mathbb{R}$, $\operatorname{Im} \rho_n \geqslant 0$ $(n = 1, 2, \ldots)$, and the infinite product converges uniformly in any compact subset of $\mathbb{C}$. Then, $W(z) + \overline{W(\overline{z})}$ (or $W(z) - \overline{W(\overline{z})}$) has at most N pair of conjugate complex zeros (counted with multiplicity).

Unfortunately, it seems that

$F(s)$ in the R-S formula has infinitely many zeros

in $\mathrm{Re}(s) > 1/2$ and also in $\mathrm{Re}(s) < 1/2$.

Recall

$$E(s) + \overline{E(1-\overline{s})},$$

or

$$W(z) + \overline{W(\overline{z})}.$$

Is there any nice representation of $\zeta(s)$ like this?

Eisenstein series

$$E_0(z;s) = \frac{1}{2} \sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}} \qquad (\mathrm{Re}\,(s) > 1),$$

where $z = x + yi,\ x \in \mathbb{R},\ y > 0.$

Fourier series

$$E_0(z;s) = \zeta(2s)y^s + \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-s}$$
$$+ 4\pi^s\sqrt{y}\sum_{n=1}^{\infty} n^{1/2-s}\sum_{d|n} d^{2s-1}\frac{K_{s-1/2}(2\pi ny)}{\Gamma(s)}\cos(2\pi nx).$$

Properties

$$\pi^{-s}\Gamma(s)E_0(z;s) = \pi^{-1+s}\Gamma(1-s)E_0(z;1-s).$$

$$E_0(i;s) = 4\zeta(s)L(s,\chi_{-4}).$$

Warning: In general, the Eisenstein series does not satisfy the analogue of the Riemann hypothesis.

Truncations

$$E_{0,N}(z;s) = \zeta(2s)y^s + \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-s} + 4\pi^s\sqrt{y}\sum_{n=1}^{N} n^{1/2-s}\sum_{d|n} d^{2s-1}\frac{K_{s-1/2}(2\pi ny)}{\Gamma(s)}\cos(2\pi nx).$$

Constant terms

$$\zeta(2s)y^s + \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-s}.$$

Theorem [Ki]

For $y \geqslant 1$, all complex zeros of the constant term are simple and on $\mathrm{Re}\, s = 1/2$.

Theorem [Ki]

All but finitely many zeros of truncations of the Eisenstein series in any strip containing the line $\mathrm{Re}\, s = 1/2$ are simple and on $\mathrm{Re}\, s = 1/2$.

If $\mathrm{Im}\, z \geqslant 1$, then all but finitely many zeros of truncations of the Eisenstein series are simple and on $\mathrm{Re}\, s = 1/2$.

Use

$$\hat{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s)\zeta(s)$$

Notations

F: a number field
$\mathbb{A} = \mathbb{A}_F$ its ring of adèles
G: a quasi-split connected reductive algebraic group over F
Z: the central subgroup of G
Fix a Borel subgroup P_0 of G over F.
Write $P_0 = M_0 U_0$ (M_0: a maximal torus, U_0: the unipotent radical of P_0).
$P \supset P_0$: a parabolic subgroup of G over F.
Write $P = MU$ ($M_0 \subset M$ the standard Levi, U the unipotent radical).
W: the Weyl group of the maximal F-split subtorus of M_0 in G

Δ_0: the set of simple roots
ρ_P: half the sum of roots in U.
K: a maximal compact subgroup of $G(\mathbb{A})$ such that $P(\mathbb{A}) \cap \mathbf{K} = (M(\mathbb{A}) \cap \mathbf{K})(U(\mathbb{A}) \cap \mathbf{K})$.
$m_P : G(\mathbb{A}) \to M(\mathbb{A})/M(\mathbb{A})^1$, $g = m \cdot n \cdot k \mapsto M(\mathbb{A})^1 \cdot m$, $g \in G(\mathbb{A})$, $m \in M(\mathbb{A})$, $n \in U_0(\mathbb{A})$, $k \in \mathbf{K}$.
Fix Haar measures on $M_0(\mathbb{A})$, $U_0(\mathbb{A})$, **K** (the induced measures on $M(F)$ and $U_0(F)$ are the counting measures and the volumes of $M(F) \setminus M(\mathbb{A})^1$, $U_0(F) \setminus U_0(\mathbb{A})$ and **K** are 1).

$X(G)_F$: for the additive group homomorphisms from G to $GL(1)$ over F.
$\mathfrak{a}_G = \mathrm{Hom}_F(X(G)_F, \mathbb{R})$.
$\mathfrak{a}_P = \mathfrak{a}_M$ ($P = MU$), $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$.
Δ_0^P: the set of simple roots in P.
Δ_P: the set of linear forms on $\mathfrak{a}_P$ obtained by restriction of elements in the complement $\Delta_0 - \Delta_0^P$.
$\widehat{\Delta}_0 = \{\varpi_\alpha : \alpha \in \Delta_0\}$: the set of simple weights.
$\widehat{\Delta}_P = \{\varpi_\alpha : \alpha \in \Delta_0 - \Delta_0^P\}$.
$\widehat{\tau}_p$: the characteristic function of the subset

$$\{t \in \mathfrak{a}_P : \varpi(t) > 0, \, \varpi \in \widehat{\Delta}_P\}.$$

Fix $T \in a_0$ with $\alpha(T) \gg 0$ for any simple root α.

Arthur's analytic truncation

$$\left(\Lambda^T\phi\right)(x) = \sum_{P\supseteq P_0} (-1)^{\dim(A_P/Z)} \sum_{\delta\in P(F)\backslash G(F)} \phi_P(\delta x)\widehat{\tau}_P(H(\delta x)-T).$$

Here $\phi \in C(G(F)\setminus G(\mathbb{A})^1)$, A_P: the central subgroup of M $(P = MU)$, $\phi_P(x) = \int_{U(F)\backslash U(\mathbb{A})} \phi(nx)dn$.

Arthur's period

For an automorphic form ϕ of G, define

$$A(\phi; T) = \int_{G(F)\backslash G(\mathbb{A})} \Lambda^T \phi(g) dg.$$

For ϕ an M-level automorphic form, we form the associated Eisenstein series

$$E(\phi, \lambda)(g) = \sum_{\delta \in P(F)\backslash G(F)} m_P(\delta g)^{\lambda+\rho_P} \phi(\delta g) \qquad (\mathrm{Re}\, \lambda \in \mathcal{C}_P^+).$$

Here $\mathcal{C}_P^+$ denotes the positive chamber in a_P and $\lambda = (\lambda_1, \ldots, \lambda_r)$, where r is the rank of the group.

The Eisenstein period $A(E(\phi;\lambda);T)$ (ϕ: a cusp form)

(1) 0 if $P \neq P_0$;

(2) $$v \sum_{w\in W} \frac{e^{\langle w\lambda-\rho_{P_0},T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho_{P_0},\alpha^\vee\rangle}\times$$

$$\int_{M_0(F)\backslash M_0(\mathbb{A})^1\times\mathbf{K}}(M(w,\lambda)\phi)(mk)\,dm\,dk, \text{ if } P=P_0.$$

Here $v = \mathrm{vol}\left(\{\sum_{\alpha\in\Delta_0} a_\alpha \alpha^\vee : a_\alpha \in [0,1)\}\right)$, $\alpha^\vee$ is the coroot associated to α and for $g \in G(\mathbb{A})$

$$(M(w,\lambda)\phi)(g) = m_{P'}(g)^{\omega\lambda+\rho_{P'}} \int_{U'(F)\cap wU(F)w^{-1}\backslash U'(\mathbb{A})} m_P(w^{-1}n'g)^{\lambda+\rho_P} dn'$$

with $M' = wMw^{-1}$ and $P' = M'U'$.

Weng

Define the period $\omega_F^G(\lambda)$ of G over F by

$$\omega_F^G(\lambda) = \sum_{w \in W} \frac{1}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho_{P_0}, \alpha^\vee \rangle} \times M(w, \lambda),$$

where

$$M(w, \lambda) = m_{P'}(e)^{\omega_\lambda + \rho_{P'}} \int_{U'(F) \cap wU(F)w^{-1} \backslash U'(\mathbb{A})} m_P(w^{-1} n')^{\lambda + \rho_P} dn'.$$

Weng's zeta functions

· G : a connected semisimple algebraic group defined over $\mathbb{Q}$ endowed with a maximal ($\mathbb{Q}$-)split torus T
· Φ : the root system with respect to (G, T)
· B : a Borel subgroup of G containing T
· Δ : the fundamental system of Φ
· $X^*(T)$: the group of characters of T defined over $\mathbb{Q}$, being a free module of rank $r = \dim T$
· $\mathfrak{a}_0^* = X^*(T) \otimes \mathrm{Re}$, $\mathfrak{a}_0 = \mathrm{Hom}(X^*(T), \mathrm{Re})$
· $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$: real vector spaces of dimension r
· Φ : a finite subset of $X^*(T)$, embedded in $\mathfrak{a}_0^*$.
· $\alpha^\vee \in \mathfrak{a}_0$: the coroot for a simple root $\alpha \in \Delta$.

Gindikin-Karpelevich

G: a classical semisimple algebraic group over $\mathbb{Q}$.

$$\omega_{\mathbb{Q}}^{G}(\lambda) = \sum_{w \in W} \frac{1}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho_{P_0}, \alpha^\vee \rangle} \times \prod_{\alpha > 0, w\alpha < 0} \frac{\hat{\zeta}(\langle \lambda, \alpha^\vee \rangle)}{\hat{\zeta}(\langle \lambda, \alpha^\vee \rangle + 1)},$$

where $\lambda \in \mathcal{C}^+$ ($\mathcal{C}^+ = \mathcal{C}_{P_0}^+$).

Weng

P: a fixed maximal parabolic subgroup of G.
α_P: the corresponding simple root in Δ_0.
Write $\Delta \setminus \{\alpha_P\} = \{\beta_1, \ldots, \beta_{r-1}\}$, $r = r(G)$: the rank of G.
Define the period $\omega_{\mathbb{Q}}^{G/P}$ for (G, P) over $\mathbb{Q}$ by

$$\omega_{\mathbb{Q}}^{G/P}(\lambda_P) = \mathrm{Res}_{\langle \lambda-\rho, \beta_{r(G)-1}^{\vee}\rangle=0} \cdots \mathrm{Res}_{\langle \lambda-\rho, \beta_1^{\vee}\rangle=0} \left(\omega_{\mathbb{Q}}^{G}(\lambda)\right)$$

with $\lambda_P \gg 0$ and the constraint of taking residues along with $(r-1)$ singular hyperplanes

$$\langle \lambda - \rho, \beta_1^{\vee}\rangle = 0, \ldots, \langle \lambda - \rho, \beta_{r(G)-1}^{\vee}\rangle = 0,$$

Weng's Zeta function

Using $\omega_{\mathbb{Q}}^{G/P}(\lambda_P)$ with necessary normalizations, we can define the Weng's zeta function:

$$\hat{\zeta}_{\mathbb{Q},P}^{(G,T)}(s)$$

Weng's Conjecture

The zeta function $\hat{\zeta}_{\mathbb{Q},P}^{(G,T)}(s)$ satisfies the analogue of the Riemann hypothesis.

Examples

Weng provides the following ten zeta functions:

one zeta function for $SL(2)$;
one zeta function for $SL(3)$;
two zeta functions, for $SL(4)$;
two zeta functions for $SL(5)$;
two zeta functions for $Sp(4)$;
two zeta functions for G_2.

$\hat{\zeta}^{SL(5)}_{\mathbb{Q},P_{2,3}}(s) =$

$$\frac{\hat{\zeta}(2)^2\hat{\zeta}(3)\cdot\hat{\zeta}(5s-1)\hat{\zeta}(5s)}{5s-5} + \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-1)\hat{\zeta}(5s)}{4(5s-3)}$$

$$+ \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-3)\hat{\zeta}(5s-2)}{(5s-1)^2(5s-5)} - \frac{\hat{\zeta}(2)\hat{\zeta}(3)\cdot\hat{\zeta}(5s-1)\hat{\zeta}(5s)}{2(5s-4)}$$

$$- \frac{\hat{\zeta}(2)^2\cdot\hat{\zeta}(5s-1)\hat{\zeta}(5s)}{3(5s-4)} - \frac{\hat{\zeta}(2)^2\cdot\hat{\zeta}(5s-1)\hat{\zeta}(5s)}{3(5s-3)}$$

$$- \frac{\hat{\zeta}(5s-2)\hat{\zeta}(5s-1)}{4(5s-3)(5s-1)} + \frac{\hat{\zeta}(2)^2\cdot\hat{\zeta}(5s-4)\hat{\zeta}(5s-3)}{3(5s-2)}$$

$$- \frac{\hat{\zeta}(5s-3)\hat{\zeta}(5s-2)}{2(5s-4)(5s-2)(5s-1)} + \frac{\hat{\zeta}(5s-4)\hat{\zeta}(5s-3)}{8(5s-3)}$$

$$- \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-2)\hat{\zeta}(5s-1)}{(5s-4)^2(5s)} - \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-4)\hat{\zeta}(5s-3)}{6(5s-2)}$$

$$
\begin{aligned}
&- \frac{\hat{\zeta}(5s-3)^2}{2(5s-5)(5s-2)^2} - \frac{\hat{\zeta}(5s-3)\hat{\zeta}(5s-1)}{4(5s-2)(5s-3)} \\
&+ \frac{\hat{\zeta}(2)\hat{\zeta}(3)\cdot\hat{\zeta}(5s-4)\hat{\zeta}(5s-3)}{2(5s-1)} + \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-3)\hat{\zeta}(5s-2)}{2(5s-5)(5s-1)} \\
&+ \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-3)\hat{\zeta}(5s-2)}{2(5s-4)(5s-1)} + \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-1)\hat{\zeta}(5s)}{6(5s-3)} \\
&+ \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-1)\hat{\zeta}(5s)}{6(5s-2)} + \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-2)\hat{\zeta}(5s-1)}{2(5s-4)(5s-1)} \\
&+ \frac{\hat{\zeta}(5s-2)^2}{(5s-4)^2(5s-1)^2} + \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-3)\hat{\zeta}(5s-1)}{3(5s-3)(5s-1)} \\
&- \frac{\hat{\zeta}(5s-1)\hat{\zeta}(5s)}{8(5s-2)} - \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-4)\hat{\zeta}(5s-3)}{6(5s-3)} \\
&- \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-4)\hat{\zeta}(5s-3)}{4(5s-2)} + \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-2)\hat{\zeta}(5s-1)}{2(5s-4)(5s)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\hat{\zeta}(5s-1)^2}{2(5s-3)^2(5s)} - \frac{\hat{\zeta}(2)\hat{\zeta}(3)\cdot\hat{\zeta}(5s-3)\hat{\zeta}(5s-1)}{(5s-5)(5s)} \\
&-\frac{\hat{\zeta}(2)^2\cdot\hat{\zeta}(5s-2)\hat{\zeta}(5s-1)}{(5s-5)(5s)} - \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-1)^2}{(5s-4)(5s-3)(5s)} \\
&+\frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-3)\hat{\zeta}(5s-1)}{3(5s-4)(5s-2)} \\
&+\frac{\hat{\zeta}(5s-2)\hat{\zeta}(5s-1)}{2(5s-4)(5s-3)(5s-1)} \\
&-\frac{\hat{\zeta}(5s-3)\hat{\zeta}(5s-2)}{4(5s-4)(5s-2)} - \frac{\hat{\zeta}(2)^2\hat{\zeta}(3)\cdot\hat{\zeta}(5s-4)\hat{\zeta}(5s-3)}{(5s)} \\
&-\frac{\hat{\zeta}(2)^2\cdot\hat{\zeta}(5s-3)\hat{\zeta}(5s-2)}{(5s-5)(5s)} + \frac{\hat{\zeta}(2)\cdot\hat{\zeta}(5s-3)^2}{(5s-5)(5s-2)(5s-1)} \\
&+\frac{\hat{\zeta}(2)^2\cdot\hat{\zeta}(5s-4)\hat{\zeta}(5s-3)}{3(5s-1)}.
\end{aligned}
$$

Theorem [Lagarias, Suzuki, Weng]

All zeros of $\hat{\zeta}^{SL(2)}_{\mathbb{Q},P_0}$, $\hat{\zeta}^{SL(3)}_{\mathbb{Q},P_{1,2}}$, $\hat{\zeta}^{Sp(4)}_{\mathbb{Q},P_{1,3}}$, $\hat{\zeta}^{G_2}_{\mathbb{Q},P_{\mathrm{long}}}$, $\hat{\zeta}^{G_2}_{\mathbb{Q},P_{\mathrm{short}}}$ are on $\mathrm{Re}\, s = 1/2$.

Theorem [Ki]

All zeros of ten Weng's zeta functions are on $\mathrm{Re}\,(s) = 1/2$ and simple.

Main Theorem

Theorem [Ki, Komori, Suzuki]

Let G be a Chevalley group defined over $\mathbb{Q}$, in other words, G is a connected semisimple algebraic group defined over $\mathbb{Q}$ endowed with a maximal ($\mathbb{Q}$-)split torus T. Let B be a Borel subgroup of G containing T. Let P be a maximal parabolic subgroup of G defined over $\mathbb{Q}$ containing B. Then all but finitely many zeros of $\hat{\zeta}^{(G,T)}_{\mathbb{Q},P/B}(s)$ are simple and on the critical line of its functional equation.

Chevalley's fundamental theorem

· G : a connected semisimple algebraic group; $\mathfrak{g}$ its Lie algebra of G
· T : a maximal torus of G; $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, \mathrm{Ad}(t)X = \alpha(t)X\}$ for each character $\alpha \in X^*(T)$
· $\Phi = \Phi(G, T) = \{\alpha \in X^*(T) \,|\, \mathfrak{g}_\alpha \neq \emptyset\}$: finite with a root system (in the vector space $X^*(T) \otimes \mathrm{Re}$)
· Conversely, for a given root system Φ, there exists a connected semisimple algebraic group $G = G(\Phi)$ defined over a prime field having Φ as its root system with respect to a split maximal torus T of G

· $G(\Phi)$: a Chevalley group of type Φ (or split group, since it has a maximal torus which is split over the prime field)

Weng zeta functions in terms of abstract root system

Thus, we can deal with Weng zeta functions for Chevalley groups defined over $\mathbb{Q}$ by using the language of abstract root systems only.

Root system and the Weyl group

Important properties:

· Φ^+ : the corresponding positive system of Φ; $\Phi^- = -\Phi^+$ so that $\Phi = \Phi^+ \cup \Phi^-$

· $\Phi_p^+ = \Phi^+ \cap \Phi_p \ (\subset \Phi^+)$: the corresponding positive system of Φ_p; $\Phi_p = \Phi_p^+ \cup \Phi_p^-$ with $\Phi_p^- = -\Phi_p^+$

· w_0 : the longest element of W; $w_0^2 = \mathrm{id}$, $w_0\Delta = -\Delta$ and $w_0\Phi^+ = \Phi^-$

· w_p : the longest element of W_p; $w_p^2 = \mathrm{id}$, $w_p\Delta_p = -\Delta_p$ and $w_p\Phi_p^+ = \Phi_p^-$

· The condition $\Delta_p \subset w^{-1}(\Delta \cup \Phi^-)$

Outline of the proof of Theorem

Define the entire function $\xi_p(s) = Q(s)\,\hat{\zeta}_p(s)$ by multiplying a suitable polynomial.

1. At first we construct an entire function $\varepsilon_p(s)$ satisfying

$$\xi_p(s) = \varepsilon_p(s) \pm \varepsilon_p(-c_p - s).$$

2. (i) the number of zeros of $\varepsilon_p(s)$ in $\Re(s) \geqslant -c_p/2$ is finitely many,
(ii) in a left half-plane, $\varepsilon_p(s)$ has no zero in a region $\Re(s) \leqslant -\kappa \log(|\Im(s)| + 10)$.

3. Essentially, Hermite-Biehler's theorem

Proposition

The number of zeros of $\varepsilon_p(s)$ lying in right half-plane $\Re(s) \geqslant -c_p/2$ is finitely many at most. Furthermore, there exists a positive function $\delta(t)$ on the real line satisfying $\delta(t) \log |t| \to \infty$ $(|t| \to \infty)$ such that the number of zeros of $\varepsilon_p(s)$ in $\Re(s) \geqslant -c_p/2 - \delta(t)$ is finitely many at most.

A crucial point of Proposition

$$\sum_{\substack{w \in \mathfrak{W}_p^{\ddagger} \\ |(w^{-1}\Delta)\setminus \Phi_p|=1}} \frac{1}{\langle \lambda_p, \alpha_w^{\vee} \rangle} C_{p,w} D_{p,w} = \prod_{\alpha \in \Phi_p^{+}} \hat{\zeta}(\mathrm{ht}\, \alpha^{\vee} + 1) \cdot \operatorname*{Res}_{\lambda=\rho_p} \omega_{\Delta_p}^{\Phi_p}(\lambda).$$

By a theorem of Weng, the right-hand side is a product of special values of the Riemann zeta function and volumes of several (truncated) domains corresponding to irreducible components of Φ_p.

Proposition

There exists a positive real number κ such that $E_p(s)$ has no zeros in the region $\Re(s) \leqslant -\kappa \log(|\Im(s)| + 10)$.

Proposition

Let $T > 1$, and $\sigma > c_p/2$. Denote by $N(T;\sigma)$ the number of zeros of $\varepsilon_p(s)$ in the region

$$-\sigma < \Re(s) < -c_p/2 - \delta(t), \quad 0 < \Im(s) < T.$$

Then there exist a positive number $\sigma_{\mathrm{L}} > 0$ such that

$$N(T;\sigma_{\mathrm{L}}) = C_1\, T \log T + C_2\, T + O(\log T)$$

for some positive real number $C_1 > 0$ and real number C_2, and

$$N(T;+\infty) = C_1\, T \log T + C_3\, T + O(\log^2 T)$$

for some real number C_3.

Proposition

Define

$$W_p(z) = \varepsilon_p(-c_p/2 + iz).$$

Then it has the product formula

$$W_p(z) = \omega\, e^{\alpha z} V(z) W_1(z) W_2(z),$$

where ω is a nonzero real number, α is a real number, $V(z)$ is a polynomial having no zeros in $\Im(z) > 0$ except for purely imaginary zeros,

Proposition [Continue]

$$W_1(z) = \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\rho_n}\right) \left(1 + \frac{z}{\bar{\rho}_n}\right) \right],$$

$$W_2(z) = \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\eta_n}\right) \left(1 + \frac{z}{\bar{\eta}_n}\right) \right]$$

with $\Re(\rho_n) > 0$, $\Re(\eta_n) > 0$ and $0 < \delta(t) < \Im(\rho_n) < \sigma_{\mathrm{L}} + 1 < \Im(\eta_n) < \kappa \log(\Re(\eta_n) + 10)$ for every $n \geqslant 1$.

Proposition

Let $W(z)$ be a function in $\mathbb{C}$. Suppose that $W(z)$ has the product factorization

$$W(z) = h(z)\, e^{\alpha z} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\rho_n}\right) \left(1 + \frac{z}{\bar{\rho}_n}\right) \right],$$

where $h(z)$ is a nonzero polynomial having N many zeros counted with multiplicity in the lower half-plane, $\alpha \in \mathrm{Re}$, $\Im(\rho_n) > 0$ $(n = 1, 2, \cdots)$, and the product converges uniformly in any compact subset of $\mathbb{C}$. Then, $W(z) + \overline{W(\bar{z})}$ and $W(z) - \overline{W(\bar{z})}$ has at most N pair of conjugate complex zeros counted with multiplicity.

Theorem [Weak Riemann Hypothesis for ξ_p]

There exists a bounded region $\mathfrak{B}_p$ such that all zeros of $\xi_p(s)$ outside $\mathfrak{B}_p$ lie on the line $\Re(s) = -c_p/2$.

Theorem [Simple zeros of ξ_p]

There exists a bounded region $\mathfrak{B}'_p(\supset \mathfrak{B}_p)$ such that all zeros of $\xi_p(s)$ outside $\mathfrak{B}'_p$ lie on the line $\Re(s) = -c_p/2$ and simple.

Thank you very much!