Cohomology and L-values

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§1. One dimensional case

G: group, M: left G-module

$C^0(G,M) = M$,

$C^n(G,M) = \{f : G^n \longrightarrow M\}$, $n \geq 1$.

$d : C^n(G,M) \longrightarrow C^{n+1}(G,M)$

$$(df)(g_1,\ldots,g_{n+1}) = g_1 f(g_2,\ldots,g_{n+1}) + (-1)^{n+1} f(g_1,\ldots,g_n) + \sum_{i=1}^{n} (-1)^i f(g_1,\ldots,g_i g_{i+1},\ldots,g_{n+1})$$

The cohomology group $H^n(G,M)$ is that of the complex

$\{C^*(G,M), d\}$: $H^n(G,M) = Z^n(G,M)/B^n(G,M)$.

For $0 \le l \in \mathbf{Z}$ and $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathbf{C}^2$, put

$$\begin{bmatrix} u \\ v \end{bmatrix}^l = {}^t(u^l \ u^{l-1}v \ldots uv^{l-1} \ v^l).$$

Define a representation $\rho_l : GL(2, \mathbf{C}) \longrightarrow GL(l+1, \mathbf{C})$ by

$$\rho_l(g) \begin{bmatrix} u \\ v \end{bmatrix}^l = (g \begin{bmatrix} u \\ v \end{bmatrix})^l.$$

$\Gamma \subset \mathrm{SL}(2, \mathbf{R})$: Fuchsian group

$\Omega \in S_k(\Gamma)$, $k \ge 2$. Put $l = k - 2$, $\rho = \rho_l$.

$$\mathfrak{d}(\Omega) = \Omega(z) \begin{bmatrix} z \\ 1 \end{bmatrix}^l dz, \qquad z \in \mathfrak{H}.$$

(V-valued differential form, $V = \mathbf{C}^{l+1}$)

$$\mathfrak{d}(\Omega) \circ \gamma = \rho(\gamma)\mathfrak{d}(\Omega), \quad \gamma \in \Gamma.$$

$z_0 \in \mathfrak{H}$: a base point. Put

$$f(\gamma) = \int_{z_0}^{\gamma z_0} \mathfrak{d}(\Omega).$$

Then

$$f(\gamma_1\gamma_2) = f(\gamma_1) + \rho(\gamma_1)f(\gamma_2) \quad \text{(1-cocycle).}$$

The class of $f \in H^1(\Gamma, V)$ doen't depend on z_0.

$\gamma \in \Gamma$: parabolic, z_0': cusp

$z_0 \mapsto z_0'$:

$$f(\gamma) = (\rho(\gamma) - 1) \int_{z_0'}^{z_0} \mathfrak{d}(f).$$

Let $\Omega \in S_k(\mathsf{SL}(2, \mathbf{Z}))$.

Put $z_0 = i\infty$, $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$$f(\sigma\tau) = -\Big(\int_0^{i\infty} \Omega(z) z^t dz\Big)_{0 \le t \le l} = -\Big(i^{t+1} R(t+1, \Omega)\Big)_{0 \le t \le l}$$

where $R(s, \Omega) = (2\pi)^{-s}\Gamma(s)L(s, \Omega)$.

$(\sigma\tau)^3 = 1$ implies

$(1 + \rho(\sigma\tau) + \rho((\sigma\tau)^2))f(\sigma\tau) = 0$.

Take $k = 12$, $\Omega = \Delta$.

$$R(8) = \frac{5}{4}R(6), \quad R(10) = \frac{12}{5}R(6), \quad \text{etc.}$$

Shimura: “Sur les intégrales attachées aux formes automorphes ”
(J. Math. Soc. Japan, 1959)

Problem: In higher dimensional cases, can we calculate L-values using cohomology in a similar manner?

Another method:

Shimura, The special values of the zeta functions associated with cusp forms (Comm. pure and applied Math., 1976)

Shimura, The special values of the zeta functions associated with Hilbert modular forms (Duke Math. J., 1978)

I will compare two methods at the end of my talk.

§2. Hilbert modular case

F: totally real, $[F : \mathbf{Q}] = n \geq 2$

$\mathcal{O}_F$: ring of intergers, $E_F = \mathcal{O}_F^\times$

σ_1, ..., σ_n: all isomorphisms of F into $\mathbf{R}$, $\xi^{(i)} = \xi^{\sigma_i}$

$\Gamma \subset \mathrm{SL}(2, \mathcal{O}_F)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\gamma^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$.

A holomorphic function Ω on $\mathfrak{H}^n$ is called a Hilbert modular form of weight $k = (k_1, \ldots, k_n) \in \mathbf{Z}^n$ with respect to Γ if Ω satisfies

$$\Omega(\gamma z) = f(\gamma^{(1)} z_1, \ldots, \gamma^{(n)} z_n) = \Omega(z) \prod_{i=1}^{n} (c^{(i)} z_i + d^{(i)})^{k_i}$$

for all $\gamma \in \Gamma$, where $z = (z_1, \ldots, z_n) \in \mathfrak{H}^n$.

Assume $k_i \geq 2$, $1 \leq i \leq n$. Put $l_i = k_i - 2$.

$\rho = \rho_{l_1} \otimes \cdots \otimes \rho_{l_n}$, $V = \mathbf{C}^{l_1+1} \otimes \cdots \otimes \mathbf{C}^{l_n+1}$ (rep. space of ρ)

Define a V-valued differential n-form on $\mathfrak{H}^n$ by

$$\mathfrak{d}(\Omega) = \Omega(z) \begin{bmatrix} z_1 \\ 1 \end{bmatrix}^{l_1} \otimes \cdots \otimes \begin{bmatrix} z_n \\ 1 \end{bmatrix}^{l_n} dz_1 \cdots dz_n, \qquad z \in \mathfrak{H}^n.$$

$\mathfrak{d}(\Omega) \circ \gamma = \rho(\gamma)\mathfrak{d}(\Omega), \qquad \gamma \in \Gamma.$

We have an explicit procedure

$$\Omega \longrightarrow f(\Omega) \in Z^n(\Gamma, V) \quad (n\text{-cocycle}).$$

(Absolute CM-periods, 2003, AMS, Chapter V, §5)

$w = (w_1, \ldots, w_n) \in \mathfrak{H}^n$: a base point. Put

$$I(z) = \int_{w_1}^{z_1} \cdots \int_{w_n}^{z_n} \mathfrak{d}(\Omega).$$

Then

$$\rho(\gamma) I(\gamma^{-1} z) = \int_{\gamma^{(1)} w_1}^{z_1} \cdots \int_{\gamma^{(n)} w_n}^{z_n} \mathfrak{d}(\Omega).$$

$\mathcal{H}$: vector space of all V-valued holomorphic functions on $\mathfrak{H}^n$

$$(\gamma\varphi)(z) = \rho(\gamma)\varphi(\gamma^{-1} z), \quad \varphi \in \mathcal{H}.$$

$\mathcal{H}$: left Γ-module

$$\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n} (\gamma I - I) = 0.$$

Assume $n = 2$:

$$\gamma I - I = g(\gamma; z_1) + h(\gamma; z_2), \quad g, h \in C^1(\Gamma, \mathcal{H}).$$

$$dg(\gamma_1, \gamma_2; z_1) + dh(\gamma_1, \gamma_2; z_2) = 0.$$

$$f(\Omega) = dg(\gamma_1, \gamma_2; z_1) \in Z^2(\Gamma, V) \quad \text{(2-cocycle)}.$$

The cohomology class of $f(\Omega)$ doesn't depend on w and the choice of $g(\gamma; z_1)$.

Explicitly ($n = 2$, $f = f(\Omega)$)

$$f(\gamma_1, \gamma_2) = \int_{\gamma_1^{(1)}\gamma_2^{(1)} w_1}^{\gamma_1^{(1)} w_1} \int_{w_2}^{\gamma_1^{(2)} w_2} \mathfrak{d}(\Omega).$$

Assume $\Omega \in S_k(\mathrm{SL}(2, \mathcal{O}_F))$, $l_1 \equiv l_2 \mod 2$.

$$\Omega(z) = \sum_{0 \ll \xi \in \mathfrak{d}_F^{-1}} a(\xi) \mathbf{e}_F(\xi z), \quad \text{(Fourier expansion)}$$

where $\mathbf{e}_F(\xi z) = \exp(2\pi i \sum_{i=1}^n \xi^{(i)} z_i)$.

$k_0 = \max(k_1, \ldots, k_n)$, $k'_i = k_0 - k_i$.

$$L(s, \Omega) = \sum_{\xi E_F^2} a(\xi) \prod_{i=1}^{n} (\xi^{(i)})^{k'_i/2} N(\xi)^{-s}.$$

$$R(s, \Omega) = (2\pi)^{2ns} \prod_{i=1}^{n} \Gamma(s - \frac{k'_i}{2}) L(s, \Omega).$$

$$\int_{\mathbf{R}_+^n/E_F^2} \Omega(iy_1, \ldots, iy_n) \prod_{i=1}^{n} y_i^{s-k'_i/2-1} dy_i = (2\pi)^{\sum_{i=1}^n k'_i/2} R(s, \Omega).$$

$$R(s, \Omega) = i^{\sum_{i=1}^n k_i} R(k_0 - s, \Omega).$$

Assume $n = 2$, $k_1 \geq k_2$ and the narrow class number of F is one. If Ω is a Hecke eigen form then

$$L(s, \Omega) = \text{const} \cdot D_F^s \prod_{\mathfrak{p}} (1 - \lambda(\mathfrak{p}) N(\mathfrak{p})^{-s} + N(\mathfrak{p})^{k_1 - 1 - 2s})^{-1}.$$

The Ramanujan conjecture for this Euler profuct is proved by Blasius (Aspects Math. 37 (2006)).

Assume that Ω is a primitive Hecke eigenform. The nature of the critical values $L(m,\Omega)$ is well understood. Put $k^0 = \min(k_1,\ldots,k_n)$, $J_F = \mathrm{Hom}(F,\mathbf{R})$. Let φ be a Hecke character of finite order of $F_A^\times$. Let m be an integer such that

$$\frac{k_0 - k^0}{2} < m < \frac{k_0 + k^0}{2}.$$

Assume that $k_i \mod 2$ is independent of i. Then for every $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$, there exists a constant $u(\epsilon,\Omega)$ such that

$$L(m,\Omega\otimes\varphi) \sim \pi^{mn} u(\epsilon,\Omega)$$

if φ satisfies

$$\varphi(x) = \prod_{\tau\in J_F} \mathrm{sgn}(x_\tau)^{\epsilon(\tau)+m}, \quad x \in F_\infty^\times.$$

(Shimura, Duke Math. J. 1978). There exist $2n$ constants $c_\tau^\pm(\Omega)$ such that (if $k_i > 2$, $\forall i$)

$$u(\epsilon,\Omega) \sim \prod_{\tau\in J_F} c_\tau^{\epsilon(\tau)}(\Omega).$$

(Yoshida, Amer. J. Math. 1995. Duke Math. J. 1994.)

§3. Elimination of the effect of coboundary

Assume $n = 2$, $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$, $k_1 \equiv k_2 \mod 2$, $k_1 \geq k_2$.

ϵ: fundamental unit

$\Omega \in S_k(\Gamma)$, $k = (k_1, k_2)$, $l_1 = k_1 - 2$, $l_2 = k_2 - 2$.

Take $w = (i\epsilon^{-1}, i\infty)$ as a base point. Then

$$f(\gamma_1, \gamma_2) = \int_{\gamma_1^{(1)}\gamma_2^{(1)} i\epsilon^{-1}}^{\gamma_1^{(1)} i\epsilon^{-1}} \int_{i\infty}^{\gamma_1^{(2)} i\infty} \mathfrak{d}(\Omega).$$

Let

$$P = \left\{ \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \middle| \; t \in E_F, u \in \mathcal{O}_F \right\} / \{\pm 1_2\}.$$

$p \cdot i\infty = i\infty$, $p \in P$ implies the parabolic condition

(P) $$f(p\gamma_1, \gamma_2) = pf(\gamma_1, \gamma_2), \quad p \in P.$$

Let

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}.$$

Then

$$f(\sigma, \mu) = -\int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \mathfrak{d}(\Omega) = (-P_{s,t})_{0 \le s \le l_1, 0 \le t \le l_2},$$

$$P_{s,t} = \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \Omega(z) z_1^s z_2^t dz_1 dz_2.$$

$$P_{m,m-(k_1-k_2)/2} = (-1)^{m+1} i^{-(k_1-k_2)/2} (2\pi)^{(k_1-k_2)/2} R(m+1, \Omega).$$

All critical values $L(m, \Omega)$

$$\frac{l_1 - l_2}{2} + 1 \leq m \leq \frac{l_1 + l_2}{2} + 1$$

appear as the components of $f(\sigma, \mu)$.

$f(\sigma, \mu)$ has $(l_1 + 1)(l_2 + 1)$-components.

Only $l_2 + 1$ components among them are related to critical values.

$\{\mathbf{e}_1, \ldots, \mathbf{e}_{l_1+1}\}$: the standard basis of $\mathbf{C}^{l_1+1}$

$\{\mathbf{e}'_1, \ldots, \mathbf{e}'_{l_2+1}\}$: the standard basis of $\mathbf{C}^{l_2+1}$

The coefficient of $\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}$ in $f(\sigma, \mu)$ is

$-P_{m-1,m-1-(l_1-l_2)/2}$.

$$\rho\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)(\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m})$$
$$= N(a)^{2m-2-l_1}(\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}).$$

We define the parabolic cohomology group as follows:

$$Z^2_P(\Gamma, V) = \{f \in Z^2(\Gamma, V) \mid f \text{ is normalized and satisfies (P)}\},$$

$$\bar{B}^2(\Gamma, V) = \{f = db \mid b \in C^1(\Gamma, V), b(1) = 0\},$$

$$B^2_P(\Gamma, V) = \bar{B}^2(\Gamma, V) \cap Z^2_P(\Gamma, V),$$

$$H^2_P(\Gamma, V) = Z^2_P(\Gamma, V)/B^2_P(\Gamma, V).$$

Theorem 1. *Let $i = 1$ or 2. Then*

$$\dim H^i(P, V) = \begin{cases} 0 & \text{if} \quad l_1 \neq l_2 \text{ or } N(\epsilon)^{l_1} = -1, \\ 1 & \text{if} \quad l_1 = l_2 \text{ and } N(\epsilon)^{l_1} = 1. \end{cases}$$

Now suppose

$$f \mapsto f', \quad f, f' \in Z^2_P(\Gamma, V),$$

$$f'(\gamma_1, \gamma_2) = f(\gamma_1, \gamma_2) + b(\gamma_1\gamma_2) - \gamma_1 b(\gamma_2) - b(\gamma_1).$$

Then

$$b(p\gamma) = pb(\gamma) + b(p), \quad p \in P, \gamma \in \Gamma.$$

$$b|P \in Z^1(P, V).$$

$$f'(\sigma, \mu) = f(\sigma, \mu) + b(\sigma\mu) - \sigma b(\mu) - b(\sigma).$$

Suppose $l_1 \neq l_2$. Then

$$b(\mu) = (\mu - 1)\mathbf{b}, \quad \exists \mathbf{b} \in V.$$

$$b(\sigma\mu) - \sigma b(\mu) - b(\sigma) = (\mu^{-1} - 1)[b(\sigma) + (1 - \sigma)\mathbf{b}].$$

$\mu^{-1} - 1$ kills the components related to critical values.

Theorem 2. *If* $l_1 \neq l_2$*, then* $H^2(\Gamma, V) = H^2_P(\Gamma, V)$*.*

Conclusion: Suppose $l_1 \neq l_2$. Then we can deduce information on critical values of $L(s, \Omega)$ once we know a parabolic 2-cocycle attached to Ω. Suppose $l_1 = l_2$. The same conclusion holds except for that we may lose information on $L(1, \Omega)$ and $L(l_1 + 1, \Omega)$, the critical values at edges.

§4. Generators and relations

Write $\Gamma = \mathcal{F}/R$, $\mathcal{F}$ is a free group. We have the exact sequence

$$0 \longrightarrow H^1(\Gamma, V) \longrightarrow H^1(\mathcal{F}, V) \longrightarrow H^1(R, V)^\Gamma \longrightarrow H^2(\Gamma, V) \longrightarrow 0.$$

Hence

$$H^2(\Gamma, V) \cong H^1(R, V)^\Gamma/\mathrm{Im}(H^1(\mathcal{F}, V)).$$

Let

$$\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\omega$$

and put

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

It is known (Vaserštein) that Γ is generated by σ, μ, τ and η. Furthermore we have the relations

(i) $$\sigma^2 = 1.$$

(ii) $$(\sigma\tau)^3 = 1.$$

(iii) $$(\sigma\mu)^2 = 1.$$

(iv) $$\tau\eta = \eta\tau.$$

(v) $$\mu\tau\mu^{-1} = \tau^A\eta^B.$$

(vi) $$\mu\eta\mu^{-1} = \tau^C\eta^D.$$

Here we put

$$\epsilon^2 = A + B\omega, \qquad \epsilon^2\omega = C + D\omega.$$

The relation (ii) follows from

(vii) $$\sigma \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} -t & 1 \\ 0 & -t^{-1} \end{pmatrix}, \quad t \in E_F.$$

We call the relation group R *minimal* if it is generated by the corresponding elements to (i) $\sim$ (vii) and their conjugates.

Theorem 3. *Let* $F = \mathbf{Q}(\sqrt{5})$, $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$. *Then* R *is minimal.*

We can prove Theorem 3 by using a theorem of Macbeath.

(Macbeath, Ann. of Math. 1964)

Swan, Advances Math. 1971:

He generalized the theorem of Macbeath and gave explicit generators and relations for $\mathrm{SL}(2, \mathcal{O}_K)$ when K are imaginary quadratic fields with small discriminants.

§5. The action of Hecke operators

We assume that l_1 and l_2 are even.

Change ρ to $\rho'_{l_1} \otimes \rho'_{l_2}$ where $\rho'_l(g) = \rho_l(g)\,\mathrm{det}(g)^{-l/2}$.

Let $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$ as before.

V is a $\mathrm{PGL}(2, \mathcal{O}_F)$-module. Let

$$\nu = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \qquad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

ν and δ act on $H^2(\Gamma, V)$.

Let Γ^* be the subgroup of $\mathrm{PGL}(2, \mathcal{O}_F)$ generated by Γ and ν.

Then $[\Gamma^* : \Gamma] = 2$.

Let $f \in Z^2_P(\Gamma, V)$, $f^* \in Z^2(\Gamma^*, V)$: the tranfer of f.

$f^+ \in Z^2(\Gamma, V)$: the restriction of f^*.

$$f^*(\sigma, \mu) = f^+(\sigma, \mu) = (1+\nu)f(\sigma, \mu).$$

Assume $\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\epsilon$. Then

$\Gamma^* = \langle \sigma, \nu, \tau \rangle$.

$\mathcal{F}^*$: the free group on three letters $\widetilde{\sigma}$, $\widetilde{\nu}$, $\widetilde{\tau}$.

$\pi^* : \mathcal{F}^* \longrightarrow \Gamma^*$: the homomorphism such that $\pi^*(\widetilde{\gamma}) = \gamma$, $\gamma = \sigma$, ν, τ.

$\Gamma^* = \mathcal{F}^*/R^*$, $R^* = \mathrm{Ker}(\pi^*)$.

We have the relations

(i) $\sigma^2 = 1$, (ii) $(\sigma\tau)^3 = 1$, (iii) $(\sigma\nu)^2 = 1$, (iv) $\tau\nu\tau\nu^{-1} = \nu\tau\nu^{-1}\tau$,

(v) $\nu^2\tau\nu^{-2} = \tau^A(\nu\tau\nu^{-1})^B$. (Here $\epsilon^2 = A + B\epsilon$.)

We have

(∗) $$H^2(\Gamma^*, V) \cong H^1(R^*, V)^{\Gamma^*}/\mathrm{Im}(H^1(\mathcal{F}^*, V)),$$

$$H^1(R^*, V)^{\Gamma^*} = \{\varphi \in \mathrm{Hom}(R^*, V) \mid \varphi(grg^{-1}) = g\varphi(r), g \in \mathcal{F}^*, r \in R^*\}.$$

Let $\varphi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to $f^* \in Z^2(\Gamma^*, V)$.

$\pi^*(g) = \bar{g}$. For $f^* \in Z^2(\Gamma^*, V)$ (normalized), $\exists a \in C^1(\mathcal{F}^*, V)$,

$$f^*(\bar{g}_1, \bar{g}_2) = a(g_1) + g_1 a(g_2) - a(g_1 g_2), \quad g_1, g_2 \in \mathcal{F}^*.$$

$$Z^2(\Gamma^*, V) \ni f \longrightarrow \varphi = a|R^* \in H^1(R^*, V)^{\Gamma^*}$$

induces the isomorphism (∗).

For every $\gamma \in \Gamma^*$, we choose $\tilde{\gamma} \in \mathcal{F}^*$ such that $\pi^*(\tilde{\gamma}) = \gamma$ and fix it. We may assume

$$f^*(\gamma_1, \gamma_2) = -\varphi(\tilde{\gamma}_1\tilde{\gamma}_2(\widetilde{\gamma_1\gamma_2})^{-1}).$$

(iv) and (v) are defining relations for $P^* = \langle P, \nu \rangle$.

By the parabolic condition we may assume that φ takes value 0 on the corresponding elements to (iv) and (v). We may also assume $\varphi(\tilde{\sigma}^2) = 0$.

Two quantities

$$A = \varphi((\tilde{\sigma}\tilde{\nu})^2), \qquad B = \varphi((\tilde{\sigma}\tilde{\tau})^3)$$

remain to be determined. We have

$$f^*(\sigma, \mu) = -(1 + \nu^{-1})A. \qquad \text{(related to } L\text{-values)}$$

We need to find constraints on A and B. If f^* is in the plus space under the action of δ, then

$$(\sigma\nu - 1)A = 0, \qquad (\delta - 1)A = 0.$$

Let $0 \ll \varpi \in \mathcal{O}_F$, $\mathfrak{p} = (\varpi)$,

$$\Gamma^* \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} \Gamma^* = \sqcup_{i=1}^{d} \Gamma^* \beta_i.$$

Then the action of the Hecke operator $T(\mathfrak{p})$ on φ is given by the following formula.

Let $\psi = T(\mathfrak{p})\varphi$. Suppose $\gamma_1\gamma_2\cdots\gamma_m = 1$ and $\gamma_j = \sigma$ or $\gamma_j \in P^*$. Then $\psi(\tilde{\gamma}_1\tilde{\gamma}_2\cdots\tilde{\gamma}_m)$ is equal to

$$c\sum_{i=1}^{d}\beta_i^{-1}\varphi(\beta_i\widetilde{\gamma_1\beta^{-1}_{q_1(i)}}\widetilde{\beta_{q_1(i)}\gamma_2\beta^{-1}_{q_2(i)}}\cdots\widetilde{\beta_{q_{m-1}(i)}\gamma_m\beta^{-1}_{q_m(i)}}(\beta_i\widetilde{\gamma_1\gamma_2\cdots\gamma_m\beta^{-1}_{q_m(i)}})^{-1}).$$

Here

$$\beta_i\gamma_j\beta^{-1}_{p_i(j)} \in \Gamma^*, \quad 1 \le i \le d, \quad p_i \in S_d,$$

$$q_1 = p_1, \quad q_k = p_k q_{k-1}, \quad 2 \le k \le m.$$

$$c = \varpi^{2l_1}(\varpi')^{l_2}.$$

This is compatible with

$$f(\Omega) \mapsto f(\Omega|T(\mathfrak{p})).$$

§6. Numerical examples

Let $F = \mathbf{Q}(\sqrt{5})$. Then (i) $\sim$ (v) are the fundamental relations for $\Gamma^* = \langle \sigma, \nu, \tau \rangle$.

We consider $T(2)$. We have

$$\Gamma^* \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \Gamma^* = \sqcup_{i=1}^{5} \Gamma^* \beta_i,$$

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \qquad \beta_3 = \begin{pmatrix} 1 & \epsilon \\ 0 & 2 \end{pmatrix},$$

$$\beta_4 = \begin{pmatrix} 1 & \epsilon^2 \\ 0 & 2 \end{pmatrix}, \qquad \beta_5 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

($\epsilon = (1 + \sqrt{5})/2$.)

Let $\varphi \in H^1(R^*, V)^{\Gamma^*}$ and put $\psi = T(2)\varphi$. Then

$$\psi((\tilde{\sigma}\tilde{\tau})^3) = c(\beta_3^{-1} Z_3 + \beta_4^{-1} Z_4),$$

where

$$Z_3 = \varphi((\widetilde{\begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix}}\tilde{\tau})^3), \qquad Z_4 = \varphi((\widetilde{\begin{pmatrix} \epsilon^2 & -\epsilon^2 \\ 2 & -\epsilon \end{pmatrix}})^3).$$

ψ vanishes on the elements corresponding to (iv) and (v) and $\psi(\tilde{\sigma}^2)$ and $\psi((\tilde{\sigma}\tilde{\nu})^2)$ can be calculated similarly.

Fact 1. Suppose $0 \le l_2 \le l_1 \le 20$. Then adding $h|R^*$, $h \in H^1(\mathcal{F}^*, V)$ to φ (keeping φ in the plus space under the action of δ), we may assume $B = 0$.

Therefore our task is to find constraints on $A = \varphi((\tilde{\sigma}\tilde{\nu})^2)$. Note that $(\sigma\nu - 1)A = 0$. We put $x = \begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix} \tau$, and

$$Z_A^+ = \{\mathbf{v} \in V \mid (\sigma\nu - 1)\mathbf{v} = 0,\ (\delta - 1)\mathbf{v} = 0,\ xZ_3 = Z_3\}.$$

The meaning of $xZ_3 = Z_3$:
(i) we must have $xZ_3 = Z_3$. (ii) Z_3 can be expressed by A.

A linear mapping

$$\zeta^+ : Z_A^+ \longrightarrow \mathbf{C}^{l_2+1}.$$

Let $\mathbf{v} \in Z_A^+$. We let the coefficient of $\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}$ in $(1+\nu^{-1})\mathbf{v}$ be equal to the $(l_1+l_2)/2+2-m$-th coefficient of $\zeta^+(\mathbf{v})$, for $(l_1 - l_2)/2 + 1 \leq m \leq (l_1 + l_2)/2 + 1$.

Example 1. We take $l_1 = 8$, $l_2 = 4$. Then $\dim S_{10,6}(\Gamma) = 1$. We find $\zeta^+(Z_A^+)$ is one dimensional and consists of scalar multiples of ${}^t(4, 0, 1, 0, 4)$. Hence we obtain

$$R(7, \Omega)/R(5, \Omega) = 4, \qquad \Omega \in S_{10,6}(\Gamma).$$

My computer calculates this example in 6 seconds.

Example 2. In the same way as in Example 1, we obtain the following numerical values.

$$R(9, \Omega)/R(7, \Omega) = 6, \qquad \Omega \in S_{14,6}(\Gamma).$$

$$R(6, \Omega)/R(4, \Omega) = \frac{25}{6}, \qquad \Omega \in S_{8,8}(\Gamma).$$

$$R(8, \Omega)/R(6, \Omega) = 7, \qquad \Omega \in S_{12,8}(\Gamma).$$

$$R(10, \Omega)/R(8, \Omega) = \frac{720}{11}, \qquad \Omega \in S_{12,10}(\Gamma).$$

The spaces of cusp forms appearing in this example are all one dimensional.

Let B_A^+ be the subspace of Z_A^+ which represents the contribution from $\mathrm{Im}(H^1(\mathcal{F}^*, V))$.

Fact 2. Suppose $0 \leq l_2 \leq l_1 \leq 20$. Then
$\dim S_{l_1+2,l_2+1}(\Gamma) = \dim Z_A^+/B_A^+$.

This fact means that the constraints posed on $A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2)$ is enough. (We expect that Fact 1 and Fact 2 always hold.)

Example 3. We take $l_1 = 12$, $l_2 = 8$. We have $\dim S_{14,10}(\Gamma) = 2$. Calculating the action of $T(2)$ on Z_A^+/B_A^+, we find that the eigenvalues are $-2560 \pm 960\sqrt{106}$. Take an eigenvector in Z_A^+/B_A^+ and map it by ζ^+. Then we find

$$R(11,\Omega)/R(7,\Omega) = 1616 - 76\sqrt{106},$$

$$R(9,\Omega)/R(7,\Omega) = \frac{58}{3} - \frac{5}{6}\sqrt{106}$$

if $0 \neq \Omega \in S_{14,10}(\Gamma)$ satisfies $\Omega|T(2) = (-2560 + 960\sqrt{106})\Omega$.

A calculation for the minus space (under ν) yields

$$R(10,\Omega)/R(8,\Omega) = 50 - \sqrt{106}.$$

Example 4. We take $l_1 = l_2 = 18$. We have $\dim S_{20,20}(\Gamma) = 7$. Calculating the action of $T(2)$ on Z_A^+/B_A^+, we find that the characteristic polynomial of $T(2)$ is (we can use Z_A^-/B_A^- which gives the same result)

$$(X - 97280)^2(X + 840640)(X^4 - 1286780X^3 + 19006483200X^2 + 2718109039083520 0X - 22979876427231395840000).$$

The irreducible factor of degree four corresponds to the base change part from $S_{20}(\Gamma_0(5), (\frac{}{5}))$; $X + 840640$ corresponds to the base change part from $S_{20}(\mathrm{SL}_2(\mathbf{Z}))$; the factor $(X - 97280)^2$ corresponds to the non base change part. Let $\Omega \in \dim S_{20,20}(\Gamma)$ be a Hecke eigenform in the non base change part. A calculation for the plus part yields the result

$$R(18,\Omega)/R(10,\Omega) = 39355680000,$$

$$R(16,\Omega)/R(10,\Omega) = 33163650,$$

$$R(14,\Omega)/R(10,\Omega) = \frac{1266460}{27},$$

$$R(12,\Omega)/R(10,\Omega) = \frac{26075}{216}.$$

A calculation for the minus part yields the result

$$R(17,\Omega)/R(11,\Omega) = \frac{111006792000}{803},$$

$$R(15,\Omega)/R(11,\Omega) = \frac{54618434}{365},$$

$$R(13,\Omega)/R(11,\Omega) = \frac{453159}{1606}.$$

Here it is remarkable that the denominators are simple.

We note that though there are two Hecke eigenforms in the non base change part, these ratios are the same for them.

We can show that the L-functions are the same for two Hecke eigenforms in the non base change part. In fact, let $\Omega \neq 0$ be a Hecke eigenform in the non base change part and let $\lambda(\mathfrak{m})$ be the eigenvalue of $T(\mathfrak{m})$ for Ω. For the nontrivial automorphism σ of F, there exists a Hecke eigenform $\Omega_\sigma \neq 0$ such that $\Omega_\sigma | T(\mathfrak{m}) = \lambda(\mathfrak{m}^\sigma)\Omega_\sigma$ (cf. my paper in Amer. J. Math. 1995). Since Ω is not a base change, we have $\lambda(\mathfrak{m}) \neq \lambda(\mathfrak{m}^\sigma)$ for some $\mathfrak{m}$. Hence Ω_σ is not a constant multiple of Ω. On the other hand, $L(s, \Omega_\sigma)$ is equal to $L(s, \Omega)$.

Let $F = \mathbf{Q}(\sqrt{13})$. We have $3 = (4+\sqrt{13})(4-\sqrt{13})$.

Put $\mathfrak{p} = (4-\sqrt{13})$, $\varpi = 4-\sqrt{13}$.

We consider $T(\mathfrak{p})$.

$$\Gamma^* \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} \Gamma^* = \sqcup_{i=1}^{4} \Gamma^* \beta_i,$$

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} 1 & 1 \\ 0 & \varpi \end{pmatrix}, \qquad \beta_3 = \begin{pmatrix} 1 & \epsilon \\ 0 & \varpi \end{pmatrix},$$

$$\beta_4 = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}.$$

($\epsilon = (3+\sqrt{13})/2$.)

Put $\psi = T(\mathfrak{p})\varphi$. We have

$$\psi((\tilde{\sigma}\tilde{\tau})^3) = c\beta_3^{-1} Z_3,$$

where

$$Z_3 = \varphi((\tilde{\sigma}\widetilde{\begin{pmatrix} \epsilon^{-1} & 2\epsilon - 7 \\ 0 & \epsilon \end{pmatrix}}\tilde{\sigma}\widetilde{\begin{pmatrix} 1 & -2\epsilon \\ 0 & 1 \end{pmatrix}})^3).$$

Let

$$x = \sigma \begin{pmatrix} \epsilon^{-1} & 2\epsilon - 7 \\ 0 & \epsilon \end{pmatrix} \sigma \begin{pmatrix} 1 & -2\epsilon \\ 0 & 1 \end{pmatrix}$$

$$Z_A^+ = \{\mathbf{v} \in V \mid (\sigma\nu - 1)\mathbf{v} = 0,\ (\delta - 1)\mathbf{v} = 0,\ xZ_3 = Z_3\}.$$

Example 5. We take $l_1 = l_2 = 10$. We have $\dim S_{12,12}(\Gamma) = 11$. We find that the characteristic polynomial of $T(\mathfrak{p})$ is

$$(X - 252)(X^4 + 252X^3 - 496198X^2 - 116604684X + 25202349477)$$
$$(X^6 + 244X^5 - 665334X^4 - 129598956X^3 + 109163403621X^2$$
$$+ 14522233287672X - 255121008509808).$$

The irreducible factor of degree four corresponds to the non base change part; $X - 252$ corresponds to the base change part from $S_{12}(\mathsf{SL}_2(\mathbf{Z}))$ and the irreducible factor of degree six corresponds to the base change from $S_{12}(\Gamma_0(13), (\frac{}{13}))$. Put

$$f(X) = X^4 + 252X^3 - 496198X^2 - 116604684X + 25202349477.$$

Let θ be a root of $f(X)$ and put $K = \mathbf{Q}(\theta)$. We find that K contains a quadratic subfield $F = \mathbf{Q}(\sqrt{7 \cdot 5167})$. Put $d = 7 \cdot 5167$. Then a root of $f(X)$ is given by

$$\psi = -(63 + \sqrt{d}) + \sqrt{223837 - 360\sqrt{d}}.$$

We have

$$N(223837 - 360\sqrt{d}) = 13 \cdot 563 \cdot 6205151.$$

This number is consistent with the table given in a paper of Doi-Hida-Ishii (Inv. Math. 134 (1998)).

For the Hecke eigenform $\Omega \in S_{12,12}(\Gamma)$ such that $\Omega | T(\mathfrak{p}) = \psi\Omega$, we find

$$R(10, \Omega)/R(6, \Omega) = \frac{3732099 + 18663\sqrt{d}}{5},$$

$$R(8, \Omega)/R(6, \Omega) = \frac{24367 + 121\sqrt{d}}{20}.$$

§7. Comparison of two methods

Method A: The method initiated by Shimura's 1976 paper.

Method B: The cohomological method initiated by Shimura's 1959 paper.

Method A is conceptually simpler and more general. For example if $[F:\mathbf{Q}] > 2$, B can't be used at present.

Suppose $[F:\mathbf{Q}] = 2$. In a well worked out case, $F = \mathbf{Q}(\sqrt{5})$ for example, we can get many examples by a single program rather quickly by method B. It can also be used to compute characterictic polynomials of Hecke operators.

Suppose $F = \mathbf{Q}$. I don't know which is faster. But modular symbols (method B) are now used by many people.

§8. A remark on periods unrelated to critical values

We can deduce some information on the components of the cocycle $f(\Omega)$ which are not related to critical values in certain cases. We use the notation of section 6 assuming $F = \mathbf{Q}(\sqrt{5})$. For simplicity, we consider the plus space assuming $l_1 \neq l_2$.

Let us recall

$$H^2(\Gamma^*, V)^+ \cong Z_A^+/B_A^+, \tag{8.1}$$

which is verified for $4 \leq l_2 < l_1 \leq 20$. We assume that (8.1) always holds.

$$\zeta^+ : Z_A^+ \longrightarrow \mathbf{C}^{l_2+1}.$$

ζ^+ picks up information on critical values.

$\zeta^+(B_A^+) = 0$: A crucial point of our calculation of L-values

ζ^+ consists of (at most) $[l_2/4] + 1$ linearly independent linear forms on Z_A^+.

We have

$$Z_A^+ \supset \mathrm{Ker}(\zeta^+) \supset B_A^+, \qquad \dim \zeta^+(Z_A^+) = \dim Z_A^+/\mathrm{Ker}(\zeta^+).$$

Put

$$g^+ = \dim \mathrm{Ker}(\zeta^+)/B_A^+, \qquad L = \mathrm{Hom}(Z_A^+/B_A^+, \mathbf{C}).$$

We regard an element of L as a linear form on Z_A^+ which is trivial on B_A^+. Let L_0 be the subspace of L spanned by the components of ζ^+.

Now our idea is very simple:

Fix l_2.

We have $g^+ > 0$ when l_1 is sufficiently large by the dimensionality reason.

(For example, $g^+ = 1$ when $(l_1, l_2) = (12, 6)$, $(18, 6)$, $(18, 8)$.)

Hence there exists $l \in L$ which does not belong to L_0. In view of (8.1), l defines the linear form of $Z^2(\Gamma^*, V)^+$ which is trivial on the coboundary space. Considering the image under l of the cocycle obtained from Ω, we can deduce information on periods which are not related to critical values.

More concretely:

χ: the system of eigenvalues of Hecke operators attached to Ω.

$(Z_A^+/B_A^+)(\chi)$: the χ-isotypic component of Z_A^+/B_A^+.

$Z_A^+(\chi)$: its pull back under the canonical homomorphism $Z_A^+ \longrightarrow Z_A^+/B_A^+$.

By the method of section 6, we can calculate $(Z_A^+/B_A^+)(\chi)$ algebraically. Take $\varphi \in Z_A^+(\chi)$ whose components are in $\overline{\mathbf{Q}}$. On the other hand, we can calculate the corresponding element $\psi \in Z_A^+(\chi)$ from values of the cocycle $f(\Omega)$. We have $\psi \equiv c\varphi \mod B_A^+$ with $c \in \mathbf{C}^\times$ and therefore

$$l(\psi) = cl(\varphi). \tag{8.2}$$

The equation (8.2) contains information on the values of $f(\Omega)$ unrelated to the critical values.