Selberg type zeta functions for the Hilbert modular group of a real quadratic field

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Contents

Introduction (1)

- $G := \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$
  - $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$, G acts on $\mathbb{H}$ by $g.z := \frac{az+b}{cz+d} \in \mathbb{H}$
  - $\Gamma \subset G$: co-compact torsion-free discrete subgroup

$\Rightarrow$ $X := \Gamma \backslash G / K$ is a compact Riemann surface of genus $g \geq 2$

- Let $\gamma \in \Gamma$ is hyperbolic $\Leftrightarrow$ $|\mathrm{tr}(\gamma)| > 2$.

$\Rightarrow$ the centralizer of γ in Γ is infinite cyclic and γ is conjugate in G to

$$\gamma \sim \pm \begin{pmatrix} N(\gamma)^{1/2} & 0 \\ 0 & N(\gamma)^{-1/2} \end{pmatrix} \quad \text{with } N(\gamma) > 1.$$

- Prim(Γ) := the set of Γ-conjugacy classes of the primitive hyperbolic elements in Γ. (i.e, not a power of other hyperbolic elements)

Introduction (2)

The Selberg zeta function for Γ (or X) is defined by

$$Z_\Gamma(s) := \prod_{p \in \mathrm{Prim}(\Gamma)} \prod_{k=0}^{\infty} \left(1 - N(p)^{-(k+s)}\right) \qquad \text{for } \mathrm{Re}(s) > 1.$$

Theorem (Selberg 1956)

1. *$Z_\Gamma(s)$ defined for $\mathrm{Re}(s) > 1$ extends meromorphically over $\mathbb{C}$ (actually entire)*
2. *$Z_\Gamma(s)$ has zeros at $s = -k$ $(k \in \mathbb{N})$ of order $(2g-2)(2k+1)$, at $s = 0$ of order $2g-1$ and at $s = 1$ of order 1 : trivial zeros*
3. *$Z_\Gamma(s)$ has zeros at $s = \frac{1}{2} \pm i r_n$: nontrivial zeros*

Here, $\{\lambda_n = 1/4 + r_n^2\}$ is the eigenvalues of the Laplacian $\Delta_0 = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ acting on $L^2(\Gamma \backslash \mathbb{H})$.

- The theorem is proved by using the Selberg trace formula.

Introduction (3)

Theorem (Functional equation)

$$Z_\Gamma(1-s) = Z_\Gamma(s)\exp\Big(-4(g-1)\pi\int_0^{s-\frac{1}{2}} r\,\tan(\pi r)\,dr\Big)$$

by Selberg 1956. We have also

$$\hat{Z}_\Gamma(1-s) = \hat{Z}_\Gamma(s) := Z_\Gamma(s)\big(\Gamma_2(s)\Gamma_2(s+1)\big)^{2g-2}.$$

- $\Gamma_2(z) := \exp(\zeta_2'(0,z))$ with $\zeta_2(s,z) = \sum_{n,m\geq 0}(n+m+z)^{-s}$

: the double Γ function

Problem

- Generalize Selberg's Theorem for $\Gamma \subset \mathrm{PSL}(2,\mathbb{R}) \Rightarrow$ "Theorem" for $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})^2$.
- Construct Selberg type zeta functions for $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})^2$.
- Study analytic properties of the above Selberg type zeta functions for $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})^2$.

Selberg type zeta functions for the Hilbert modular group of a real quadratic field

Notation

- $K/\mathbb{Q}$: a real quadratic field with class number one
  - σ : the generator of $\mathrm{Gal}(K/\mathbb{Q})$
  - $a' := \sigma(a)$ for $a \in K$
  - $\mathcal{O}_K$: the ring of integers of K
  - $\gamma' := \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathcal{O}_K)$
  - $\Gamma_K := \{(\gamma, \gamma') \,|\, \gamma \in \mathrm{PSL}(2, \mathcal{O}_K)\}$: the Hilbert modular group

$\Rightarrow$

  - $\Gamma_K \subset \mathrm{PSL}(2, \mathbb{R})^2$: an irreducible discrete subgroup
  - Γ_K acts on $\mathbb{H}^2$ (product of two upper half planes) by linear fractional transformation
  - Γ_K have only one cusp (∞, ∞) (Γ_K-inequivalent parabolic fixed point)
  - $X_K := \Gamma_K \backslash \mathbb{H}^2$: the Hilbert modular surface
- Let $(\gamma, \gamma') \in \Gamma_K$ be hyperbolic-elliptic, i.e, $|\mathrm{tr}(\gamma)| > 2$ and $|\mathrm{tr}(\gamma')| < 2$

$\Rightarrow$ the centralizer of hyperbolic-elliptic (γ, γ') in Γ_K is infinite cyclic.

- Fix $m \geq 4$: even integer

Definition (Selberg type zeta function for Γ_K)

$$Z_K(s;m) := \prod_{(p,p')} \prod_{k=0}^{\infty} \Big(1 - e^{i(m-2)\omega}\, N(p)^{-(k+s)}\Big)^{-\kappa} \quad \text{for } \mathrm{Re}(s) \gg 0$$

Here, (p, p') run through the set of primitive hyperbolic-elliptic Γ_K-conjugacy classes of Γ_K, and (p, p') is conjugate in $\mathrm{PSL}(2,\mathbb{R})^2$ to

$$(p,p') \sim \left(\begin{pmatrix} N(p)^{1/2} & 0 \\ 0 & N(p)^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \right).$$

- $N(p) > 1$, $\omega \in (0, \pi)$ and $\omega \notin \pi\mathbb{Q}$.
- $\kappa \in \mathbb{N}$ such that $\kappa\,\zeta_K(-1) \in \mathbb{N}$, ($\zeta_K(s)$:Dedekind zeta function of K) and $\kappa\nu_j^{-1} \in \mathbb{N}$ $(1 \leq j \leq N)$ $\{\nu_1, \nu_2, \ldots, \nu_N\}$: the orders of primitive elliptic elements in Γ_K

Problem

1. Analytic properties of $Z_K(s;m)$
2. Functional equation of $Z_K(s;m)$

Analytic properties of $Z_K(s;m)$

Theorem 1

For an even integer $m \geq 4$, $Z_K(s;m)$ a priori defined for $\mathrm{Re}(s) \gg 0$ has a meromorphic extension over the complex plane $\mathbb{C}$.

Theorem 2

$Z_K(s,m)$ has the following "essential" zeros and poles at

- $s = \frac{1}{2} \pm i\mu_j \quad j = 0, 1, 2, \cdots$: zeros
- $s = \frac{1}{2} \pm i\nu_k \quad k = 0, 1, 2, \cdots$: poles

Here,

- $\{\frac{1}{4} + \mu_j^2 \,|\, j = 0, 1, 2, \cdots\} = \mathrm{Spec}(\Delta_0^{(1)}|_{\mathrm{Ker}(\Lambda_m^{(2)})})$
- $\{\frac{1}{4} + \nu_k^2 \,|\, k = 0, 1, 2, \cdots\} = \mathrm{Spec}(\Delta_0^{(1)}|_{\mathrm{Ker}(\Lambda_{m-2}^{(2)})})$

are the sets of eigenvalues of the Laplacian $\Delta_0^{(1)}$ acting on "Hilbert-Maass forms" of weight $(0,m)$ or $(0,m-2)$ and $\Lambda_m^{(2)}$, $\Lambda_{m-2}^{(2)}$ are "Maass operators".

Functional equation of $Z_K(s;m)$

- $Z_K(s,m)$ has another series of zeros and poles coming from the idenity, elliptic, "type 2 hyperbolic" conjugacy classes of Γ_K and "scattering terms".

Theorem 3

$Z_K(s,m)$ satisfies the following functional equation

$$\hat{Z}_K(s;m) = \hat{Z}_K(1-s;m).$$

Here the completed zeta function $\hat{Z}_K(s,m)$ is given by

$$\hat{Z}_K(s;m) := Z_K(s;m)\Big(Z_{\mathrm{id}}(s)\, Z_{\mathrm{ell}}(s) Z_{\mathrm{sct/hyp2}}(s)\Big)^{\kappa}$$

with

Gamma and local factors

Gamma and local factors of $Z_K(s;m)$

$$Z_{\rm id}(s) := \Big(\Gamma_2(s)\Gamma_2(s+1)\Big)^{2\zeta_K(-1)}$$

$$Z_{\rm ell}(s) := \prod_{j=1}^{N}\prod_{l=0}^{\nu_j-1}\Gamma\big(\tfrac{s+l}{\nu_j}\big)^{\frac{\nu_j-1-\xi_l(m,j)}{\nu_j}}$$

$$Z_{\rm sct/hyp2}(s) := \zeta_\varepsilon(s-\tfrac{m}{2}-1)\zeta_\varepsilon(s-\tfrac{m}{2}-2)^{-1}$$

- $\{\nu_1,\nu_2,\ldots,\nu_N\}$: the orders of primitive elliptic elements in Γ_K
- $\xi_l(m,j)\in\{0,1,\cdots,2\nu_j-2\}$
- $\zeta_\varepsilon(s) := (1-\varepsilon^{-2s})^{-1}$ • ε : the fundamental unit of K

The zeros and poles of $Z_{\rm id}(s)$, $Z_{\rm ell}(s)$ and $Z_{\rm sct/hyp2}(s)$ are easily calculated.
$\Rightarrow$ All zeros and poles of $Z_K(s;m)$ are determined !

• These analytic properties and functional equation of $Z_K(s;m)$ are obtained by the "differences" of the Selberg trace formula for Hilbert modular groups.

What is the differences of the Selberg trace formula ? (short sketch)

Maass forms of weight m $(m \in 2\mathbb{Z}_{\geq 0})$

- $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$: discrete subgroup,
- $\Delta_m := -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + im\,y\frac{\partial}{\partial x}$

$$L^2(\Gamma\backslash\mathbb{H}\,;\,m) := \Big\{ f\colon \mathbb{H} \to \mathbb{C},\ C^\infty \,\Big|\, \bullet\, f(\gamma z) = \Big(\frac{cz+d}{|cz+d|}\Big)^m f(z)\ \forall\gamma\in\Gamma$$
$$\bullet\, \Delta_m f(z) = \lambda f(z) \quad \bullet\, ||f||^2 = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{f(z)}\,\frac{dxdy}{y^2} < \infty. \Big\}$$

STF for $L^2(\Gamma\backslash\mathbb{H}\,;\,m)$, h: test function, G: "Fourier trans." of h

$$\sum_{\lambda\in\mathrm{Spec}(\Delta_m)} h(\lambda) = \sum_{[\gamma]\in\mathrm{Conj}(\Gamma)} G(\gamma).$$

- By considering the "differences" of the above STF, we have

$$m_\Gamma(\lambda_{\min})\,h(\lambda_{\min}) = \sum_{[\gamma]\in S} G(\gamma). \quad \exists S \subset \mathrm{Conj}(\Gamma).$$

Hilbert Maass forms of weight (m_1, m_2) $(m_j \in 2\mathbb{Z}_{\geq 0})$

- $\Gamma \subset \mathrm{SL}(2,\mathbb{R})^2$: discrete subgroup,
- $\Delta_{m_j} := -y_j^2\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) + i m_j\, y_j \frac{\partial}{\partial x_j}$

$$L^2(\Gamma\backslash\mathbb{H}^2\,;\,(m_1,m_2)) := \Big\{ f\colon \mathbb{H}^2 \to \mathbb{C},\, C^\infty \;\Big|\; \bullet\, f \text{ is weight } (m_1,m_2) \text{ w.r.t } \Gamma$$
$$\bullet\, \Delta_{m_1} f = \lambda^{(1)} f \quad \bullet\, \Delta_{m_2} f = \lambda^{(2)} f \quad \bullet\, \|f\|^2 < \infty. \Big\}$$

STF for $L^2(\Gamma\backslash\mathbb{H}\,;\,(m_1,m_2))$, h: test function

$$\sum_{(\lambda^{(1)},\lambda^{(2)}) \in \mathrm{Spec}(\Delta_{m_1},\Delta_{m_2})} h(\lambda^{(1)},\lambda^{(2)}) = \sum_{[\gamma]\in \mathrm{Conj}(\Gamma)} G(\gamma).$$

- By considering the "differences" of the above STF, we have

$$\sum_{(\lambda^{(1)},\lambda^{(2)}_{\min}) \in \mathrm{Spec}(\Delta_{m_1},\Delta_{m_2})} h(\lambda^{(1)},\lambda^{(2)}_{\min}) = \sum_{[\gamma]\in S} G(\gamma). \quad \exists S \subset \mathrm{Conj}(\Gamma).$$

Differences of the Selberg trace formula for compact Riemann surfaces

Notation

First of all, we recall the differences of the Selberg trace formula for compact Riemann surfaces.

- $G := \mathrm{SL}(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \det g = 1 \right\}.$

- $G = NAK$: the Iwasawa decomposition, $M :=$ the centralizer of A in K
  $\Rightarrow N \simeq (\mathbb{R},+), \quad A \simeq \mathbb{R}^{\times}_{>0}, \quad K = \mathrm{SO}(2), \quad M = \{\pm I_2\}$
- $G/K \simeq \mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$: the upper half plane
- G acts on $\mathbb{H}$ by $g.z := \frac{az+b}{cz+d} \in \mathbb{H}$
- $\Gamma \subset G$: discrete subgroup

1. γ is hyperbolic $\Leftrightarrow |\mathrm{tr}(\gamma)| > 2 \Leftrightarrow$ Fix$(\gamma) = \{\alpha, \alpha^{-1}\} \subset \mathbb{R} \cup \{\infty\}$
2. γ is elliptic $\Leftrightarrow |\mathrm{tr}(\gamma)| < 2 \Leftrightarrow$ Fix$(\gamma) = \{\alpha, \bar{\alpha}\}$, $\alpha \in \mathbb{H}$
3. γ is parabolic $\Leftrightarrow |\mathrm{tr}(\gamma)| = 2 \Leftrightarrow$ Fix$(\gamma) = \{\alpha\} \subset \mathbb{R} \cup \{\infty\}$

$\Gamma \subset G$: co-compact discrete subgroup

- $\Gamma\backslash\mathbb{H}$ is compact. $\Leftrightarrow$ Γ has no parabolic ellements.

Assumption on Γ

- $\Gamma \subset G$: co-compact discrete subgroup
  $\Rightarrow X := \Gamma\backslash G/K$ is a compact Riemann surface

- $\gamma \in \Gamma$ is hyperbolic $\Rightarrow$ γ is conjugate in G to
  $\gamma \sim \begin{pmatrix} N(\gamma)^{1/2} & 0 \\ 0 & N(\gamma)^{-1/2} \end{pmatrix}$ with $N(\gamma) > 1$.
- $\gamma \in \Gamma$ is elliptic $\Rightarrow$ γ is conjugate in G to
  $\gamma \sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2)$

Selberg trace formula for compact Riemann surfaces

- Fix $m \in 2\mathbb{Z}_{\geq 0}$: weight
- $j_\gamma(z) := \frac{cz+d}{|cz+d|}$ for $\gamma \in \Gamma$
- $\Delta_m := -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + im\, y \frac{\partial}{\partial x}$: the Laplacian acting on

$$L^2(\Gamma\backslash\mathbb{H}\,;\, m) := \left\{ f : \mathbb{H} \to \mathbb{C},\ C^\infty \,\middle|\, \bullet\, f(\gamma z) = j_\gamma(z)^m f(z)\ \forall \gamma \in \Gamma \right.$$
$$\left. \bullet\, \Delta_m\, f(z) = \lambda f(z) \quad \bullet\, ||f||^2 = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{f(z)}\, \frac{dxdy}{y^2} < \infty. \right\}$$

Let $\{\lambda_n = 1/4 + r_n^2\}$ is the eigenvalues of the Laplacian Δ_m acting on $L^2(\Gamma\backslash\mathbb{H}\,;\, m)$ enumerated as $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$

- $h(r) = h(-r)$: test function, analytic on $|\operatorname{Im}(r)| < \max\{\frac{m-1}{2}, \frac{1}{2}\} + \delta$ $(\exists \delta > 0)$ and $|h(r)| \leq A[1 + |r|]^{-2-\delta}$
- $g(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iru}\, dr$

Selberg trace formula for $L^2(\Gamma\backslash\mathbb{H}\,;\,m)$ (Γ : co-compact, $m \in 2\mathbb{Z}_{\geq 0}$)

$$\sum_{n=0}^{\infty} h(r_n) = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}\Big\{\int_{-\infty}^{\infty} r h(r)\tanh(\pi r)\,dr + \sum_{k=0}^{m/2-1}(m-1-2k)\,h\Big(\frac{i(m-1-2k)}{2}\Big)\Big\}$$
$$+ \sum_{\gamma\in\Gamma_{\mathsf{hyp}}} \frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} g(\log N(\gamma))$$
$$+ \sum_{R\in\Gamma_{\mathsf{ell}}} \frac{1}{\nu_R \sin\theta_R}\Big\{\frac{1}{4}\int_{-\infty}^{\infty}\frac{\cosh((\pi-2\theta)r)}{\cosh\pi r}h(r)\,dr + \sum_{k=0}^{m/2-1}\frac{ie^{i(m-1-2k)\theta}}{2}\,h\Big(\frac{i(m-1-2k)}{2}\Big)\Big\}$$

- Γ_{hyp} (resp. Γ_{ell}) : hyperbolic (resp. elliptic) Γ-conjugacy classes
- m_R : the order of the elliptic element R, $0 < \theta_R < \pi$

Maass operators

Definition (Maass operators ($m \in 2\mathbb{Z}$))

$$K_m := iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{m}{2} : L^2(\Gamma\backslash\mathbb{H}\,;\,m) \to L^2(\Gamma\backslash\mathbb{H}\,;\,m+2)$$

$$\Lambda_m := iy\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{m}{2} : L^2(\Gamma\backslash\mathbb{H}\,;\,m) \to L^2(\Gamma\backslash\mathbb{H}\,;\,m-2)$$

$\Rightarrow$ $\Delta_{m+2}K_m = K_m\Delta_m$ and $\Delta_{m-2}\Lambda_m = \Lambda_m\Delta_m$

- Let $L^2(\Gamma\backslash\mathbb{H}\,;\,\lambda,\,m)$ be a eigen-subspace with the eigenvalue λ.

Proposition

- $\Lambda_m[L^2(\Gamma\backslash\mathbb{H}\,;\,\lambda,\,m)] = L^2(\Gamma\backslash\mathbb{H}\,;\,\lambda,\,m-2)$ whenever $\lambda \neq \frac{m}{2}(1-\frac{m}{2})$
- $K_m[L^2(\Gamma\backslash\mathbb{H}\,;\,\lambda,\,m)] = L^2(\Gamma\backslash\mathbb{H}\,;\,\lambda,\,m+2)$ whenever $\lambda \neq -\frac{m}{2}(1+\frac{m}{2})$
- $\Lambda_m[L^2(\Gamma\backslash\mathbb{H}\,;\,\lambda,\,m)] = 0$ when $\lambda = \frac{m}{2}(1-\frac{m}{2})$
- $K_m[L^2(\Gamma\backslash\mathbb{H}\,;\,\lambda,\,m)] = 0$ when $\lambda = -\frac{m}{2}(1+\frac{m}{2})$

$\Rightarrow$ • $\{\lambda_j(\Delta_m)\} = \{\frac{m}{2}(1-\frac{m}{2})\}_{k=1}^{d} \cup \{\lambda_j(\Delta_{m-2}) \mid \lambda_j(\Delta_{m-2}) \neq \frac{m}{2}(1-\frac{m}{2})\}$

• $\frac{m}{2}(1-\frac{m}{2}) = \frac{1}{4} + \left(\frac{i(m-1)^2}{2}\right)^2$

Let $m \geq 2$ be an even integer.

Differnce of STF for $L^2(\Gamma\backslash\mathbb{H}\,;\,m) - L^2(\Gamma\backslash\mathbb{H}\,;\,m-2)$

$$\Big(d(m) - \delta_{2,m}\Big)\, h\Big(\frac{i(m-1)}{2}\Big) = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}(m-1)\, h\Big(\frac{i(m-1)}{2}\Big) + \sum_{R\in\Gamma_{\mathrm{ell}}} \frac{ie^{i(m-1)\theta}}{2\nu_R \sin\theta_R}\, h\Big(\frac{i(m-1)}{2}\Big)$$

•$d(m) - \delta_{2,m}$: the multiplicity of the eigenvalue $\lambda = \frac{m}{2}(1-\frac{m}{2})$ of Δ_m on $L^2(\Gamma\backslash\mathbb{H}\,;\,m)$

Dimension formula for the holomorphic modular forms of weight m

$$d(m) = \delta_{2,m} + \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}(m-1) + \sum_{R\in\Gamma_{\mathrm{ell}}} \frac{ie^{i(m-1)\theta}}{2\nu_R \sin\theta_R}$$

Differences of the Selberg trace formula for the Hilbert modular group

Preliminaries

- $G := \mathrm{PSL}(2,\mathbb{R})^2 = \Big(\mathrm{SL}(2,\mathbb{R})/\{\pm I\}\Big)^2$

- G acts on $\mathbb{H}^2$ by $(g_1, g_2).(z_1, z_2) := \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}\right) \in \mathbb{H}^2$
- $\Gamma \subset G$: irreducible discrete subgroup i.e, not comensurable with any direct product $\Gamma_1 \times \Gamma_2$ of two discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$

Classification of the elements of irreducible Γ

1. $\gamma = (I, I)$ is the identity
2. $\gamma = (\gamma_1, \gamma_2)$ is hyperbolic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| > 2$ and $|\mathrm{tr}(\gamma_2)| > 2$
3. $\gamma = (\gamma_1, \gamma_2)$ is elliptic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| < 2$ and $|\mathrm{tr}(\gamma_2)| < 2$
4. $\gamma = (\gamma_1, \gamma_2)$ is hyperbolic-elliptic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| > 2$ and $|\mathrm{tr}(\gamma_2)| < 2$
5. $\gamma = (\gamma_1, \gamma_2)$ is elliptic-hyperbolic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| < 2$ and $|\mathrm{tr}(\gamma_2)| > 2$
6. $\gamma = (\gamma_1, \gamma_2)$ is parabolic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| = |\mathrm{tr}(\gamma_2)| = 2$

- There are no other types in Γ. (parabolic-elliptic etc.) (Cf. Shimizu 63)

Hilbert modular group of a real quadratic field

- K : real quadratic field of the class number 1
  - D : the discriminiant of K
  - $\mathcal{O}_K \subset K$: the ring of integers
  - ε : the fundamental unit
  - $a' = \sigma(a)$, σ is the nontrivial element of $\mathrm{Gal(K/\mathbb{Q})}$
  - $N(a) := aa'$

Hilbert modular group

$$\Gamma_K := \left\{ (\gamma, \gamma') = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathcal{O}_K) \right\}.$$

$\Rightarrow$ • Γ_K is an irreducible discrete subgroup of $G = \mathrm{PSL}(2,\mathbb{R})^2$ with the only one cusp $\infty := (\infty, \infty)$.

Selberg trace formula for Hilbert modular surfaces

- Fix $(m_1, m_2) \in (2\mathbb{Z}_{\geq 0})^2$: weight
- $j_\gamma(z_j) := \frac{cz_j+d}{|cz_j+d|}$ for $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ $(j = 1, 2)$
- $\Delta^{(j)}_{m_j} := -y_j^2(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}) + im_j\, y_j \frac{\partial}{\partial x_j} \quad (j = 1, 2)$

$$L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(m_1, m_2)) := \Big\{ f\colon \mathbb{H}^2 \to \mathbb{C},\ C^\infty \ \Big|$$

- $f((\gamma, \gamma')(z_1, z_2)) = j_\gamma(z_1)^{m_1} j_{\gamma'}(z_2)^{m_2} f(z_1, z_2) \quad \forall(\gamma, \gamma') \in \Gamma_K$
- $\Delta^{(1)}_{m_1} f(z_1, z_2) = \lambda^{(1)} f(z_1, z_2), \quad \Delta^{(2)}_{m_2} f(z_1, z_2) = \lambda^{(2)} f(z_1, z_2)$ $\exists(\lambda^{(1)}, \lambda^{(2)}) \in \mathbb{R}^2$
- $||f||^2 = \int_{\Gamma_K\backslash\mathbb{H}^2} f(z)\overline{f(z)}\, d\mu(z) < \infty. \Big\}$

- $d\mu(z) = \frac{dx_1 dy_1}{y_1^2} \frac{dx_2 dy_2}{y_2^2}$ for $z = (z_1, z_2) \in \mathbb{H}^2$

Proposition

We have a direct sum decomposition:

$$L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(m_1,m_2)) = L^2_{\text{dis}}(\Gamma_K\backslash\mathbb{H}^2\,;\,(m_1,m_2)) \oplus L^2_{\text{con}}(\Gamma_K\backslash\mathbb{H}^2\,;\,(m_1,m_2))$$

and there is an ortonormal basis $\{\phi_j\}_{j=0}^{\infty}$ of $L^2_{\text{dis}}(\Gamma_K\backslash\mathbb{H}^2\,;\,(m_1,m_2))$.

- Let $(\lambda_j^{(1)}, \lambda_j^{(2)}) \in \mathbb{R}^2$ such that

$$\Delta^{(1)}_{m_1}\phi_j = \lambda_j^{(1)}\phi_j \quad \text{and} \quad \Delta^{(2)}_{m_2}\phi_j = \lambda_j^{(2)}\phi_j$$

- Let $\mathrm{Spec}(m_1,m_2) := \left\{(r_j^{(1)}, r_j^{(2)})\right\}_{j=0}^{\infty} \subset \mathbb{R}^2$. (discrete subset)

Here, we write $\lambda_j^{(l)} = \frac{1}{4} + (r_j^{(l)})^2$. $(l = 1,2)$
Now we can say about the Selberg trace formula:

- $h(r_1,r_2) = h(\pm r_1, \pm r_2)$: test function (satisfying certain analytic conditions)
- $g(u_1,u_2) := \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(r_1,r_2)e^{-i(r_1u_1+r_2u_2)}\,dr_1dr_2$

- γ is type 1 hyperbolic $\Leftrightarrow$ γ is hyperbolic and whose all fixed points are not fixed by parabolic elements.
- γ is type 2 hyperbolic $\Leftrightarrow$ γ is hyperbolic and not type 1 hyperbolic.

Selberg trace formula for $L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(m_1,m_2))$ $((m_1,m_2)\in(2\mathbb{Z}_{\geq 0})^2)$: Zograf 82, Efrat 87 for $(m_1,m_2)=(0,0)$

$$\sum_{j=0}^{\infty} h(r_j^{(1)},r_j^{(2)}) \quad - \text{(Contribution from "Eisenstein series")}$$

$$= \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2}\iint_{\mathbb{R}^2}\frac{\frac{\partial^2}{\partial u_1\partial u_2}g(u_1,u_2)}{\sinh(u_1/2)\sinh(u_2/2)}e^{-\frac{m_1}{2}u_1}e^{-\frac{m_2}{2}u_2}\,du_1du_2$$

$$+\sum_{\gamma\in\Gamma_{\text{hyp}1}}\frac{\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\,g(\log N(\gamma),\log N(\gamma'))}{(N(\gamma)^{1/2}-N(\gamma)^{-1/2})(N(\gamma')^{1/2}-N(\gamma')^{-1/2})}$$

$$+\sum_{R\in\Gamma_{\text{ell}}}E(m_1,m_2;R)+\sum_{\gamma\in\Gamma_{\text{hyp-ell}}}HE(m_1,m_2;\gamma)+\sum_{\gamma\in\Gamma_{\text{ell-hyp}}}EH(m_1,m_2;\gamma)$$

$$+P(m_1,m_2)+\sum_{\gamma\in\Gamma_{\text{hyp}2}}H_2(m_1,m_2;\gamma)$$

- Hereafter, we assume that $h(r_1,r_2)=h_1(r_1)\,h_2(r_2)$.
- $\Lambda_m^{(2)} := iy_2\frac{\partial}{\partial x_2}-y_2\frac{\partial}{\partial y_2}+\frac{m}{2} : L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m))\to L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-2))$

- Let $\{\frac{1}{4}+\rho_j^2\}_{j=0}^{\infty} := \mathrm{Spec}(\Delta_0^{(1)}|_{\mathrm{Ker}(\Lambda_m^{(2)})})$ and recall that $\mathrm{Ker}(\Lambda_m^{(2)}) = L^2\big(\Gamma_K\backslash\mathbb{H}^2\,;\,\big(*,\frac{m}{2}(1-\frac{m}{2})\big),(0,m)\big)$ i.e. $\lambda_2 = \frac{m}{2}(1-\frac{m}{2})$-eigenspace

Differences of STF for $L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m)) - L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-2))$

$$
\begin{aligned}
&\sum_{j=0}^{\infty} h_1(\mu_j)\, h_2(\tfrac{i(m-1)}{2}) - \delta_{m,2}\, h_1(\tfrac{i}{2}) h_2(\tfrac{i}{2}) \\
&= (m-1) h_2(\tfrac{i(m-1)}{2}) \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2} \int_{-\infty}^{\infty} r_1 h_1(r_1) \tanh(\pi r_1)\, dr_1 \\
&\quad + \sum_{R(\theta_1,\theta_2)\in\Gamma_{\text{ell}}} \frac{i e^{(m-1)\theta_2}}{8\nu_R \sin\theta_1 \sin\theta_2} h_2(\tfrac{i(m-1)}{2}) \int_{-\infty}^{\infty} \frac{\cosh((\pi-2\theta_1)r_1)}{\cosh \pi r_1} h_1(r_1)\, dr_1 \\
&\quad + \sum_{(\gamma,\omega)\in\Gamma_{\text{hyp-ell}}} \frac{\log N(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} g_1(\log N(\gamma)) \frac{i e^{i(m-1)\omega}}{2\sin\omega} h_2(\tfrac{i(m-1)}{2}) \\
&\quad - \log\varepsilon\, g_1(0)\, h_2(\tfrac{i(m-1)}{2}) - 2\log\varepsilon\, h_2(\tfrac{i(m-1)}{2}) \sum_{k=1}^{\infty} g_1(2k\log\varepsilon)\varepsilon^{-k(m-1)}
\end{aligned}
$$

- We write the above formula as $L(m) - L(m-2)$ for $m \geq 2$.

- Next we consider (for $m \geq 4$) :

$\bigl(L(m) - L(m-2)\bigr) h_2(\frac{i(m-1)}{2})^{-1} - \bigl(L(m-2) - L(m-4)\bigr) h_2(\frac{i(m-3)}{2})^{-1}$

Theorem (Double differences of STF for $L^2(\Gamma_K \backslash \mathbb{H}^2\,;\,(0,m))$)

Let $m \in 2\mathbb{N}$ and $m \geq 4$. We have

$$
\begin{aligned}
\sum_{j=0}^{\infty} h_1(\rho_j) - \sum_{k=0}^{\infty} h_1(\nu_k) + \delta_{m,4} h_1(\tfrac{i}{2}) \;&=\; \frac{\mathrm{vol}(\Gamma_K \backslash \mathbb{H}^2)}{8\pi^2} \int_{-\infty}^{\infty} r h_1(r) \tanh(\pi r)\, dr \\
&- \sum_{R(\theta_1,\theta_2)\in\Gamma_{\text{ell}}} \frac{e^{i(m-2)\theta_2}}{4\nu_R \sin\theta_1} \int_{-\infty}^{\infty} \frac{\cosh((\pi - 2\theta_1)r)}{\cosh \pi r} h_1(r)\, dr \\
&- \sum_{(\gamma,\omega)\in\Gamma_{\text{hyp-ell}}} \frac{\log N(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} g_1(\log N(\gamma))\, e^{i(m-2)\omega} \\
&- 2\log\varepsilon \sum_{k=1}^{\infty} g_1(2k\log\varepsilon)\, (\varepsilon^{-k(m-1)} - \varepsilon^{-k(m-3)}).
\end{aligned}
$$

Test function $h(r_1, r_2) = h_1(r_1)h_2(r_2)$

Here, $\{\mu_j\}, \{\nu_k\}$ are given by

- $\{\frac{1}{4} + \mu_j^2\}_{j=0}^{\infty} = \mathrm{Spec}(\Delta_0^{(1)}|_{\mathrm{Ker}(\Lambda_m^{(2)})})$ with

$\Lambda_m^{(2)}: L^2\big(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m)\big) \to L^2\big(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-2)\big)$. Note that
$\mathrm{Ker}(\Lambda_m^{(2)}) = L^2\big(\Gamma_K\backslash\mathbb{H}^2\,;\,\big(*, \frac{m}{2}(1-\frac{m}{2})\big),(0,m)\big)$ i.e. $\lambda_2 = \frac{m}{2}(1-\frac{m}{2})$-eigenspace

- $\{\frac{1}{4} + \nu_k^2\}_{k=0}^{\infty} = \mathrm{Spec}(\Delta_0^{(1)}|_{\mathrm{Ker}(\Lambda_{m-2}^{(2)})})$ with

$\Lambda_{m-2}^{(2)}: L^2\big(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-2)\big) \to L^2\big(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-4)\big)$. Note that
$\mathrm{Ker}(\Lambda_{m-2}^{(2)}) = L^2\big(\Gamma_K\backslash\mathbb{H}^2\,;\,\big(*, \frac{m-2}{2}(2-\frac{m}{2})\big),(0,m)\big)$ i.e.
$\lambda_2 = \frac{m-2}{2}(2-\frac{m}{2})$-eigenspace

Let us consider the following test function $h(r_1, r_2) = h_1(r_1)h_2(r_2)$:

- $h_1(r) = \dfrac{1}{r^2 + (s-\frac{1}{2})^2} - \dfrac{1}{r^2+\beta^2} \quad \Rightarrow \quad g_1(u) = \dfrac{1}{2s-1}e^{-(s-\frac{1}{2})|u|} - \dfrac{1}{2\beta}e^{-\beta\,|u|}$

(or $\left(\frac{1}{2s-1}\frac{d}{ds}\right)^n h_1(r)$ for $n \gg 0$)

- $h_2(r)$ such that $h_2(\frac{i(m-1)}{2}) \neq 0$ and $h_2(\frac{i(m-3)}{2}) \neq 0$

We consider DD-STF for the above $h(r_1, r_2)$

Analytic continuation of $Z_K(s;m)$

DD-STF for the above test function h_1 and h_2

$$\kappa\sum_{j=0}^{\infty}\Big[\frac{1}{\mu_j^2+(s-\frac{1}{2})^2}-\frac{1}{\mu_j^2+\beta^2}\Big]-\kappa\sum_{k=0}^{\infty}\Big[\frac{1}{\nu_k^2+(s-\frac{1}{2})^2}-\frac{1}{\nu_k^2+\beta^2}\Big]$$

$$=\kappa\,\zeta_K(-1)\sum_{k=0}^{\infty}\Big[\frac{1}{s+k}-\frac{1}{\beta+\frac{1}{2}+k}\Big]$$

$$+\frac{1}{2s-1}\frac{Z_K'(s)}{Z_K(s)}-\frac{1}{2\beta}\frac{Z_K'(\frac{1}{2}+\beta)}{Z_K(\frac{1}{2}+\beta)}+\frac{\kappa}{2s-1}\frac{Z_{\mathrm{ell}}'(s)}{Z_{\mathrm{ell}}(s)}-\frac{\kappa}{2\beta}\frac{Z_{\mathrm{ell}}'(\frac{1}{2}+\beta)}{Z_{\mathrm{ell}}(\frac{1}{2}+\beta)}$$

$$+\frac{\kappa}{2s-1}\frac{d}{ds}\log\left\{\frac{(1-\varepsilon^{-(2s+m-4)})}{(1-\varepsilon^{-(2s+m-2)})}\right\}-\frac{\kappa}{2\beta}\left\{(s-\tfrac{1}{2})\to\beta \text{ in the left}\right\}.$$

- $\kappa\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{8\pi^2}=\kappa\,\zeta_K(-1)\in\mathbb{N}$.

$\Rightarrow$ Analytic continuation and functional equation of $\frac{d}{ds}\log Z_K(s;m)$.
$\Rightarrow$ Analytic continuation and functional equation of $Z_K(s;m)$.

Remark

• We remark that the scattering and type 2 hyperbolic components of $Z_K(s;m)$ are local Selberg zeta functions for $\mathrm{PSL}(2,\mathbb{Z})$:

$$Z_{\mathrm{sct/hyp2}}(s) = \zeta_\varepsilon(s+\tfrac{m}{2}-1)\zeta_\varepsilon(s+\tfrac{m}{2}-2)^{-1}$$

with $\quad \zeta_\varepsilon(s) = (1-\varepsilon^{-2s})^{-1}$

• ε : the fundamental unit of K

Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. The Selberg (Ruelle) zeta function for Γ is given by

$$\zeta_\Gamma(s) := \prod_{p\in\mathrm{Prim}(\Gamma)} \left(1-N(p)^{-s}\right)^{-1} \quad \Rightarrow \quad \zeta_\Gamma(s) = \prod_K (1-\varepsilon(K)^{-2s})^{-h(K)},$$

where, K run through "all" real quadratic fields over $\mathbb{Q}$ and $\varepsilon(K)$ and $h(K)$ are the fundamental unit and the class number of K.