FOCUSING ON SYMMETRY CHARACTERIZATION THEOREMS FOR HOMOGENEOUS SIEGEL DOMAINS

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1. Introduction

In the paper [82] whose Russian original was published in 1959, Piatetski-Shapiro constructed non-symmetric homogeneous Siegel domains in $\mathbb{C}^4$ and $\mathbb{C}^5$. Since Siegel domains are holomorphically equivalent to bounded domains, these examples revealed the existence of non-symmetric homogeneous bounded domains.

More than 20 years earlier than this discovery, E. Cartan had pursued systematic studies on homogeneous bounded domains, and established a theorem in [12] that any homogenous bounded domain in $\mathbb{C}^2$ or in $\mathbb{C}^3$ is symmetric. Thus it was quite natural to pose a question what happens if the dimension is higher than 3. Although it is by no means in order to solve this Cartan's problem[1] that Piatetski-Shapiro introduced the notion of Siegel domains in [81] (Russian original in 1957), it resulted that, as, for instance, we nowadays know the existence of continuum cardinal of mutually inequivalent non-symmetric homogeneous bounded domains for dimension higher than or equal to 7, a lot of non-symmetric homogeneous bounded domains have been found out. The conjecture which Cartan wrote in [12, p. 118] that discovery of such non-symmetric domains would have to be based on a fresh idea thus turned out perfectly correct.

Now the holomorphic automorphism group $\mathrm{Hol}(\mathcal{D})$ of a homogeneous bounded domain $\mathcal{D}$ is a finite-dimensional Lie group (H. Cartan), and it is semisimple if $\mathcal{D}$ is symmetric. Conversely, Borel [10] and Koszul [61] proved that $\mathcal{D}$ is symmetric if it is a homogeneous space of a semisimple Lie group. Moreover, Hano [37] weakened the semisimplicity assumption to the unimodularity assumption, that is, the assumption that left Haar measure is right invariant. Reductive Lie groups which include semisimple Lie groups, and nilpotent Lie groups are unimodular. At that time, Hano's result was considered as an almost conclusive answer, and through Gindikin's essay [35] we can imagine the atmosphere of those days that there might be no non-symmetric homogeneous bounded domains.

With that research current appeared non-symmetric Siegel domains, but both of Piatetski-Shapiro's examples in $\mathbb{C}^4$ and $\mathbb{C}^5$ mentioned above were of the second kind. Construction of non-symmetric Siegel domains of the first kind, that is, construction of non-symmetric tube domains had to wait discovery of non-selfdual open convex cones. This was brought to light by Vinberg [97] in which a theory of homogeneous open convex cones was built up, and the 5-dimensional, that is, the lowest

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[1]See [80, p. 10].

dimensional, non-symmetric homogeneous open convex cone was explicitly given. In addition, it became clear that there exist continuum cardinal of mutually linearly non-equivalent non-selfdual homogeneous open convex cones if the dimension is higher than or equal to 11.

The theme of characterizing symmetric domains among homogeneous Siegel domains gets even more interesting with the above historical circumstances. The focus of this article will be on recent results obtained by the present author's research and by a collaboration with C. Kai. Some of the related preceding works are also included. Theorems by the author establish an equivalence of the symmetry of the domain with properties of operators or functions such as Berezin transforms or Poisson–Hua kernel. The tools used there are geometric norm equalities which describe the shapes of the bounded domains obtained as the images of homogeneous Siegel domains under the Cayley transforms. Collaboration with Kai has to do with properties of these Cayley transforms, and the decisive result has been obtained by Kai himself in [48]. In all cases, what attracts our interest is the fact that some of the well-known properties in symmetric domains are actually symmetry characterizations.

The organization of this article is as follows:

2. Definition of Siegel domains

We start with the definition of Siegel domains. Consider a finite-dimensional real vector space V, and let Ω be a regular open convex cone in V. Here by regularity, we mean that Ω contains no entire line at all. $W = V_{\mathbb{C}}$ is the complexification of V, and we denote by $w \mapsto w^*$ the conjugation in W relative to the real form V. Take another finite-dimensional complex vector space U, and let $Q : U \times U \to W$ be a Hermitian sesqui-linear (complex linear in the first variable and anti-linear in the second) map which is Ω-positive. Thus we have

$$Q(u', u) = Q(u, u')^* \quad (\forall u, u' \in U), \qquad Q(u, u) \in \overline{\Omega} \setminus \{0\} \quad (0 \neq \forall u \in U).$$

With these data we define a Siegel domain $D = D(\Omega, Q)$ as follows[2]:

$$D := \{(u, w) \in U \times W \; ; \; w + w^* - Q(u, u) \in \Omega\}. \tag{2.1}$$

[2]We take a generalized right half-space instead of a more familiar upper half-space. This is due to author's current preference and there is no serious mathematical reason for that.

Here we do not exclude the possibility of $U = \{0\}$. If $U = \{0\}$, then we have $D = \Omega + iV$ which is called *a Siegel domain of the first kind* (*a tube domain*). When $U \neq \{0\}$, we call D *a Siegel domain of the second kind.*

In this article we always assume that D is homogeneous. This means that we have the Lie group $\mathrm{Hol}(D)$ of the holomorphic automorphisms of D acting transitively on D. When we write "a Siegel domain D" in what follows, we will understand, without any explicit mentioning, the above usage of the symbols for the defining data of D, that is, Ω is an open convex cone in V, $W = V_{\mathbb{C}}$, and $Q : U \times U \to W$.

A Siegel domain D is said to be *symmetric* if for any $z \in D$, there exists $\sigma_z \in \mathrm{Hol}(D)$ with σ_z^2 equal to the identity map such that z is an isolated fixed point of σ_z.

Example 2.1. Let $V = \mathbb{R}$, $\Omega := \{t \in \mathbb{R}\ ;\ t > 0\}$, $U = \mathbb{C}^m$ and $W = V_{\mathbb{C}} = \mathbb{C}$. We denote by $z \cdot \overline{w}$ the canonical inner product of U. Since $Q(u_1, u_2) := 2\, u_1 \cdot \overline{u}_2$ is Hermitian sesqui-linear and Ω-positive (positive definite in the usual sense), we get a Siegel domain D by (2.1). This D is holomorphically equivalent to the open unit ball B in $\mathbb{C}^{m+1} = \mathbb{C}^m \times \mathbb{C}$ via the following Cayley transform $\mathcal{C}$:

$$\mathcal{C}(u, w) := \left(\frac{2u}{w+1},\ \frac{w-1}{w+1} \right) \qquad ((u, w) \in D). \tag{2.2}$$

D is symmetric. In fact at the base point $\mathsf{e} := (0, 1)$, we have

$$\sigma_{\mathsf{e}}(u, w) = (-w^{-1}u,\ w^{-1}).$$

Of course this corresponds to the symmetry $z \mapsto -z$ at the origin $0 = \mathcal{C}(\mathsf{e})$ of B.

3. Symmetry conditions

The results of Borel [10], Koszul [61] and Hano [37] already referred to in Introduction can be considered as symmetry conditions of Siegel domains. However, what is the most fundamental would be those which are given in terms of defining data of Siegel domains. In this regard, there are deep researches done by Satake [88] and Dorfmeister [21]. Though some of their results are used in the proof of our theorems, we leave them until later for the time being, since Jordan algebras and Jordan triple systems are necessary to fully understand them (cf. section 11).

There is also a work of D'Atri and Dotti [17] where a characterization of symmetric Siegel domains is given by the condition that the sectional curvatures by the Bergman metric[3] are all non-positive. We use not the main theorem of [17] but their quasisymmetry condition for irreducible Siegel domains — some uniformity of root multiplicities — for the proof of our main theorems (cf. Proposition 11.3). We defer giving the definitions of quasisymmetry and roots until they are actually necessary. The discussion using this condition of root multiplicities appears also in Azukawa [6] where a symmetry characterization theorem is proven by the number of distinct eigenvalues of the curvature operator. We finally mention the work of Vey [96] in which it is shown that a Siegel domain D is symmetric if there is a discrete subgroup Γ of $\mathrm{Hol}(D)$ acting on D properly such that $\Gamma K = D$ for some compact subset K of

[3]See section 6 for definition.

D. The proof given there is done by showing that $\mathrm{Hol}(D)$ is unimodular, and the symmetry of D is concluded by the above Hano's theorem.

Now for a homogeneous Siegel domain D we denote by $\mathbf{G}$ the identity component of $\mathrm{Hol}(D)$. We fix a base point $\mathsf{e} \in D$, and let $\mathbf{K}$ be the stabilizer of $\mathbf{G}$ at e. We know that $\mathbf{K}$ is a maximal compact subgroup of $\mathbf{G}$. The group $\mathbf{K}$ has a natural linear action on the tangent space $T_{\mathsf{e}}(D)$ at e, and this linear action is called the isotropy representation. It is D'Atri, Dorfmeister and Zhao who conducted detailed researches on this isotropy representation. We summarize some of their results as follows.

Theorem 3.1 (D'Atri–Dorfmeister–Zhao [16])**.** *The following four conditions are mutually equivalent.*

(1) *D is symmetric.*
(2) *The almost complex structure on $T_{\mathsf{e}}(D)$ is given by an operator of the infinitesimal representation of the isotropy representation.*
(3) *There is no non-trivial $\mathbf{G}$-invariant vector field.*
(4) *The algebra $\mathbf{D}(D)^{\mathbf{G}}$ of $\mathbf{G}$-invariant differential operators on D is commutative.*

Here, in general, an operator T acting on a function space over D is said to be $\mathbf{G}$-*invariant* if T commutes with the $\mathbf{G}$-action, that is, if we have $TL(g) = L(g)T$ for any $g \in G$, where $L(g)f(x) := f(g^{-1} \cdot x)$ $(x \in D)$.

The statement (2) of Theorem 3.1 is well-known for Hermitian symmetric spaces [38, Chap. VIII]. Moreover, (4) is well-known for Riemannian symmetric spaces and actually a stronger fact is established that $\mathbf{D}(D)^{\mathbf{G}}$ is isomorphic to a polynomial algebra with the number of algebraically independent generators equal to the rank of the symmetric space [39, Chap. II]. Finally differential operators of even degrees form a set of generators of $\mathbf{D}(D)^{\mathbf{G}}$, so that we have (3) in particular. Like this, in the case of homogeneous Siegel domains, some of the well-known properties in symmetric spaces turn out to be characteristic of symmetric domains. Main theorems of this article too make this kind of big differences more conspicuous between symmetric and non-symmetric domains.

We now state our main theorems in a somewhat rough form, where yet undefined terms come in. Introducing a standard, but more general than the Bergman, metric in a homogeneous Siegel domain $D = D(\Omega, Q)$, we denote by $\mathcal{L}$ the corresponding Laplace–Beltrami operator. To simplify the description, we suppose that D is irreducible. This means that the ingredients V, U, Ω, Q of D do not have the decomposition below by which D is written as a direct product of two Siegel domains $D(\Omega_1, Q_1)$ and $D(\Omega_2, Q_2)$ (see [51] for details including the relevance with the irreducibility as a Kähler manifold):

$$V = V_1 \oplus V_2, \quad U = U_1 \oplus U_2, \quad \Omega = \Omega_1 \times \Omega_2 \ \ (\Omega_1 \subset V_1,\ \Omega_2 \subset V_2),$$
$$Q(U_j, U_j) \subset (V_j)_{\mathbb{C}} \text{ (hence we put } Q_j := Q|_{U_j \times U_j},\ j = 1, 2), \quad Q(U_1, U_2) = 0.$$

Theorem 3.2 ([72])**.** *$\mathcal{L}$ commutes with the Berezin transform if and only if D is symmetric and the metric considered coincides with the Bergman metric up to a positive constant multiple.*

Theorem 3.3 ([74])**.** *Poisson–Hua kernel is annihilated by $\mathcal{L}$, that is, $\mathcal{L}$-harmonic if and only if D is symmetric and the metric considered coincides with the Bergman metric up to a positive constant multiple.*

Remark 3.4. Berezin transforms are integral operators which play an important role in Berezin's quantization [7], and $\mathbf{G}$-invariant bounded selfadjoint operators on the L^2 space of the $\mathbf{G}$-invariant measure on D. When D is symmetric, one obtains an explicit spectral resolution of the Berezin transform through the Fourier analysis on symmetric spaces developed by Helgason [40] (see Arazy–Zhang [3], Berezin [8], Unterberger–Upmeier [95] etc.). The point is that one can compute explicitly the Fourier transform image of the Berezin kernel, the integral kernel of the Berezin transform. In particular, the Berezin transform is a function of algebraically independent generators of $\mathbf{D}(D)^{\mathbf{G}}$ (cf. [95, 3.43]), and thus commutes with the Laplace–Beltrami operator. Theorem 3.2 shows, however, that if D is no longer symmetric, the spectral resolution of the Berezin transform inhabits a world different from that of the Laplace–Beltrami operator. A recent paper Arazy–Upmeier [2] analyzes the Berezin transform by using the non-unimodular Plancherel theorem of the solvable Lie group acting simply transitively on D, but it does not seem to give an information to the original spectral resolution of the Berezin transform which is a selfadjoint operator. Finally we would like to mention that there are various research activities on the Berezin transform on symmetric spaces. We just refer the reader to the following literature as well as references cited therein: [18], [19], [20], [29], [64], [66], [67], [69], [75], [104], [105].

Remark 3.5. "The inverse problem" such that a property of the underlying space or manifold is deduced from a property of operators or functions appears in various fields of mathematics and is undoubtedly a fascinating problem. As for the Berezin transform, we cite an earlier paper Engliš [28] in which a condition on the curvature of the domain in $\mathbb{C}$ is deduced from the commutativity of the Berezin transform with the Laplace–Beltrami operator.

Remark 3.6. If we only consider the Bergman metric in Theorem 3.3 from the beginning, the theorem is due to Hua–Look [42], Korányi [57] and Xu [101]. It is especially due to [101] that the harmonicity of the Poisson–Hua kernel implies the symmetry of D. But it seems difficult at least for the present author to follow completely its technically complicated proof.

Remark 3.7. If we restrict ourselves to the case of tube domains $T_\Omega := \Omega + iV$, the symmetry of T_Ω is equivalent to the selfduality (see right before Theorem 8.2 for definition) of the open convex cone Ω. Concerning characterizations of selfdual open convex cones, we mention the work of Vinberg [98] for some uniformity condition of root multiplicities, and the works of Shima [89], Tsuji [93], [94]. The result stated later (Theorem 8.2) is also a characterization of symmetric tube domains.

Remark 3.8. Penney [78] shows for tube domains T_Ω that the symmetry of T_Ω is necessary and sufficient in order for the set of boundary functions annihilated by a system of differential operators of Hua-type to be dense in $L^\infty(iV)$. The result of [101] that we touched on in Remark 3.6 is quoted for the proof of the necessity.

4. Piatetski-Shapiro algebra

A Piatetski-Shapiro algebra is a Lie algebra usually called a normal j-algebra in the literature. In this article we denote the complex structure exclusively by the capital J, but if accordingly the algebra is called a J-algebra, then there is a strong possibility of confusing this with a Jordan algebra which also shows up later. Thus we decide to name the algebra a Piatetski-Shapiro algebra.

For a complex Euclidean space Z, let $\mathrm{Aff}(Z)$ denote the group of the complex affine automorphisms of Z. For a homogeneous Siegel domain $D = D(\Omega, Q)$, we set

$$\mathrm{Aff}(D) := \{g \in \mathrm{Aff}(U \times W) \ ; \ g(D) = D\}.$$

Clearly we have $\mathrm{Aff}(D) \subset \mathrm{Hol}(D)$, and it is known ([83]) that we can find a split solvable (the adjoint representation Ad is triangularizable over $\mathbb{R}$) subgroup G in $\mathrm{Aff}(D)$, called an Iwasawa subgroup, such that G acts on D simply transitively, in other words, without fixed points. The Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ of G has a structure of *Piatetski-Shapiro algebra*. This is an algebraic translation of the Kähler structure of D, and the following two conditions are satisfied:

(1) There is an integrable (Nijenhuis tensor $\equiv 0$) almost complex structure J on $\mathfrak{g}$,
(2) There exists $\omega \in \mathfrak{g}^*$ such that $\langle\, x \,|\, y \,\rangle_\omega := \langle\, [Jx, y], \omega \,\rangle$ defines a J-invariant positive definite inner product on $\mathfrak{g}$.

In general, the linear forms ω with the above property (2) are said to be *admissible*. As an example of admissible linear form, we have the following *Koszul form* β:

$$\langle\, x, \beta \,\rangle := \mathrm{tr}\big(\mathrm{ad}(Jx) - J\,\mathrm{ad}(x)\big) \qquad (x \in \mathfrak{g}). \tag{4.1}$$

Indeed, Koszul [61] shows that $\langle\, x \,|\, y \,\rangle_\beta$ coincides, up to a positive number multiple, with the real part of the Hermitian inner product obtained by the Bergman metric of D under the identification of $\mathfrak{g}$ with $T_{\mathsf{e}}(D)$.

The Lie algebra $\mathfrak{g}$ has a structure very similar to the solvable subalgebra that is an Iwasawa constituent of a semisimple Lie algebra. We now describe it briefly. The Lie algebra $\mathfrak{g}$ is written as a semidirect product $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{n}$ of a commutative subalgebra $\mathfrak{a}$ and the derived ideal $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$, and $\mathfrak{n}$ is a sum of the root subspaces. For $\alpha \in \mathfrak{a}^*$, we put

$$\mathfrak{n}_\alpha := \{x \in \mathfrak{n} \ ; \ [a, x] = \langle\, a, \alpha \,\rangle x \ \text{ for all } a \in \mathfrak{a}\}.$$

Then there is a finite subset Δ in $\mathfrak{a}^* \setminus \{0\}$ such that $\mathfrak{n}_\alpha \neq \{0\}$ for $\alpha \in \Delta$ and $\mathfrak{g} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{n}_\alpha$. An element in Δ is called a *root* of the Piatetski-Shapiro algebra $\mathfrak{g}$. The number $r := \dim \mathfrak{a}$ is said to be the *rank* of $\mathfrak{g}$. We can choose a basis $H_1, \ldots, H_r$ of $\mathfrak{a}$ so that with $E_j := -JH_j (\in \mathfrak{n})$ we have $[H_j, E_k] = \delta_{jk} E_k$. Let $\alpha_1, \ldots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $H_1, \ldots, H_r$. Then the elements of Δ are of the following forms (not all possibilities need occur):

$$\frac{1}{2}(\alpha_k \pm \alpha_j) \ \ (j < k), \quad \alpha_k \ \ (k = 1, \ldots, r), \quad \frac{1}{2}\alpha_k \ \ (k = 1, \ldots, r).$$

Moreover we have $\mathfrak{n}_{\alpha_k} = \mathbb{R}E_k$ $(\forall k)$, and we note that the root subspaces are mutually orthogonal with respect to the inner product $\langle\, x \,|\, y \,\rangle_{\omega'}$ defined by any admissible linear form ω'. For details of the above fundamental structures, see [83], [85], [86], [90] etc.

For $k = 1, \ldots, r$, we define $E_k^* \in \mathfrak{g}^*$ by requiring $\langle E_k, E_k^* \rangle = 1$ and $E_k^* = 0$ on $\mathfrak{a}$ and $\mathfrak{n}_\alpha$ ($\alpha \neq \alpha_k$). Furthermore we set $E_{\mathbf{s}}^* := s_1 E_1^* + \cdots + s_r E_r^*$ for every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$. If $s_k > 0$ for all $k = 1, \ldots, r$, we write $\mathbf{s} > 0$.

Proposition 4.1. *The set of the admissible linear forms on* $\mathfrak{g}$ *coincides with*

$$\mathfrak{a}^* + \{E_{\mathbf{s}}^* \,;\, \mathbf{s} > 0\}.$$

Thus there is no loss of generality even if we assume that the inner products obtained by admissible linear forms are parametrized by $E_{\mathbf{s}}^*$ ($\mathbf{s} > 0$), since they are through the commutation product. In what follows we set $\langle x \,|\, y \rangle_{\mathbf{s}} := \langle [Jx, y], E_{\mathbf{s}}^* \rangle$ for simplicity. This inner product on the Lie algebra $\mathfrak{g}$ then defines a left-invariant Riemannian metric on the group manifold G, and we denote by $\mathcal{L}_{\mathbf{s}}$ the Laplace–Beltrami operator relative to this metric.

Example 4.2. Let β be the Koszul form (4.1). Then $\langle E_k, \beta \rangle = 2d_k + b_k$, where

$$d_k := 1 + \frac{1}{2} \sum_{j \neq k} \dim \mathfrak{n}_{(\alpha_k + \alpha_j)/2}, \qquad b_k := \frac{1}{2} \dim \mathfrak{n}_{\alpha_k/2}. \tag{4.2}$$

5. Compound power functions

Compound power functions were introduced by Gindikin [33]. They play a basic role in analysis on Siegel domains, tube domains and open convex cones. These are multi-variable and matrix variable analogues of the power functions t^λ on the half-line $\{t \in \mathbb{R} \,;\, t > 0\}$. On the cone of positive definite real symmetric matrices, they are

$$\Delta_{\mathbf{s}}(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \cdots \Delta_r(x)^{s_r}$$

for the parameter $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$. Here $\Delta_k(x)$ stands for the k-th principal minor of the matrix x for $k = 1, \ldots, r$.

Consider the open convex cone Ω in the defining data of the Siegel domain $D = D(\Omega, Q)$ which we are dealing with. We can find a subgroup H of the simply transitive split solvable group G such that H acts on Ω linearly and simply transitively. The Lie algebra $\mathfrak{h}$ of H is expressed as a semidirect product $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}(0)$, where $\mathfrak{a}$ is the commutative subalgebra appearing in the structure description of Piatetski-Shapiro algebra in section 4 and $\mathfrak{n}(0) := \sum_{m>k} \mathfrak{n}_{(\alpha_m - \alpha_k)/2}$. For each $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we define a one-dimensional representation $\chi_{\mathbf{s}}$ of $A := \exp \mathfrak{a}$ by the formula

$$\chi_{\mathbf{s}}\Big(\exp \sum_{j=1}^r t_j H_j\Big) = \exp\Big(\sum_{j=1}^r s_j t_j\Big) \qquad (t_1 \in \mathbb{R}, \ldots, t_r \in \mathbb{R}),$$

and extend it to a one-dimensional representation of $H = A \ltimes N(0)$ by setting identically 1 on $N(0) := \exp \mathfrak{n}(0)$. We denote this extension still by the same symbol $\chi_{\mathbf{s}}$.

Now we take base points $\mathsf{e} \in D$ and $E \in \Omega$ in such a way that $\mathsf{e} = (0, E)$, and consider the diffeomorphic orbit map $H \ni h \mapsto hE \in \Omega$. We define functions $\Delta_{\mathbf{s}}$ on Ω by $\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h)$ $(h \in H)$, and call them *compound power functions.*

Theorem 5.1 (Gindikin [34], Ishi [44]). *The functions $\Delta_{\mathbf{s}}$ are analytically continued to holomorphic functions on $\Omega + iV$.*

These analytic continuations are obtained as the Laplace transforms of distributions, called Riesz distributions, on the dual cone Ω^*, where

$$\Omega^* := \bigl\{\xi \in V^* \ ; \ \langle\, x, \xi\,\rangle > 0 \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}\bigr\}. \tag{5.1}$$

6. Berezin transforms

Generally speaking, the Hilbert space of holomorphic functions f on a domain D in a complex Euclidean space Z such that f is square integrable with respect to the Euclidean measure has a reproducing kernel $\kappa(z_1, z_2)$, as is well-known, and is easily seen[4]. This function κ is called the *Bergman kernel.* The Bergman kernel has the following covariance property: for any holomorphic automorphism g of D,

$$\kappa(z_1, z_2) = \det g'(z_1)\, \kappa(g \cdot z_1,\ g \cdot z_2)\, \overline{\det g'(z_2)} \qquad (z_1, z_2 \in D). \tag{6.1}$$

The Hermitian metric

$$\partial_{z_1} \overline{\partial}_{z_2} \log \kappa(p, p) \qquad (z_1, z_2 \in Z,\ p \in D)$$

obtained by the Bergman kernel κ is named the *Bergman metric.*

Now, since our Siegel domain $D = D(\Omega, Q)$ admits a transitive action of the split solvable Lie group G, we have an explicit expression of the Bergman kernel of D in terms of (the analytic continuation of) a compound power function by making use of (6.1). In fact we have

$$\kappa(z_1, z_2) = \Delta_{-2\mathbf{d}-\mathbf{b}}\bigl(w_1 + w_2^* - Q(u_1, u_2)\bigr) \qquad (z_j = (u_j, w_j) \in D; j = 1, 2) \tag{6.2}$$

up to a positive number multiple, where $\mathbf{d} := (d_1, \ldots, d_r)$ and $\mathbf{b} := (b_1, \ldots, b_r)$ with d_k and b_k as in (4.2). Denoting by $dm(w)$ and $dm(u)$ the Euclidean measures on $W = V_{\mathbb{C}}$ and U respectively, we see that the measure $d\mu$ on D which is invariant under $\mathrm{Hol}(D)$ is given by

$$d\mu(u, w) := \Delta_{-2\mathbf{d}-\mathbf{b}}\bigl(w + w^* - Q(u, u)\bigr)\, dm(u) dm(w).$$

Next we consider, for each $\lambda \in \mathbb{R}$, the measure

$$d\mu_\lambda(u, w) := \Delta_{-2\mathbf{d}-\mathbf{b}}\bigl(w + w^* - Q(u, u)\bigr)^{-\lambda+1}\, dm(u) dm(w)$$

on D, and denote by $\mathcal{H}^2_\lambda(D)$ the Hilbert space of holomorphic functions on D that are square integrable for $d\mu_\lambda$. This Hilbert space is called a *weighted Bergman space.* For simplicity we put

$$\lambda_0 := \max_{1 \leqq k \leqq r} \frac{b_k + d_k + \frac{1}{2} p_k}{b_k + 2d_k}, \quad \text{where } p_k := \sum_{m > k} \dim \mathfrak{n}_{(\alpha_m + \alpha_k)/2}.$$

[4] In the case of Siegel domain, this Hilbert space has a non-zero function, because the domain is holomorphically equivalent to a bounded domain.

We note that $0 < \lambda_0 < 1$ by (4.2).

Proposition 6.1 (Rossi–Vergne [86], Ishi [43])**.** $\mathcal{H}^2_\lambda(D) \neq \{0\} \iff \lambda > \lambda_0$.

Now we suppose $\lambda > \lambda_0$. Then normalizing the relevant measures appropriately, we see that the reproducing kernel κ_λ of $\mathcal{H}^2_\lambda(D)$ is equal to a power of κ:

$$\kappa_\lambda(z_1, z_2) = \Delta_{-2\mathbf{d}-\mathbf{b}}\big(w_1 + w_2^* - Q(u_1, u_2)\big)^\lambda.$$

The *Berezin kernel* A_λ associated to the reproducing kernel subspace $\mathcal{H}^2_\lambda(D)$ of $L^2(D, d\mu_\lambda)$ is defined as (cf. [68])

$$A_\lambda(z_1, z_2) := \frac{|\kappa_\lambda(z_1, z_2)|^2}{\kappa_\lambda(z_1, z_1)\kappa_\lambda(z_2, z_2)}.$$

The covariance property (6.1) of the Bergman kernel yields the invariance property of the Berezin kernel:

$$A_\lambda(g \cdot z_1,\ g \cdot z_2) = A_\lambda(z_1, z_2) \qquad (g \in \mathrm{Hol}(D)). \tag{6.3}$$

The *Berezin transform* is an integral operator B^D_λ on $L^2(D) := L^2(D, d\mu)$ with the Berezin kernel A_λ as the integral kernel:

$$B^D_\lambda f(z) := \int_D A_\lambda(z, z') f(z')\, d\mu(z') \qquad (f \in L^2(D)).$$

B^D_λ is a bounded non-negative selfadjoint operator which is $\mathrm{Hol}(D)$-invariant, hence in particular, G-invariant.

Since G is diffeomorphic to D by the orbit map $g \mapsto g \cdot \mathsf{e}$, we transfer the Berezin transform to an operator on $L^2(G)$ of the left Haar measure. For this purpose we put $a_\lambda(g) := A_\lambda(g \cdot \mathsf{e}, \mathsf{e})$ $(g \in G)$. Since we are supposing $\lambda > \lambda_0$, it is easy to see that $a_\lambda \in L^1(G)$. Now the transferred Berezin transform B_λ is an operator of the convolution by a_λ from the right:

$$B_\lambda f(x) = \int_G f(y) a_\lambda(y^{-1}x)\, dy = f * a_\lambda(x) \qquad (f \in L^2(G)). \tag{6.4}$$

The fact that the integral converges absolutely follows from a standard argument for $L^2 * L^1$ in the integration theory course. Let us recall the Laplace–Beltrami operator $\mathcal{L}_\mathbf{s}$ on G defined just before Example 4.2.

Theorem 6.2 ([72])**.** *Suppose that D is irreducible, and fix $\lambda > \lambda_0$. The Berezin transform B_λ commutes with $\mathcal{L}_\mathbf{s}$ if and only if D is symmetric and $\mathbf{s}$ is a positive number multiple of $2\mathbf{d} + \mathbf{b}$. In this case, it holds that $s_1 = \cdots = s_r$.*

7. Poisson–Hua kernel

Let us start this section with the introduction of the Hardy space $H^2(D)$ over the Siegel domain $D = D(\Omega, Q)$. The Euclidean measure on V is denoted by dx and as in section 6 let $dm(u)$ be the Euclidean measure on U. Then $H^2(D)$ is the Hilbert space of holomorphic functions F on D satisfying the following condition:

$$\|F\|^2 = \sup_{t \in \Omega} \int_U dm(u) \int_V \big|F\big(u,\, t + \tfrac{1}{2}Q(u,u) + ix\big)\big|^2\, dx < \infty.$$

As is well-known (see Damek–Hulanicky–Penney [13], Gindikin [33], Korány–Stein [59] etc.) $H^2(D)$ has a reproducing kernel $S(z_1, z_2)$. This function S is called the *Szegö kernel*. Here also, by the homogeneity of D, the Szegö kernel can be written explicitly. We have, up to a positive number multiple,

$$S(z_1, z_2) = \Delta_{-\mathbf{d}-\mathbf{b}}\big(w_1 + w_2^* - Q(u_1, u_2)\big) \qquad (z_j := (u_j, w_j) \in D). \tag{7.1}$$

On the other hand, it is known that the Shilov boundary Σ of D is described in the following way (see Kaneyuki–Sudo [54, Theorem 1.1], or Rossi [85, Lemma 3.25]):

$$\Sigma = \{(u, w) \in U \times W \ ; \ 2\,\mathrm{Re}\, w = Q(u, u)\}.$$

If D is a tube domain $\Omega + iV$, then we have $\Sigma = iV$. Now in the formula (7.1) we see that $S(z, \zeta)$ ($z \in D$, $\zeta \in \Sigma$) has a meaning, so that we define, after Hua [41], the Poisson kernel $P(z, \zeta)$ as follows[5]:

$$P(z, \zeta) := \frac{|S(z, \zeta)|^2}{S(z, z)} \qquad (z \in D,\ \zeta \in \Sigma). \tag{7.2}$$

The Poisson kernel is also transferred to a function on G:

$$P_\zeta^G(g) := P(g \cdot \mathsf{e},\ \zeta) \qquad (g \in G,\ \zeta \in \Sigma).$$

Theorem 7.1 ([74]). *Suppose that D is irreducible. Then $\mathcal{L}_{\mathbf{s}} P_\zeta^G = 0$ for any $\zeta \in \Sigma$ if and only if D is symmetric and $\mathbf{s}$ is a positive number multiple of $\mathbf{d} + \mathbf{b}$. In this case one has $s_1 = \cdots = s_r$.*

8. Cayley transforms of Siegel domains

By introducing Cayley transforms of a Siegel domain $D = D(\Omega, Q)$, we can disclose geometric backgrounds that make Theorems 6.2 and 7.1 true. Our Cayley transforms will be modeled on (2.2), and we only need something like denominator in order for $(w + E)^{-1}$ ($\mathrm{Re}\, w \in \Omega$) to make sense in the general case. Indeed, we will define a denominator for each of the parameter $\mathbf{s} > 0$. The Cayley transforms thus defined with parameter have been introduced in [73]. If D is symmetric and the parameter $\mathbf{s} = (s_1, \ldots, s_r)$ satisfies $s_1 = \cdots = s_r$, this Cayley transform is essentially the same as the inverse of the one introduced by Korányi–Wolf [60].

As motivation we note the following relation: if x, v are real symmetric matrices, and if x positive definite, then

$$-\frac{d}{dt} \log \det(x + tv)^{-1} \Big|_{t=0} = \mathrm{tr}(x^{-1} v).$$

Here it should be noticed that on the right-hand side the inverse matrix x^{-1} appears. Based on the fact that compound power functions generalize powers of the determinant function, we now define, in the case $\mathbf{s} > 0$, elements $\mathcal{I}_{\mathbf{s}}(x) \in V^*$ ($x \in \Omega$) through the formula

$$\langle\, v, \mathcal{I}_{\mathbf{s}}(x)\, \rangle = -D_v \log \Delta_{-\mathbf{s}}(x) \qquad (v \in V), \tag{8.1}$$

[5]In the case of the open unit disk or the upper half-plane in $\mathbb{C}$, it should be noted that the relation (7.2) holds between the Szegö kernel and the ordinary Poisson kernel.

where D_v stands for the directional derivative in the direction v:

$$D_v f(x) := \frac{d}{dt} f(x+tv)\Big|_{t=0}.$$

$\mathcal{I}_{\mathbf{s}}(x)$ will serve as a denominator and is called the *pseudoinverse* of x. The $*$-map of Vinberg (cf. [97]) is the case $\mathbf{s} = \mathbf{d}$ in (8.1), and the pseudoinverse used to define the Cayley transform associated to the Bergman kernel in the analysis of the Berezin transforms is the case $\mathbf{s} = 2\mathbf{d}+\mathbf{b}$ (see (6.2) and section 9). The pseudoinverse for the Cayley transform associated to the Szegö kernel in the analysis of the Poisson–Hua kernel is the case $\mathbf{s} = \mathbf{d} + \mathbf{b}$ (see (7.1) and section 10).

Let us recall the group H introduced in section 5 that acts on Ω linearly and simply transitively. A simple computation shows that $\mathcal{I}_{\mathbf{s}}$ has the H-covariance property: $\mathcal{I}_{\mathbf{s}}(hx) = h \cdot \mathcal{I}_{\mathbf{s}}(x)$ $(h \in H)$, where the right-hand side is the contragredient H-action on V^*. We have in particular $\mathcal{I}_{\mathbf{s}}(\lambda x) = \lambda^{-1}\mathcal{I}_{\mathbf{s}}(x)$ $(\lambda > 0)$. Properties of the pseudoinverse map $\mathcal{I}_{\mathbf{s}} : x \to \mathcal{I}_{\mathbf{s}}(x)$ are summarized in Proposition 8.1 below. We recall that Ω^* denotes the dual cone of Ω (see (5.1)).

Proposition 8.1 ([73])**.** *Suppose* $\mathbf{s} > 0$.

(1) *For any* $x \in \Omega$, *one has* $\mathcal{I}_{\mathbf{s}}(x) \in \Omega^*$, *and* $\mathcal{I}_{\mathbf{s}}$ *gives a bijection of* Ω *onto* Ω^*.
(2) $\mathcal{I}_{\mathbf{s}}(E) = E^*_{\mathbf{s}}$.
(3) $\mathcal{I}_{\mathbf{s}}$ *is analytically continued to a rational map* $W \to W^*$.
(4) *Taking* $E^*_{\mathbf{s}}$ *as a base point of the dual cone* Ω^*, *and then introducing compound power functions* $\Delta^*_{\mathbf{s}}$ *associated to* Ω^*, *one obtains anew a pseudo-inverse map* $\mathcal{I}^*_{\mathbf{s}} : \Omega^* \to \Omega^{**} = \Omega$. *This* $\mathcal{I}^*_{\mathbf{s}}$, *too, continues analytically to a rational function* $W^* \to W$. *Furthermore, one has* $\mathcal{I}^*_{\mathbf{s}} = \mathcal{I}^{-1}_{\mathbf{s}}$, *so that* $\mathcal{I}_{\mathbf{s}}$ *is a birational map.*
(5) $\mathcal{I}_{\mathbf{s}} : \Omega + iV \to \mathcal{I}_{\mathbf{s}}(\Omega + iV)$ *is a biholomorphic bijection.*

In general, we have $\mathcal{I}_{\mathbf{s}}(\Omega + iV) \not\subseteq \Omega^* + iV^*$, and actually we have Theorem 8.2 below. Here we say that Ω is *selfdual* if the dual cone Ω^* coincides with Ω under the identification of V^* with V by an appropriate inner product in V.

Theorem 8.2 ([49])**.** $\mathcal{I}_{\mathbf{s}}(\Omega + iV) = \Omega^* + iV^*$ *if and only if* $\mathbf{s}$ *is a positive number multiple of* $\mathbf{d}$, *and* Ω *is selfdual. In this case one has* $s_1 = \cdots = s_r$.

We would like to mention here the holomorphic domain of the pseudoinverse map. Let $\Delta_1, \ldots, \Delta_r$ be the basic relative invariants associated to the cone Ω introduced in Ishi [45]. If Ω is the open convex cone of real positive definite symmetric matrices, then $\Delta_k(x)$ is the k-th principal minor of the matrix x. Regarding these $\Delta_1, \ldots, \Delta_r$ naturally as holomorphic polynomial functions on W, we denote by $\mathcal{N}_k$ the set of zeros of Δ_k $(k = 1, \ldots, r)$. Then, $\mathcal{I}_{\mathbf{s}}$ is holomorphic outside of the union of these $\mathcal{N}_k$ $(k = 1, \ldots, r)$ (see [73, Lemma 3.17] and [46]).

We now introduce the Cayley transform after [73] with the pseudoinverse as "denominator". First we put

$$C_{\mathbf{s}}(w) := \mathcal{I}_{\mathbf{s}}(E) - 2\,\mathcal{I}_{\mathbf{s}}(w + E) \qquad (w \in W). \tag{8.2}$$

The closure of the tube domain $\Omega + iV$ is contained in the holomorphic domain of $C_{\mathbf{s}}$, and $C_{\mathbf{s}}(\Omega + iV)$ is a domain in W^*. Our Cayley transforms $\mathcal{C}_{\mathbf{s}}$ are defined in the following way:

$$\mathcal{C}_{\mathbf{s}}(z) := \big(2\,\langle\, Q(u,\cdot), \mathcal{I}_{\mathbf{s}}(w+E)\,\rangle,\, C_{\mathbf{s}}(w)\big), \qquad (z=(u,w)\in D). \tag{8.3}$$

The Cayley transform introduced by Penney [77] has Vinberg's $*$ map as denominator, so that it is the case $\mathbf{s}=\mathbf{d}$. Since the characteristic function $\phi(x)$ of the cone Ω defined by

$$\phi(x) := \int_{\Omega^*} e^{-\langle\, x,\lambda\,\rangle}\,d\lambda \qquad (x\in\Omega)$$

is written as $\phi(x) = \Delta_{-\mathbf{d}}(x)$ up to a positive number multiple, Penney's Cayley transform can be named the Cayley transform associated to the characteristic function of Ω.

Proposition 8.3 ([73])**.** *$\mathcal{C}_{\mathbf{s}}$ is a birational and biholomorphic bijection of D onto $\mathcal{C}_{\mathbf{s}}(D)$, and the inverse $\mathcal{C}_{\mathbf{s}}^{-1}$ can be written down explicitly.*

Actually it can be seen from (8.2) and (8.3) that the closure of D is contained in the holomorphic domain of $\mathcal{C}_{\mathbf{s}}$.

Theorem 8.4 ([73])**.** *The image $\mathcal{C}_{\mathbf{s}}(D)$ is bounded in $U^\dagger\oplus W^*$, where $U^\dagger$ denotes the complex vector space of all anti-linear forms on U.*

If the parameter $\mathbf{s}$ is generic, we do not have the convexity of the image $\mathcal{C}_{\mathbf{s}}(D)$ even if D is symmetric. We had established a preliminary result in [50], and the conclusive result has been proved by Kai recently.

Theorem 8.5 (Kai [48])**.** *$\mathcal{C}_{\mathbf{s}}(D)$ is convex if and only if D is symmetric and the parameter $\mathbf{s}$ satisfies $s_1=\cdots=s_r$.*

9. Geometric norm equality I

Recalling the J-invariant inner product $\langle\,\cdot\,|\,\cdot\,\rangle_{\mathbf{s}}$ on $\mathfrak{g}$ introduced immediately after Proposition 4.1, we transfer it to a Hermitian inner product on the tangent space $T_{\mathsf{e}}(D)=U+W$ of D through the diffeomorphism given by the orbit map $G\ni g\mapsto g\cdot\mathsf{e}\in D$. Then we carry it into the dual vector space $U^\dagger+W^*$ naturally, and obtain a Hermitian inner product $(\,\cdot\,|\,\cdot\,)_{\mathbf{s}}$ there. The corresponding norm will be denoted by $\|\cdot\|_{\mathbf{s}}$.

We take an element $\Psi_{\mathbf{s}}\in\mathfrak{g}$ so that we have $\operatorname{tr}\operatorname{ad}(x) = \langle\, x\,|\,\Psi_{\mathbf{s}}\,\rangle_{\mathbf{s}}$ for all $x\in\mathfrak{g}$. It is shown that $\Psi_{\mathbf{s}}\in\mathfrak{a}$. By using the Laplace–Beltrami operator $\mathcal{L}_{\mathbf{s}}$, the Berezin kernel a_λ transferred to G, and the Cayley transform $\mathcal{C}_{2\mathbf{d}+\mathbf{b}}$ (the Cayley transform $\mathcal{C}_{\mathbf{s}}$ for the parameter $\mathbf{s}=2\mathbf{d}+\mathbf{b}$), we obtain Proposition 9.1 below which acts as a mediator between the Berezin transform B_λ and the geometry of Siegel domains. We set $\alpha_{\mathbf{s}} := \sum_{j=1}^r s_j\alpha_j \in\mathfrak{a}^*$ for $\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{R}^r$ in what follows.

Proposition 9.1 ([72])**.** *One has*

$$\mathcal{L}_{\mathbf{s}}a_\lambda(g) = \lambda a_\lambda(g)\big(-\lambda\|\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(g\cdot\mathsf{e})\|_{\mathbf{s}}^2 + \langle\,\Psi_{\mathbf{s}},\alpha_{2\mathbf{d}+\mathbf{b}}\,\rangle\big) \qquad (\forall g\in G).$$

Since the selfadjointness of the integral operator (6.4) yields $a_\lambda(g)=a_\lambda(g^{-1})$ for any $g\in G$, and since in addition we have

$$B_\lambda \text{ commutes with } \mathcal{L}_{\mathbf{s}} \iff \mathcal{L}_{\mathbf{s}}a_\lambda(g) = \mathcal{L}_{\mathbf{s}}a_\lambda(g^{-1}) \ \ (\forall g\in G),$$

we see that the commutativity of B_λ with $\mathcal{L}_{\mathbf{s}}$ is equivalent to the validity of the following norm equality for any $g \in G$:

$$\|\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(g \cdot \mathsf{e})\|_{\mathbf{s}} = \|\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(g^{-1} \cdot \mathsf{e})\|_{\mathbf{s}}. \tag{9.1}$$

These observations together with the following Theorem 9.2 imply Theorem 6.2.

Theorem 9.2 ([71])**.** *Suppose that D is irreducible. In order that the norm equality* (9.1) *holds for any $g \in G$, it is necessary and sufficient that D is symmetric and that the parameter* $\mathbf{s} > 0$ *is a positive number multiple of* $2\mathbf{d} + \mathbf{b}$.

Since $\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(\mathsf{e}) = 0$ by definition, Theorem 9.2 can be rephrased as follows:

Theorem 9.3. *Suppose that D is irreducible. Then the norm equality $\|h \cdot 0\|_{\mathbf{s}} = \|h^{-1} \cdot 0\|_{\mathbf{s}}$ holds for any $h \in \mathcal{C}_{2\mathbf{d}+\mathbf{b}} \circ G \circ \mathcal{C}_{2\mathbf{d}+\mathbf{b}}^{-1}$ if and only $\mathcal{D}_{2\mathbf{d}+\mathbf{b}} := \mathcal{C}_{2\mathbf{d}+\mathbf{b}}(D)$ is symmetric and the parameter* $\mathbf{s} > 0$ *is a positive number multiple of* $2\mathbf{d} + \mathbf{b}$.

When $\mathcal{D}_{2\mathbf{d}+\mathbf{b}}$ is symmetric, it is shown that $\mathcal{D}_{2\mathbf{d}+\mathbf{b}}$ is essentially the Harish-Chandra model of a non-compact Hermitian symmetric space. In particular it is circular, that is, it is invariant under the multiplication by complex numbers with absolute value one. The if part of Theorem 9.3 is proved by using this circularity, the semisimplicity of the Lie group $\mathrm{Hol}(\mathcal{D}_{2\mathbf{d}+\mathbf{b}})$, and its Cartan decomposition.

As for the picture that Theorem 9.3 gives in the case of the unit disk in $\mathbb{C}$, we refer the reader to [71, §12].

10. Geometric norm equality II

Recall the explicit expression (7.1) of the Szegö kernel. In this section we will use the Cayley transform $\mathcal{C}_{\mathbf{d}+\mathbf{b}}$ associated to the Szegö kernel. As a counterpart of Proposition 9.1 we have

Proposition 10.1. $\mathcal{L}_{\mathbf{s}} P^G_\zeta = \big(-\|\mathcal{C}_{\mathbf{d}+\mathbf{b}}(\zeta)\|^2_{\mathbf{s}} + \langle\, \Psi_{\mathbf{s}}, \alpha_{\mathbf{d}+\mathbf{b}}\,\rangle\big) P^G_\zeta \quad (\forall \zeta \in \Sigma).$

Therefore, we see that $\mathcal{L}_{\mathbf{s}} P^G_\zeta = 0$ for any $\zeta \in \Sigma$ is equivalent to the validity of the following norm equality for all $\zeta \in \Sigma$:

$$\|\mathcal{C}_{\mathbf{d}+\mathbf{b}}(\zeta)\|^2_{\mathbf{s}} = \langle\, \Psi_{\mathbf{s}}, \alpha_{\mathbf{d}+\mathbf{b}}\,\rangle. \tag{10.1}$$

This is the condition that the Cayley transform image of the Shilov boundary Σ of the Siegel domain D lies on the sphere centered at the origin with a prescribed radius.

These observations together with the next Theorem 10.2 lead us to Theorem 7.1.

Theorem 10.2 ([74])**.** *Suppose that D is irreducible. A necessary and sufficient condition for the norm equality* (10.1) *to be true for any $\zeta \in \Sigma$ is that D is symmetric and that the parameter* $\mathbf{s} > 0$ *is a positive number multiple of* $\mathbf{d} + \mathbf{b}$.

If D is symmetric, then $\mathcal{D} := \mathcal{C}_{\mathbf{d}+\mathbf{b}}(D)$ is also essentially the Harish-Chandra model of a non-compact Hermitian symmetric space. Let G be the identity component of $\mathrm{Hol}(\mathcal{D})$, and K the stabilizer of G at the origin. Then G is a semisimple Lie group, and K is a maximal compact subgroup of G. By using the fact that the Shilov boundary is a G-orbit as well as a K-orbit (see for example Korányi [58] or Wolf [100]), the norm equality (10.1) is proved.

Example 10.3. Let us consider the case of the right half-plane D_0 and the open unit disk $\mathbb{D}$ in $\mathbb{C}$. The Cayley transform $w \mapsto (w-1)/(w+1)$ maps the imaginary axis, which is the Shilov boundary of D_0, to the unit circle with the point 1 removed. On the other hand, the stabilizer at the origin in $\mathbb{D}$ of the group

$$\mathsf{G} = SU(1,1) := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \ ; \ \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\}$$

is the group of rotations

$$\mathsf{K} := \left\{ \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \ ; \ 0 \leqq \theta < 4\pi \right\}.$$

The Shilov boundary of $\mathbb{D}$ is the unit circle, and it is clearly a K-orbit $\mathsf{K} \cdot 1$, and also a G-orbit $\mathsf{G} \cdot 1$ as can be seen from a direct computation.

11. Outline of analysis of norm equalities

In this section we exhibit some of the propositions used to prove that the validity of the norm equality (9.1) for any $g \in G$ (or the validity of the norm equality (10.1) for any $\zeta \in \Sigma$) implies the symmetry of D.

Recalling the expression (6.2) of the Bergman kernel κ, we introduce an inner product in V by

$$\langle\, x \,|\, y \,\rangle_\kappa := D_x D_y \log \Delta_{-2\mathbf{d}-\mathbf{b}}(E) \qquad (x, y \in V). \tag{11.1}$$

Definition 11.1. The Siegel domain $D = D(\Omega, Q)$ is said to be *quasisymmetric* if the open convex cone Ω is selfdual with respect to the inner product (11.1).

Our definition of quasisymmetry has seemingly less requirements than that given in Satake [88], but a result of Dorfmeister quoted later (Proposition 11.4) together with [88, Proposition V.4.1] shows the legitimacy of Definition 11.1. Moreover, we emphasize that in Definition 11.1 the inner product by which the open convex cone is required to be selfdual is specified. For instance, it can happen[6] that the open convex cone in the defining data of Siegel domain is the set of positive definite real symmetric matrices, but that it is not selfdual relative to the inner product (11.1).

Now we introduce a non-associative product, that is, a product which does not demand the associativity low, through the formula

$$\langle\, xy \,|\, z \,\rangle_\kappa := -\frac{1}{2} D_x D_y D_z \log \Delta_{-2\mathbf{d}-\mathbf{b}}(E). \tag{11.2}$$

Since the function $\log \Delta_{-2\mathbf{d}-\mathbf{b}}$ is smooth, the product is commutative, and the multiplication operator $L(x) : y \mapsto xy$ is selfadjoint for any $x \in V$ with respect to the inner product (11.1).

Proposition 11.2 (Dorfmeister [22]). *A necessary and sufficient condition for $D(\Omega, Q)$ to be quasisymmetric is that the product* (11.2) *makes V a Jordan algebra, that is, that $x^2(xy) = x(x^2y)$ holds for all $x, y \in V$ in addition to the commutativity.*

[6]This is actually the case for the 4-dimensional (the lowest dimensional) non-symmetric Siegel domain.

Concerning this proposition we refer the reader also to the connection algebra of Vinberg [97] as well as D'Atri–Dotti [17] and Tsuji [94]. Proposition 11.2 shows that if $D(\Omega, Q)$ is quasisymmetric, then V has a structure of Euclidean Jordan algebra in the sense of Faraut–Korányi [32].

When the Siegel domain is irreducible, we have a quasisymmetry criterion in terms of certain uniformity condition for root multiplicities of the corresponding Piatetski-Shapiro algebra. It is this criterion that we use to reduce D to a quasisymmetric domain starting with the validity of the norm equality (9.1) (or (10.1)).

Proposition 11.3 (D'Atri–Dotti [17])**.** *An irreducible Siegel domain is quasisymmetric if and only if the following two conditions are satisfied for the root subspaces of the corresponding Piatetski-Shapiro algebra*:

(1) $\dim \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$ $(k > j)$ *is independent of* k, j.
(2) $\dim \mathfrak{n}_{\alpha_m/2}$ *is also independent of* m.

In a quasisymmetric Siegel domain $D(\Omega, Q)$, the algebraic structure becomes much richer. First we note that, as the complexification of V, the space W has a structure of complex Jordan algebra. Next we extend the inner product (11.1) to a complex bilinear form on $W \times W$, and denote it by the same symbol $\langle \cdot \,|\, \cdot \rangle_\kappa$. Then

$$(u_1 \,|\, u_2)_\kappa := \langle\, Q(u_1, u_2) \,|\, E \,\rangle_\kappa \qquad (u_1, u_2 \in U)$$

defines a Hermitian inner product on U. Using this inner product, we define, for each $w \in W$, a complex linear operator $\varphi(w)$ on U as follows:

$$(\varphi(w)u_1 \,|\, u_2)_\kappa = \langle\, Q(u_1, u_2) \,|\, w \,\rangle_\kappa \qquad (u_1, u_2 \in U).$$

Clearly $\varphi(E)$ is the identity operator, and we see easily that $\varphi(w^*) = \varphi(w)^*$ (the adjoint operator of $\varphi(w)$) for any $w \in W$. The following proposition comes with some surprise (see also [70] for a proof).

Proposition 11.4 (Dorfmeister [22])**.** *If D is quasisymmetric, then the linear map $\varphi : w \mapsto \varphi(w) \in \mathrm{End}_{\mathbb{C}}(U)$ is a $*$-representation of the complex Jordan algebra W. In other words, one has, with $\varphi(w^*) = \varphi(w)^*$ $(w \in W)$, the following identity*:

$$\varphi(w_1 w_2) = \frac{1}{2}\big(\varphi(w_1)\varphi(w_2) + \varphi(w_2)\varphi(w_1)\big) \qquad (w_1, w_2 \in W).$$

We now describe our way of reducing quasisymmetric domains to symmetric ones and its backgrounds. As seen above, a quasisymmetric Siegel domain consists of a Euclidean Jordan algebra and a $*$-representation of its complexified Jordan algebra. On the other hand, symmetric Siegel domains correspond in a one-to-one way to positive Hermitian Jordan triple systems (cf. [63], [88, Chapter V] etc.). We would like to present here formulas and propositions that clarify the gap between these two types of domains.

Definition 11.5. A real or complex vector space Z is called a *Jordan triple system*, if there is given a trilinear map $\{\cdot, \cdot, \cdot\} : Z \times Z \times Z \to Z$ such that

(1) $\{x, y, z\} = \{z, y, x\}$,
(2) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$.

In a Jordan triple system, we define, for each x, y, a linear operator $x \,\square\, y$ by $x \,\square\, y : z \mapsto \{x, y, z\}$. If a complex vector space Z forms a real Jordan triple system, and if $\{x, y, z\}$ is $\mathbb{C}$-linear in x, z, and anti-linear in y, then we call Z a *Hermitian Jordan triple system.* In addition, if the sesquilinear form $\operatorname{tr}(x \,\square\, y)$ obtained by taking the trace of the $\mathbb{C}$-linear operator $x \,\square\, y$ defines a Hermitian inner product in Z, then the Hermitian Jordan triple system Z is said to be *positive.*

Example 11.6. A typical example of Jordan triple system is the space of rectangular matrices $Z := \operatorname{Mat}(p, q, \mathbb{C})$. Put

$$\{x, y, z\} := \frac{1}{2}(xy^*z + zy^*x) \qquad (y^* := {}^t\overline{y}).$$

Then we see $\operatorname{tr}(x \,\square\, y) = \frac{1}{2}(p + q)\ \operatorname{tr}(xy^*)$, so that Z is a positive Hermitian Jordan triple system.

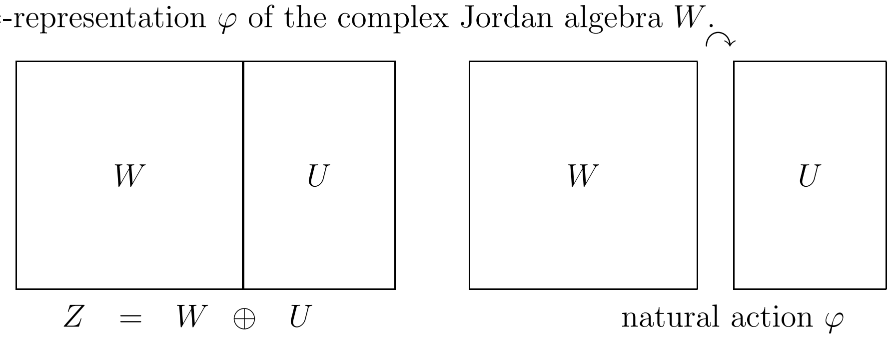
Example 11.7. Let V be a Euclidean Jordan algebra, and W its complexified Jordan algebra. We denote by $L(w)$ the multiplication operator on W by $w \in W$, and $w \mapsto w^*$ the conjugation in W relative to the real form V. If we define a triple product by

$$\{x, y, z\} := (xy^*)z + x(y^*z) - y^*(xz),$$

then we see from $\operatorname{tr}(x \,\square\, y) = \operatorname{tr} L(xy^*)$ that W is a positive Hermitian Jordan triple system.

In general, a positive Hermitian Jordan triple system Z brings us a Euclidean Jordan algebra V and a $*$-representation φ of its complexified Jordan algebra W in a canonical way. However, it would be too technical to explain this setup, and we hope that the reader can get a feeling through the process for the above Example 11.6 together with the figure below.

We suppose $p < q$ to fix the discussion, and consider the space $Z = \operatorname{Mat}(p, q, \mathbb{C})$. We extract a maximal square, here from the $(1, 1)$-entry to the (p, p)-entry, and denote by W the subspace of these square matrices. W is naturally a complex semisimple Jordan algebra, and as a consequence, it has a Euclidean real form V (a procedure exactly reverse to Example 11.7). The remaining part U has a natural action of W (in the figure below, this is the left matrix multiplication). The action gives a $*$-representation φ of the complex Jordan algebra W.



According to the decomposition of the underlying vector space $Z = U \oplus W$, the original Jordan triple product is written, together with the natural $*$-representation

φ of W on U, as follows: if $z_j = (u_j, w_j) \in Z$ $(j = 1, 2, 3)$, then we have $\{z_1, z_2, z_3\} := z = (u, w)$, where

$$\tag{11.3}\begin{cases} u := \frac{1}{2}\varphi(w_3)\varphi(w_2^*)u_1 + \frac{1}{2}\varphi(w_1)\varphi(w_2^*)u_3 \\ \qquad\qquad + \frac{1}{2}\varphi\big(Q(u_1, u_2)\big)u_3 + \frac{1}{2}\varphi\big(Q(u_3, u_2)\big)u_1, \\ w := (w_1w_2^*)w_3 + w_1(w_2^*w_3) - w_2^*(w_1w_3) \\ \qquad\qquad + \frac{1}{2}Q(u_1, \varphi(w_3^*)u_2) + \frac{1}{2}Q(u_3, \varphi(w_1^*)u_2). \end{cases}$$

Now we return to a quasisymmetric Siegel domain $D = D(\Omega, Q)$. In order for D to be symmetric, it is necessary and sufficient that the ingredients V and φ, a Euclidean Jordan algebra and a $*$-representation of the complexification W of V respectively, both come from the process described above starting with a positive Hermitian Jordan triple system. Therefore, by introducing a triple product in the ambient vector space $Z := U \oplus W$ of D by (11.3), we have to show that it is a Jordan triple product.

Proposition 11.8 (Satake [88])**.** *In the ambient vector space $Z = U \oplus W$ of a quasisymmetric Siegel domain $D = D(\Omega, Q)$, a necessary and sufficient condition for the product defined by* (11.3) *to be a Jordan triple product is that one has the following identity:*

$$\tag{11.4}\varphi(w)\varphi\big(Q(u, u')\big)u = \varphi\big(Q(\varphi(w)u, u')\big)u \qquad (u, u' \in U,\ w \in W).$$

It is known that (11.4) is equivalent to each of the following identities (see [88, Lemma V.4.6]):

$$\varphi\big(Q(u, u')\big)\varphi(w)u = \varphi\big(Q(u, \varphi(w^*)u')\big)u \qquad (u, u' \in U,\ w \in W),$$
$$Q\big(\varphi(Q(u, u'))u, u''\big) = Q\big(u, \varphi(Q(u'', u))u'\big) \qquad (u, u', u'' \in U).$$

In our analysis of norm equalities, the following criterion (see D'Atri–Dorfmeister [15] for a proof) is applied to finally get to a symmetric domain.

Proposition 11.9 (Dorfmeister)**.** *An irreducible quasisymmetric Siegel domain $D(\Omega, Q)$ is symmetric if and only if there exists a complete orthogonal system of primitive idempotents* (*Jordan frame*) $f_1, \dots, f_r$ in V such that if one puts $U_k := \varphi(f_k)U$, then one has $\varphi\big(Q(u_1, u_2)\big)u_1 = 0$ for all $u_1 \in U_1$ and $u_2 \in U_2$.

The fact that the automorphism group of the Jordan algebra V acts transitively on the set of Jordan frames has an effect on Proposition 11.9.

12. Epilogue

Since the paper Dorfmeister–Nakajima [25] solved the fundamental conjecture raised in Vinberg–Gindikin [99] without any assumptions on the automorphism group, it seems that there has been a break in the stream of geometric research on homogeneous Siegel domains. The fundamental conjecture states that every homogeneous Kähler manifold is a holomorphic fiber bundle over a homogeneous bounded domain in which the fiber is the product of a flat homogeneous Kähler manifold and a compact simply connected homogeneous Kähler manifold. The paper Penney [79] seems to take this structure into account. On the other hand, the present author's motivation for approaching analysis on homogeneous Siegel domains was rather an easy idea that it might yield a generalization of analysis on Damek–Ricci spaces ([1], [5], [9], [14], [87]) to higher rank cases[7]. Damek–Ricci spaces generalize rank one non-compact Riemannian symmetric spaces, with the rank kept one, to non-symmetric spaces, and actually they are exactly the underlying manifolds of certain split solvable Lie groups. Damek–Ricci spaces, as they are *harmonic spaces* (Riemannian manifolds in which the Laplacian admits a fundamental solution depending locally only on the geodesic distance), present many examples of non-compact harmonic spaces that are not symmetric. The problem of determining if the harmonic spaces are symmetric had been long-standing since it was mentioned and left in the last paragraph of the paper Lichnerowicz [62].

The present author's research as brought together here obtains geometric results such as symmetry characterization theorems of the domain starting with analytic problems, but it yet remains to be done to develop harmonic analysis on homogeneous Siegel domains that covers harmonic analysis on Hermitian symmetric spaces, especially to get the decomposition of the L^2-space (Plancherel formula). That was the original research motivation. It is not the non-unimodular Plancherel formula ([26], [27], [91]) of the simply transitive split solvable Lie group, but the decomposition of L^2-space as established for Damek–Ricci spaces that we would like to have. Including all of these things, there are still many analytic studies to be pursued for homogeneous Siegel domains. Geometric results in a new direction might be also produced from the approach to analytic problems.

References

1. J.-P. Anker, E. Damek and C. Yacoub, Spherical analysis on harmonic AN groups, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **23** (1996), 643–679.
2. J. Arazy and H. Upmeier, Berezin transform for solvable groups, Acta Math. Appl., **81** (2004), 5–28.
3. J. Arazy and G. Zhang, L^q-estimates of spherical functions and an invariant mean-value property, Integral Equations Operator Theory, **23** (1995), 123–144.
4. H. Asano, On the irreducibility of homogeneous convex cones, J. Fac. Sci. Univ. Tokyo, **15** (1968), 201–208.
5. F. Astengo, R. Camporesi and B. Di Blasio, The Helgason Fourier transform on a class of nonsymmetric harmonic spaces, Bull. Austral. Math. Soc., **55** (1997), 405–424.
6. K. Azukawa, Curvature operator of the Bergman metric on a homogeneous bounded domain, Tohoku Math. J. (2), **37** (1985), 197–223.
7. F. A. Berezin, Quantization, Math. USSR Izv., **8** (1974), 1109–1165.
8. F. A. Berezin, A connection between the co- and contravariant symbols of operators on classical complex symmetric spaces, Soviet Math. Dokl., **19** (1978), 786–789.
9. J. Berndt, F. Tricerri and L. Vanhecke, Generalized Heisenberg groups and Damek-Ricci harmonic spaces, Lecture Notes in Math., **1598**, Springer, Berlin, 1995.
10. A. Borel, Kählerian coset spaces of semisimple Lie groups, Proc. Nat. Acad. Sci, USA, **40** (1954), 1147–1151.

[7]The only homogeneous Siegel domains that are also Damek–Ricci spaces are those given in Example 2.1.

11. D. Buraczewski, The Hua system on irreducible Hermitian symmetric spaces of nontube type, Ann. Inst. Fourier (Grenoble), **54** (2004), 81–127.
12. E. Cartan, Sur les domaines bornés homogènes de l'espace de n variables complexes, Abh. Math. Sem. Univ. Hamburg, **11** (1935), 116–162; Œuvres complètes I (1984), 1259–1305.
13. E. Damek, A. Hulanicki and R. Penney, Admissible convergence for the Poisson–Szegö integrals, J. Geom. Anal., **5** (1995), 49–76.
14. E. Damek and F. Ricci, Harmonic analysis on solvable extensions of H-type groups, J. Geom. Anal., **2** (1992), 213–248.
15. J. E. D'Atri and J. Dorfmeister, Flat totally geodesic submanifolds of quasisymmetric Siegel domains, Geom. Dedicata, **28** (1988), 321–336.
16. J. E. D'Atri, J. Dorfmeister and Y. D. Zhao, The isotropy representation for homogeneous Siegel domains, Pacific J. Math., **120** (1985), 295–326.
17. J. E. D'Atri and I. Dotti Miatello, A characterization of bounded symmetric domains by curvature, Trans. Amer. Math. Soc., **276** (1983), 531–540.
18. G. van Dijk, On canonical representations and Berezin kernels, Banach Center Publ., **55** (2002), 253–268.
19. G. van Dijk, A new approach to Berezin kernels and canonical representations, In: Asymptotic combinatorics with application to mathematical physics, NATO Sci. Ser. II Math. Phys. Chem., **77**, Kluwer Acad. Publ., Dordrecht, 2002, 279–305.
20. G. van Dijk and M. Pevzner, Berezin kernels of tube domains, J. Funct. Anal., **181** (2001), 189–208.
21. J. Dorfmeister, Homogene Siegel-Gebiete, Habilitationsschrift, Univ. Münster, 1978.
22. J. Dorfmeister, Quasisymmetric Siegel domains and the automorphisms of homogeneous Siegel domains, Amer. J. Math., **102** (1980), 537–563.
23. J. Dorfmeister, Homogeneous Siegel domains, Nagoya Math. J., **86** (1982), 39–83.
24. J. Dorfmeister and M. Koecher, Reguläre Kegel, Jber. Deutsch. Math.-Verein., **81** (1979), 109–151.
25. J. Dorfmeister and K. Nakajima, The fundamental conjecture for homogeneous Kähler manifolds, Acta Math., **161** (1988), 23–70.
26. M. Duflo and C. C. Moore, On the regular representation of a nonunimodular locally compact group, J. Funct. Anal., **21** (1976), 209–243.
27. M. Duflo and M. Raïs, Sur l'analyse harmonique sur les groupes de Lie résolubles, Ann. Sci. Éc. Norm. Sup. (4), **9** (1976), 107–144.
28. M. Engliš, Berezin transform and the Laplace–Beltrami operator, St. Petersburg Math. J., **7** (1996), 633–647.
29. M. Engliš, Invariant operators and the Berezin transform on Cartan domains, Math. Nachr., **195** (1998), 61–75.
30. M. Engliš, A characterization of symmetric domains, to appear in J. Math. Kyoto Univ.
31. J. Faraut, S. Kaneyuki, A. Korányi, Q. Lu and G. Roos, Analysis and geometry on complex homogeneous domains, Birkhäuser, Boston, 2000.
32. J. Faraut and A. Korányi, Analysis on symmetric cones, Clarendon Press, Oxford, 1994.
33. S. Gindikin, Analysis in homogeneous domains, Russian Math. Surveys, **19-4** (1964), 1–89.
34. S. Gindikin, Invariant generalized functions in homogeneous domains, Funct. Anal. Appl., **9** (1975), 50–52.
35. S. Gindikin, Siegel domains etc..., Israel Math. Conf. Proc., **2** (1990), 5–17.
36. S. Gindikin, I. Pjatecckiĭ-Šapiro and È. B. Vinberg, Homogeneous Kähler manifolds, In: Geometry of homogeneous bounded domains, CIME, Edizioni Cremonese, Rome, 1968, 3–87.
37. J. Hano, On Kaehlerian homogeneous spaces of unimodular Lie groups, Amer. J. Math., **79** (1957), 885–900.
38. S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, New York, 1978.
39. S. Helgason, Groups and geometric analysis, Academic Press, New York, 1984.

40. S. Helgason, Geometric analysis on symmetric spaces, AMS, Providence, RI, 1994.
41. L. K. Hua, Harmonic analysis of functions of several complex variables in classical domains, Translations of Math. Monographs, Amer. Math. Soc., Providence, RI, 1963.
42. L. K. Hua and K. H. Look, Theory of harmonic functions in classical domains, Scientia Sinica, **8** (1959), 1031–1094.
43. H. Ishi, Representations of the affine transformation groups acting simply transitively on Siegel domains, J. Funct. Anal., **167** (1999), 425–462.
44. H. Ishi, Positive Riesz distributions on homogeneous cones, J. Math. Soc. Japan, **52** (2000), 161–186.
45. H. Ishi, Basic relative invariants associated to homogeneous cones and applications, J. Lie Theory, **11** (2001), 155–171.
46. H. Ishi and T. Nomura, Tube domain and an orbit of a complex triangular group, preprint.
47. C. Kai, Symmetry characterization of quasisymmetric Siegel domains by convexity of Cayley transform images, J. Lie Theory, **16** (2006), 47–56.
48. C. Kai, A characterization of symmetric Siegel domains by convexity of Cayley transform images, to appear in Tohoku Math. J.
49. C. Kai and T. Nomura, A characterization of symmetric cones through pseudoinverse maps, J. Math. Soc. Japan, **57** (2005), 195–215.
50. C. Kai and T. Nomura, A characterization of symmetric tube domains by convexity of Cayley transform images, Diff. Geom. Appl., **23** (2005), 38–54.
51. S. Kaneyuki, On the automorphism groups of homogeneous bounded domains, J. Fac. Sci. Univ. Tokyo, **14** (1967), 89–130.
52. S. Kaneyuki, Homogeneous bounded domains and Siegel domains, Lecture Notes in Math., **241**, Springer, Berlin, 1971.
53. S. Kaneyuki, Geometry of complex bounded domains, Lecture Notes Series **3** (1978), Sophia University (in Japanese).
54. S. Kaneyuki and M. Sudo, On Šilov boundaries of Siegel domains, J. Fac. Sci. Univ. Tokyo, **15** (1968), 131–146.
55. S. Kaneyuki and T. Tsuji, Classification of homogeneous bounded domains of lower dimension, Nagoya Math. J., **53** (1974), 1–46.
56. M. Koecher, The Minesota notes on Jordan algebras and their applications, Lecture Notes in Math., **1710**, Springer, Berlin, 1999.
57. A. Korányi, The Poisson integral for generalized half-planes and bounded symmetric domains, Ann. of Math. (2), **82** (1965), 332–350.
58. A. Korányi, Holomorphic and harmonic functions on bounded symmetric domains, In Geometry of homogeneous bounded domains, CIME, Edizioni Cremonese, Rome 1968, 127–197.
59. A. Korányi and E. M. Stein, H^2 spaces of generalized half-planes, Studia Math., **44** (1972), 379–388.
60. A. Korányi and J. A. Wolf, Realization of Hermitian symmetric spaces as generalized half-planes, Ann. of Math. (2), **81** (1965), 265–288.
61. J. L. Koszul, Sur la forme hermitienne canonique des espaces homogènes complexes, Canad. J. Math., **7** (1955), 562–576.
62. A. Lichnerowicz, Sur les espaces riemanniens complètement harmoniques, Bull. Soc. Math. France, **72** (1944), 146–168.
63. O. Loos, Bounded symmetric domains and Jordan pairs, Lecture Notes, Univ. California at Irvine, 1977.
64. V. F. Molchanov, Quantization on para-hermitian symmetric spaces, Amer. Math. Soc. Transl. Ser. (2), **175** (1996), 81–95.
65. S. Murakami, On automorphisms of Siegel domains, Lecture Notes in Math., **286**, Springer, Berlin, 1972.
66. Y. A. Neretin, Matrix analogues of the beta function and Plancherel's formula for Berezin kernel representations, Sb. Math., **191** (2000), 683–715.

67. Y. A. Neretin, Plancherel formula for Berezin deformation of L^2 on Riemannian symmetric space, J. Funct. Anal., **189** (2002), 336–408.
68. T. Nomura, Berezin transforms and group representations, J. Lie Theory, **8** (1998), 433–440.
69. T. Nomura, Invariant Berezin transforms, In: Harmonic analysis and integral geometry, CRC Res. Notes Math., **422** (2001), 19–40.
70. T. Nomura, On Penney's Cayley transform of a homogeneous Siegel domain, J. Lie Theory, **11** (2001), 185–206.
71. T. Nomura, A characterization of symmetric Siegel domains through a Cayley transform, Transform. Groups, **6** (2001), 227–260.
72. T. Nomura, Berezin transforms and Laplace–Beltrami operators on homogeneous Siegel domains, Diff. Geom. Appl., **15** (2001), 91–106.
73. T. Nomura, Family of Cayley transforms of a homogeneous Siegel domain parametrized by admissible linear forms, Diff. Geom. Appl., **18** (2003), 55–78.
74. T. Nomura, Geometric norm equality related to the harmonicity of the Poisson kernel, J. Funct. Anal., **198** (2003), 229–267.
75. B. Ørsted and G. Zhang, Weyl quantization and tensor products of Fock and Bergman spaces, Indiana Univ. Math. J., **43** (1994), 551–583.
76. J. Peetre, The Berezin transform and Ha-plitz operators, J. Operator Theory, **24** (1990), 165–186.
77. R. Penney, The Harish-Chandra realization for non-symmetric domains in $\mathbb{C}^n$, In: Topics in geometry in memory of Joseph D'Atri, (ed. S. Gindikin), Birkhäuser, Boston, 1996, 295–313.
78. R. Penney, The Paley–Wiener theorem for the Hua system, J. Funct. Anal., **162** (1999), 323–345.
79. R. Penney, van den Ban–Schlichtkrull–Wallach asymptotic expansions on nonsymmetric domains, Ann. of Math. (2), **158** (2003), 711–768.
80. I. Piatetski-Shapiro, Étude on life and automorphic forms in the Soviet Union, In: Selected works of Ilya Piatetski-Shapiro, (ed. J. Cogdell et al.), Amer. Math. Soc., Providence, RI,2000, 3–15.
81. I. Piatetski-Shapiro, On an estimate of the dimension of the space of automorphic forms for certain types of discrete groups, Ibid., 77–80.
82. I. Piatetski-Shapiro, On a problem proposed by E. Cartan, Ibid., 193–195.
83. I. Pyatetskii-Shapiro, Automorphic functions and the geometry of classical domains, Gordon and Breach, New York, 1969.
84. G. Roos and J.-P. Vigué, Systèmes triples de Jordan et domaines symétriques, Travaux en Cours, **43**, Hermann, Paris, 1992.
85. H. Rossi, Lectures on representations of groups of holomorphic transformations of Siegel domains, Lecture Notes, Brandeis Univ., 1972.
86. H. Rossi and M. Vergne, Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group, J. Funct. Anal., **13** (1973), 324–389.
87. F. Rouvière, Espaces de Damek–Ricci, géométrie et analyse, Sémin. Congr., **7** (2003), 45–100.
88. I. Satake, Algebraic structures of symmetric domains, Iwanami Shoten and Princeton Univ. Press, Tokyo–Princeton, 1980.
89. H. Shima, A differential geometric characterization of homogeneous self dual cones, Tsukuba J. Math., **6** (1982), 79–88.
90. M. Takeuchi, Homogeneous Siegel domains, Publ. Study Group of Geometry, **7**, Yoshida College, Kyoto University, 1973.
91. N. Tatsuuma, Plancherel formula for non-unimodular locally compact groups, J. Math. Kyoto Univ., **12** (1972), 179–261.
92. B. Trojan, Asymptotic expansions and Hua-harmonic functions on bounded homogeneous domains, Math. Ann., **336** (2006), 73–110.
93. T. Tsuji, A characterization of homogeneous self-dual cones, Tokyo J. Math., **5** (1982), 1–12.

94. T. Tsuji, On connection algebras of homogeneous convex cones, Tsukuba J. Math., **7** (1983), 69–77.
95. A. Unterberger and H. Upmeier, The Berezin transform and invariant differential operators, Comm. Math. Phys., **164** (1994), 563–597.
96. J. Vey, Sur la division des domaines de Siegel, Ann. Sci. Éc. Norm. Sup. (4), **3** (1970), 479–506.
97. È. B. Vinberg, The theory of convex homogeneous cones, Trans. Moscow Math. Soc., **12** (1963), 340–403.
98. È. B. Vinberg, The structure of the group of automorphisms of a homogeneous convex cone, Trans. Moscow Math. Soc., **13** (1965), 63–93.
99. È. B. Vinberg and S. G. Gindikin, Kählerian manifolds admitting a transitive solvable automorphism group, Mat. Sb., **74** (1967), 333–351.
100. J. A. Wolf, Fine structure of Hermitian symmetric spaces, In Symmetric spaces, Pure and App. Math., **8**, Dekker, New York, 1972, 271–357.
101. Y. Xu, On the Bergman kernel function of homogeneous bounded domains, Scientia Sinica, Special Issue (II) (1979), 80–90.
102. Y. Xu, On the classification of the homogeneous bounded domains, In Advances in science of China, Mathematics, **2**, Wiley-Intersci. Publ., Wiley, New York, 1986, 105–137.
103. Y. Xu, Theory of complex homogeneous bounded domains, Kluwer, Dordrecht, 2005.
104. G. Zhang, Berezin transform on line bundles over bounded symmetric domains, J. Lie Theory, **10** (2000), 111–126; Erratum, Idid., **11** (2001), 255.
105. G. Zhang, Berezin transform on real bounded symmetric domains, Trans. Amer. Math. Soc., **353** (2001), 3769–3787.

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