Fourier coefficients of theta lifts and central value of some Rankin-Selberg L-functions

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Let $1 < \kappa_1 < \kappa_2 + 2$ and let D be a divisor of the discriminant d_B of B. For $(f, f') \in S_{\kappa_1}(\Gamma_0(D)) \times \mathcal{A}_{\kappa_2}(B_{\mathbb{A}}^\times)$, we let $\mathcal{L}(f, f')$ be their theta lift. For $\xi \in B^- \setminus \{0\}$, we denote by X_ξ the set of Hecke characters of $\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E^\times$ with $E = \mathbb{Q}(\xi)$. For an automorphic form F denote by F_ξ^χ the Fourier coefficient of F indexed by $\xi \in B^- \setminus \{0\}$ and $\chi \in X_\xi$.

Theorem 1. *Suppose that $\xi \neq 0$ is "primitive" and χ is unramified at every finite prime $p|d_B$. Let f be a primitive form and f' be a Hecke eigenform with the same signature as f, and assume that the weight of χ is $-\kappa_1$. Then we have*

$$\frac{||\mathcal{L}(f, f')_\xi^\chi(g_0)||^2}{\langle f, f\rangle\langle f', f'\rangle} = C(f, f', \xi, \chi) L(\mathcal{L}(f, f'), \chi^{-1}, \frac{1}{2}),$$

where if $\pi(f')_p|_{E_p^\times} = \chi_p$ for $p|d_B$ ramified in E and if every $p|D$ is ramified in E and some sign condition of f and χ holds at such p,

$$C(f, f', \xi, \chi) = \frac{2^{|\delta(D)|-4}(\kappa_2+1)|d_E|^{\frac{3}{2}}\mathbf{w}(E)^2}{\mathbf{h}(E)^2 A(\chi)^4 D^{\frac{3}{2}} L(\pi(f), \mathrm{Ad}, 1) L(\mathrm{JL}(\pi(f')), \mathrm{Ad}, 1)}$$

$$\prod_{p \not| d_B} p^{4\mu_p}(1 - \delta(i_p(\chi) > 0)\left(\frac{d_E}{p}\right)p^{-1})^{-2} \prod_{p|d_B} r_p p^{-1} \prod_{p|D}(p+1)^{-2} \cdot W_{\xi,\infty}^{\kappa_1,\kappa_2}(1)^2 \cdot \frac{(f'_{\infty,\kappa_2}, f'_{\infty,\kappa_2})_{\kappa_2}}{(f'_\infty, f'_\infty)_{\kappa_2}},$$

and $C(f, f', \xi, \chi) = 0$ otherwise.

Notation

- $\mathbf{w}(E) := \sharp\{\text{roots of unity in } E\}$, $\mathbf{h}(E) :=$ class number of E,
  $d_E :=$the discriminant of E, $\delta(D) :=$the set of prime divisors of D,
  $i_p(\chi) :=$the conductor of χ at p, $A(\chi) := \prod_{p<\infty} p^{i_p(\chi)}$,
  $r_p :=$the ramification index of p for $E/\mathbb{Q}$, $\mu_p := \frac{\mathrm{ord}_p(2\xi)^2 - \mathrm{ord}_p(d_E)}{2}$.
- For $\eta_\infty \in \mathbb{R}_+^\times$,

$$W_{\xi,\infty}^{\kappa_1,\kappa_2}\left(\begin{pmatrix} \eta_\infty & 0 \\ 0 & \eta_\infty^{-1}\end{pmatrix}\right) := 2^{\kappa_1} n(\xi)^{\frac{\kappa_2+2}{4}}(-\sqrt{-1})^{\kappa_2-\kappa_1}\eta_\infty^{\frac{\kappa_2}{2}+2}\exp(-4\pi\eta_\infty\sqrt{n(\xi)})$$
$$\sum_{i=0}^{\frac{\kappa_2-\kappa_1}{2}} \sum_{j=0}^{\frac{\kappa_2-\kappa_1}{2}-i} \binom{\frac{\kappa_2-\kappa_1}{2}}{i}\binom{\frac{\kappa_2-\kappa_1}{2}-i}{j} \frac{2^{(\kappa_2-\kappa_1)-2(i+j)}\Gamma(i+\frac{1}{2})\Gamma(j+\frac{1}{2})}{(2\pi\eta_\infty\sqrt{n(\xi)})^{i+j+1}}.$$

- $g_{0,f} = (g_{0,p})_{p<\infty} \in G_{\mathbb{A}_f}$ is given by

$$g_{0,p} := \begin{cases} \mathrm{diag}(p^{i_p(\chi)-\mu_p}, p^{2(i_p(\chi)-\mu_p)}, 1, p^{i_p(\chi)-\mu_p}) & (p \not| d_B), \\ 1_2 & (p|d_B). \end{cases}$$