# Tube domain and an orbit of a complex triangular group

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## **Known facts**:

For $w = (w_{ij}) \in \mathrm{Mat}(r, \mathbb{C})$

$$\Delta_k(w) := \det \begin{pmatrix} w_{11} & \cdots & w_{1k} \\ \vdots & & \vdots \\ w_{k1} & \cdots & w_{kk} \end{pmatrix} \quad \text{(the $k$-th principal minor).}$$

$\Delta_0(w) \equiv 1,\ \Delta_r(w) = \det w.$

**Proposition 1.1.** *Let* $w \in \mathrm{Sym}(r, \mathbb{C})$.
*If* $\mathrm{Re}\, w \gg 0 \implies \mathrm{Re}\, \dfrac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, 2, \ldots, r).$

To prove Proposition 1.1 we need two lemmas.

**Lemma 1.2.** *Let* $w \in \mathrm{Sym}(r, \mathbb{C})$ *and* $\Delta_k(w) \neq 0$ *for* $k = 1, \ldots, r$. *Then* $w = na\,{}^t n$, *where*

$$n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}, \quad a = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{pmatrix}$$

*with* $a_k = \dfrac{\Delta_k(w)}{\Delta_{k-1}(w)}$ $(k = 1, \ldots, r)$.

(Easy induction)

**Lemma 1.3.** $\mathrm{Re}\,(na\,{}^t n) \gg 0$ $(n, a$: *as above*$)$
$\implies \mathrm{Re}\, a_1 > 0, \ldots, \mathrm{Re}\, a_r > 0.$

**Lemma 1.3.** $\operatorname{Re}(na\,{}^t\!n) \gg 0$ ($n, a$: as above)
$\implies \operatorname{Re} a_1 > 0, \dots, \operatorname{Re} a_r > 0$.

*Proof.* Induction on $r$. Clear for $r = 1$. Let $r > 1$ and assume the truth of the lemma for $r - 1$. Writing

$$n = \underbrace{\left(\begin{array}{c|c} n' & 0 \\ \hline {}^t\xi & 1 \end{array}\right)}_{r-1 \,|\, 1}, \quad a = \underbrace{\left(\begin{array}{c|c} a' & 0 \\ \hline 0 & a_r \end{array}\right)}_{r-1 \,|\, 1}, \quad \begin{array}{l} \updownarrow r-1 \\ \updownarrow 1 \end{array}$$

we obtain

$$na\,{}^t\!n = \left(\begin{array}{c|c} n'a'\,{}^t\!n' & n'a'\xi \\ \hline {}^t\xi\, a'\,{}^t\!n' & {}^t\xi\, a'\xi \,+\, a_r \end{array}\right).$$

Since $\operatorname{Re}(na\,{}^t\!n) \gg 0$, we have $\operatorname{Re}(n'a'\,{}^t\!n') \gg 0$. Then induction hypothesis yields $\operatorname{Re} a_1 > 0, \dots, \operatorname{Re} a_{r-1} > 0$. To get $\operatorname{Re} a_r > 0$, we note $\operatorname{Re}({}^t n^{-1}a^{-1}n^{-1}) \gg 0$. Then we compute:

$$\begin{aligned} {}^t n^{-1}a^{-1}n^{-1} &= \left(\begin{array}{c|c} {}^t n'^{-1} & \eta \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} a'^{-1} & 0 \\ \hline 0 & a_r^{-1} \end{array}\right)\left(\begin{array}{c|c} n'^{-1} & 0 \\ \hline {}^t\eta & 1 \end{array}\right) \\ &= \left(\begin{array}{c|c} {}^t n'^{-1}a'^{-1}n'^{-1} + \eta\, a_r^{-1}\,{}^t\eta & \eta\, a_r^{-1} \\ \hline a_r^{-1}\,{}^t\eta & a_r^{-1} \end{array}\right). \end{aligned}$$

Thus $\operatorname{Re} a_r^{-1} > 0$. Obviously this implies $\operatorname{Re} a_r > 0$. $\square$

$V := \mathrm{Sym}(r,\mathbb{R})$, $\quad \Omega := \{x \in V \ ; \ x \gg 0\}$.
$\Omega + iV$ : the corresponding tube domain in $W := V_{\mathbb{C}} = \mathrm{Sym}(r,\mathbb{C})$.
$GL(r,\mathbb{C})$ acts on $W$ by $(g,w) \mapsto gw{}^t g$.

Consider

$$A_{\mathbb{C}} := \left\{ a = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{pmatrix} \ ; \ a_1 \in \mathbb{C}^{\times}, \dots, a_r \in \mathbb{C}^{\times} \right\},$$

$$N_{\mathbb{C}} := \left\{ n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ n_{21} & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ n_{r-1,1} & n_{r-1,2} & & 1 & 0 \\ n_{r1} & n_{r2} & \cdots & n_{r,r-1} & 1 \end{pmatrix} \ ; \ n_{ij} \in \mathbb{C} \right\}.$$

$T_{\mathbb{C}} := N_{\mathbb{C}} \ltimes A_{\mathbb{C}}$

**Facts**: (1) $\Omega + iV \subset T_{\mathbb{C}} \cdot I_r$ $\quad$ ($I_r$: the $r \times r$ identity matrix).
(2) $\Delta_k(w) \neq 0 \ (k = 1, \dots, r) \iff w \in T_{\mathbb{C}} \cdot I_r$.

## Reformulation:

$$\boxed{w \in \Omega + iV \implies \mathrm{Re}\, \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \dots, r).}$$

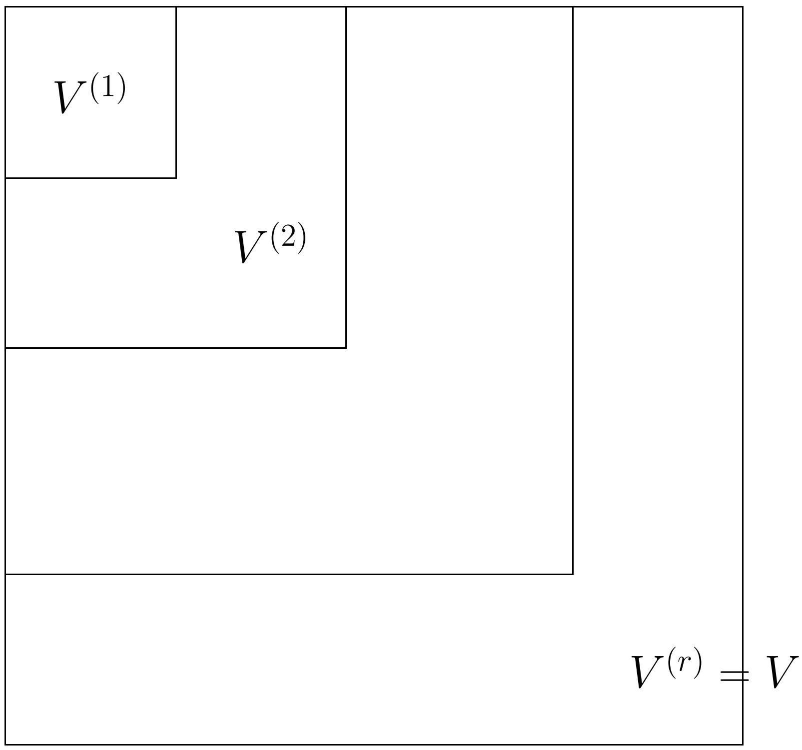
This is true for general symmetric cones $\Omega$,
$\Delta_k(w)$: Jordan algebra principal minors.

## The case of general symmetric cone

$V$ : a simple Euclidean Jordan algebra of rank $r$ with $e$.
$L(x)$ : the multiplication operator by $x \in V$ : $L(x)y := xy$.
$\langle\, x \,|\, y \,\rangle = \mathrm{tr}(xy)$ : (trace) inner product of $V$.
$c_1, \ldots, c_r$ : Jordan frame, so that $e = c_1 + \cdots + c_r$.
(Complete System of Orthogonal Primitive Idempotents)

$V^{(k)} := V(c_1 + \cdots + c_k; 1)$ : Peirce 1-space for $c_1 + \cdots + c_k$.
(the 1-eigenspace of $L(c_1 + \cdots + c_k)$).
We have $V^{(1)} \subset V^{(2)} \subset \cdots \subset V^{(r)} = V$.

$P_k$ : the orthogonal projector $V \to V^{(k)}$.
$\Delta_k(x) := \det{}^{(k)}(P_k x)$ : the $k$-th principal minor of $x$.
($\det^{(k)}$ is the determinant function of the Jordan algebra $V^{(k)}$. )




- For $\mathrm{Sym}(r, \mathbb{R})$, Jordan product is: $\frac{1}{2}(AB + BA) =: L(A)B$

Jordan frame $c_1, \dots, c_r$ gives $V = \bigoplus_{j \leqq k} V_{jk}$ with $V_{ii} = \mathbb{R}c_i$.

| | | | |
|---|---|---|---|
| $V_{11}$ | $V_{12}$ | $\cdots$ | $V_{1r}$ |
| $V_{12}$ | $V_{22}$ | $\cdots$ | $V_{2r}$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $V_{1r}$ | $V_{2r}$ | $\cdots$ | $V_{rr}$ |

$\Omega := \mathrm{Int}\{x^2 \ ; \ x \in V\}$ : the symmetric cone of $V$.
$\mathfrak{g} := \mathrm{Lie}\, G(\Omega)$, $\quad \mathfrak{k} := \mathrm{Der}\, V$, $\quad \mathfrak{p} := \{L(x) \ ; \ x \in V\}$.
Then, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (Cartan decomposition). $\quad \theta X = -{}^tX$.

$R := \mathbb{R}c_1 \oplus \cdots \oplus \mathbb{R}c_r$.
$\mathfrak{a} := \{L(a) \ ; \ a \in R\}$ : abelian and maximal in $\mathfrak{p}$.
$\alpha_1, \dots, \alpha_r$ : the basis of $\mathfrak{a}^*$ dual to $L(c_1), \dots, L(c_r)$.
The positive $\mathfrak{a}$-roots of $\mathfrak{g}$ are $\frac{1}{2}(\alpha_k - \alpha_j)$ $(k > j)$ and

$$\mathfrak{g}_{(\alpha_k - \alpha_j)/2} = \{z \,\Box\, c_j \ ; \ z \in V_{jk}\} =: \mathfrak{n}_{kj},$$

where $a \,\Box\, b := L(ab) + [\, L(a), L(b) \,]$.

Putting $\mathfrak{n} := \sum_{j<k} \mathfrak{n}_{kj}$, we get

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} \quad \text{(Iwasawa decomposition)}.$$

$A := \exp \mathfrak{a}$, $N := \exp \mathfrak{n}$.

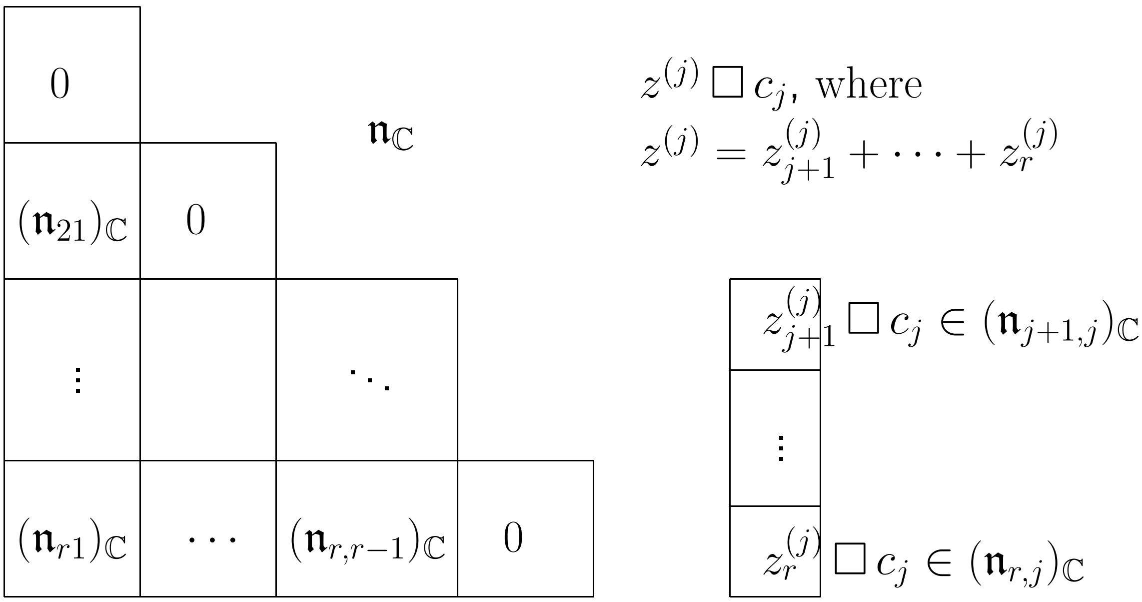
$W := V_{\mathbb{C}}$. $\quad$ Then, $W = \bigoplus_{j<k} W_{jk}$ with $W_{jk} := (V_{jk})_{\mathbb{C}}$.
$A_{\mathbb{C}}$, $N_{\mathbb{C}}$ : complexifications of $A, N$. Note $A_{\mathbb{C}}, N_{\mathbb{C}} \subset GL(W)$.
$\Delta_k$ : considered as holomorphic polynomial function on $W$.

If $c$ is an idempotent, we put

$$\tau_c(z) := \exp(2z \,\Box\, c) \quad (z \in W(c; \tfrac{1}{2})).$$

The operators $\tau_c(z)$ are called the Frobenius operators. They are unipotent.

> **Lemma 1.4.** *Suppose $w \in W$ satisfies $\Delta_k(w) \neq 0$ for any $k = 1, \ldots, r$. Then*
>
> $$\exists 1\ z^{(j)} \in \bigoplus_{m=j+1}^{r} W_{jm}\ (1 \leqq j \leqq r-1),\ a_1 \in \mathbb{C}^\times, \ldots, a_r \in \mathbb{C}^\times$$
>
> *such that*
>
> $$w = \tau_{c_1}(z^{(1)})\tau_{c_2}(z^{(2)}) \cdots \tau_{c_{r-1}}(z^{(r-1)})(a_1 c_1 + \cdots + a_r c_r).$$
>
> *Note that $\tau_{c_j}(z^{(j)}) \in N_{\mathbb{C}}$ for $j = 1, \ldots, r-1$.*



**Lemma 1.5.** *In Lemma 1.4, the numbers* $a_1, \dots, a_r$ *are given by*

$$a_k = \frac{\Delta_k(w)}{\Delta_{k-1}(w)} \qquad (k = 1, \dots, r),$$

*where* $\Delta_0(w) \equiv 1$.

**Lemma 1.6.** *Let* $n \in N_{\mathbb{C}}$ *and* $a_1 \in \mathbb{C}^\times, \dots, a_r \in \mathbb{C}^\times$. *If* $w := n \cdot (a_1 c_1 + \cdots + a_r c_r) \in \Omega + iV$, *then*

$$\operatorname{Re} a_1 > 0, \dots, \operatorname{Re} a_r > 0.$$

Lemmas 1.5 and 1.6 $\Longrightarrow$

**Proposition 1.7.** *Let* $w \in W$ *and suppose* $w \in \Omega + iV$. *Then*

$$\operatorname{Re} \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \qquad (k = 1, \dots, r).$$

**Question 1.8.**
Is Proposition 1.7 characteristic of symmetric cones?

Needs generalization of $\Delta_k$ to a homogenous convex cone $\Omega \rightsquigarrow$

$\Delta_k(x)$ : basic relative invariants associated to $\Omega$ (Ishi, 2001).
(polynomial functions on the ambient vector space $V$).
$\Delta_k(w)$ : considered as holomorphic polynomial fctns on $W := V_{\mathbb{C}}$.

- If $\Omega$ is symmetric, then $\Delta_k$ are the JA principal minors.

Now Question 1.8 can be formulated in the following way:

**Problem 1.9.** *With the above notation, the implication for $w \in V_{\mathbb{C}}$ that*

$$(*) \qquad \operatorname{Re} w \in \Omega \implies \operatorname{Re} \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \qquad (k = 1, \dots, r)$$

*is equivalent to the symmetry of $\Omega$.*

- The answer is NO.
- For non-symmetric $\Omega$, the assertion $(*)$ is generically false.

In fact, for non-symmetric $\Omega$ with dimension up to $10$, there is only one $\Omega$ for which $(*)$ holds.

$$\boxed{\begin{array}{l}\exists \text{ non-symmetric homogeneous cones } \Omega \text{ for which} \\ \operatorname{Re} w \in \Omega \implies \operatorname{Re} \dfrac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \quad (k = 1, \ldots, r).\end{array}}$$

$I_n$ : $n \times n$ unit matrix

$$V := \left\{ x = \begin{pmatrix} x_{11} I_n & x_{21} I_n & \boldsymbol{y} \\ x_{21} I_n & x_{22} I_n & \boldsymbol{z} \\ {}^t\boldsymbol{y} & {}^t\boldsymbol{z} & x_{33} \end{pmatrix} \; ; \; \begin{array}{c} \boldsymbol{y} \in \mathbb{R}^n, \\ \boldsymbol{z} \in \mathbb{R}^n, \end{array} \; x_{ij} \in \mathbb{R} \right\}.$$

Note $V \subset \mathrm{Sym}(2n+1, \mathbb{R})$.

Assuming $n \geqq 2$, we take

$$\boxed{\Omega := \{ x \in V \; ; \; x \gg 0 \}}.$$

We have $\boxed{\dim \Omega = 2n + 4}$.

Let

$$A := \left\{ a = \begin{pmatrix} a_1 I_n & 0 & 0 \\ 0 & a_2 I_n & 0 \\ 0 & 0 & a_3 \end{pmatrix} \; ; \; a_1 > 0, \; a_2 > 0, \; a_3 > 0 \right\},$$

$$N := \left\{ n = \begin{pmatrix} I_n & 0 & 0 \\ \xi I_n & I_n & 0 \\ {}^t\boldsymbol{n}_1 & {}^t\boldsymbol{n}_2 & 1 \end{pmatrix} \; ; \; \xi \in \mathbb{R}, \; \begin{array}{c} \boldsymbol{n}_1 \in \mathbb{R}^n \\ \boldsymbol{n}_2 \in \mathbb{R}^n \end{array} \right\}.$$

Then $N \ltimes A \curvearrowright \Omega$ by

$$(N \rtimes A) \times \Omega \ni (h, x) \mapsto h \, x \, {}^t h \in \Omega$$

The action is simply transitive.
Indeed, if $x \in \Omega$, then the equation $x = na{}^t n$ with $a \in A$ and $n \in N$ is solved as

$$a_1 = \Delta_1(x), \qquad a_2 = \frac{\Delta_2(x)}{\Delta_1(x)}, \qquad a_3 = \frac{\Delta_3(x)}{\Delta_2(x)},$$
$$\xi = \frac{x_{21}}{\Delta_1(x)}, \qquad \boldsymbol{n}_1 = \frac{\boldsymbol{y}}{\Delta_1(x)}, \qquad \boldsymbol{n}_2 = \frac{x_{11}\boldsymbol{z} - x_{21}\boldsymbol{y}}{\Delta_2(x)}.$$

Here, $\Delta_1, \Delta_2, \Delta_3$ are the polynomial functions on $V$ given by

$$\begin{cases} \Delta_1(x) = x_{11}, \\ \Delta_2(x) = x_{11}x_{22} - x_{21}^2, \\ \Delta_3(x) = x_{11}x_{22}x_{33} + 2x_{21}\boldsymbol{y}\cdot\boldsymbol{z} - x_{33}x_{21}^2 - x_{22}\|\boldsymbol{y}\|^2 - x_{11}\|\boldsymbol{z}\|^2, \end{cases}$$

$\boldsymbol{y}\cdot\boldsymbol{z}$ : the canonical inner product in $\mathbb{R}^n$,
$\|\cdot\|$ : the corresponding norm.

- $\Delta_1(x), \Delta_2(x), \Delta_3(x)$ are the basic relative invariants ass. to $\Omega$.

If $\delta_k(x)$ $(k = 1, \dots, 2n+1)$ stands for the $k$-th principal minor of the matrix $x = \begin{pmatrix} x_{11}I_n & x_{21}I_n & \boldsymbol{y} \\ x_{21}I_n & x_{22}I_n & \boldsymbol{z} \\ {}^t\boldsymbol{y} & {}^t\boldsymbol{z} & x_{33} \end{pmatrix} \in V$, of order $2n+1$, then

$$\delta_k(x) = \begin{cases} \Delta_1(x)^k & (1 \leqq k \leqq n), \\ \Delta_1(x)^{2n-k}\Delta_2(x)^{k-n} & (n+1 \leqq k \leqq 2n), \\ \Delta_2(x)^{n-1}\Delta_3(x) & (k = 2n+1). \end{cases}$$

Therefore

$$\boxed{x \in \Omega \iff \Delta_j(x) > 0 \text{ for any } j = 1, 2, 3.}$$

(This is true for general homogeneous cones by Ishi (2001).)

# **<u>The dual cone of $\Omega$</u>**:

$$V' := \left\{ x' = \begin{pmatrix} x'_{11} & x'_{21} & {}^t\boldsymbol{y}' \\ x'_{21} & x'_{22} & {}^t\boldsymbol{z}' \\ \boldsymbol{y}' & \boldsymbol{z}' & x'_{33} I_n \end{pmatrix} \; ; \; \begin{array}{c} \boldsymbol{y}' \in \mathbb{R}^n, \\ \boldsymbol{z}' \in \mathbb{R}^n, \end{array} \; x'_{ij} \in \mathbb{R} \right\}$$

Note $V' \subset \mathrm{Sym}(n+2, \mathbb{R})$.

$\Omega'$ is the set of positive definite ones in $V'$:

$$\boxed{\Omega' := \{x \in V' \; ; \; x \gg 0\}}.$$

The duality mapping between $V$ and $V'$ is given by

$$\langle x, x' \rangle = \sum_{j=1}^{3} x_{jj} x'_{jj} + 2\, \boldsymbol{y} \cdot \boldsymbol{y}' + 2\, \boldsymbol{z} \cdot \boldsymbol{z}' + 2 x_{21} x'_{21}$$

for $x = \begin{pmatrix} x_{11} I_n & x_{21} I_n & \boldsymbol{y} \\ x_{21} I_n & x_{22} I_n & \boldsymbol{z} \\ {}^t\boldsymbol{y} & {}^t\boldsymbol{z} & x_{33} \end{pmatrix} \in V$ and $x' = \begin{pmatrix} x'_{11} & x'_{21} & {}^t\boldsymbol{y}' \\ x'_{21} & x'_{22} & {}^t\boldsymbol{z}' \\ \boldsymbol{y}' & \boldsymbol{z}' & x'_{33} I_n \end{pmatrix} \in V'$.

Consider

$$A' := \left\{ a' = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 I_n \end{pmatrix} \; ; \; a_1 > 0, \; a_2 > 0, \; a_3 > 0 \right\},$$

$$N' := \left\{ n' = \begin{pmatrix} 1 & \xi & {}^t\boldsymbol{n}_1 \\ 0 & 1 & {}^t\boldsymbol{n}_2 \\ 0 & 0 & I_n \end{pmatrix} \; ; \; \xi \in \mathbb{R}, \; \begin{array}{c} \boldsymbol{n}_1 \in \mathbb{R}^2 \\ \boldsymbol{n}_2 \in \mathbb{R}^2 \end{array} \right\}.$$

$N' \rtimes A' \curvearrowright \Omega'$ by $(N' \rtimes A' \times \Omega') \ni (h, x) \mapsto h\, x\, {}^t h \in \Omega'$.
The action is simply transitive.

The basic relative invariants $\Delta'_k(x')$ $(k = 1, 2, 3)$ are

$$\begin{cases} \Delta'_1(x') := x'_{33} \\ \Delta'_2(x') := x'_{22}x'_{33} - \|\boldsymbol{z}'\|^2, \\ \Delta'_3(x') := (x'_{11}x'_{33} - \|\boldsymbol{y}'\|^2)(x'_{22}x'_{33} - \|\boldsymbol{z}'\|^2) - (x'_{21}x'_{33} - \boldsymbol{y}' \cdot \boldsymbol{z}')^2 \end{cases}$$

Note that $\deg \Delta'_3(x') = 4$.
If $\delta^*_k(x')$ $(k = 1, 2, \ldots, n+2)$ denotes the $k$-th principal minor taken from the right-lower corner of the matrix $x'$, then

$$\delta^*_k(x') = \begin{cases} \Delta'_1(x')^k & (1 \leqq k \leqq n), \\ \Delta'_1(x')^{n-1}\Delta'_2(x') & (k = n+1), \\ \Delta'_1(x')^{n-2}\Delta'_3(x') & (k = n+2). \end{cases}$$

Thus $\boxed{x' \in \Omega' \iff \Delta'_k(x') > 0 \quad \text{for any } k = 1, 2, 3}$.

For $n \in N$ and $a \in A$, let $n' \in N'$ and $a' \in A'$ be the elements with the same parameters $a_j$ $(j = 1, 2, 3)$, $\xi$, $\boldsymbol{n}_k$ $(k = 1, 2)$.
Put $h = na$ and $h' = a'n'$

• The fact that $\Omega'$ is dual to $\Omega$, that is,

$$\boxed{\Omega' = \left\{ x' \in \Omega' \; ; \; \langle x, x' \rangle > 0 \quad \forall x \in \overline{\Omega} \setminus \{0\} \right\}}$$

follows from $\langle hx{}^th, x' \rangle = \langle x, h'x'{}^th' \rangle$ $(x \in V,\ x' \in V')$.

• Typical phenomenon:

$w = \begin{pmatrix} 1+i & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & (1+2i)I_n \end{pmatrix} \in \Omega' + iV'$ gives

$$\mathrm{Re}\,\frac{\Delta'_3(w)}{\Delta'_2(w)} = \mathrm{Re}\{(1+i)(1+2i)\} = -1 < 0.$$

# Return to $\Omega$

Regard $\boldsymbol{y} \cdot \boldsymbol{z}$ as $\mathbb{C}$-bilinear on $\mathbb{C}^n \times \mathbb{C}^n$.
Write $\nu(\boldsymbol{y}) := \boldsymbol{y} \cdot \boldsymbol{y}$ instead of $\|\boldsymbol{y}\|^2$ (and similarly for $\nu(\boldsymbol{z})$).

$$\begin{cases} \Delta_1(x) = x_{11}, \\ \Delta_2(x) = x_{11}x_{22} - x_{21}^2, \\ \Delta_3(x) = x_{11}x_{22}x_{33} + 2x_{21}\boldsymbol{y} \cdot \boldsymbol{z} - x_{33}x_{21}^2 - x_{22}\nu(\boldsymbol{y}) - x_{11}\nu(\boldsymbol{z}), \end{cases}$$

$A_{\mathbb{C}}, N_{\mathbb{C}}$ : the complexifications of $A, N$, respectively.
We know

$$\Omega + iV \subset N_{\mathbb{C}} A_{\mathbb{C}} \cdot I_{2n+1},$$

Note $I_{2n+1} \in \Omega$. Hence $\Delta_k(x) \neq 0$ on $\Omega + iV$ for $k = 1, 2, 3$.

**Proposition 2.1.** *Suppose $w \in V_{\mathbb{C}}$ and $w \in \Omega + iV$. Then*

$$\operatorname{Re} \frac{\Delta_k(w)}{\Delta_{k-1}(w)} > 0 \qquad (k = 1, 2, 3).$$

*Proof.* Let $a \in A_{\mathbb{C}}$ and $n \in N_{\mathbb{C}}$. Then

$$na{}^t n = \begin{pmatrix} a_1 I_n & a_1 \xi I_n & a_1 \boldsymbol{n}_1 \\ \xi a_1 I_n & (\xi^2 a_1 + a_2) I_n & \xi a_1 \boldsymbol{n}_1 + a_2 \boldsymbol{n}_2 \\ a_1 {}^t\boldsymbol{n}_1 & a_1 \xi \, {}^t\boldsymbol{n}_1 + a_2 {}^t\boldsymbol{n}_2 & a_1 \nu(\boldsymbol{n}_1) + a_2 \nu(\boldsymbol{n}_2) + a_3 \end{pmatrix}.$$

Hence, given $w \in \Omega + iV$, we can solve the equation $w = na{}^t n$ for $a \in A_{\mathbb{C}}$ and $n \in A_{\mathbb{C}}$, so that

$$a_k = \frac{\Delta_k(w)}{\Delta_{k-1}(w)} \qquad (k = 1, 2, 3).$$

Since $\Omega \subset \operatorname{Pos}(2n+1, \mathbb{R})$, we can apply Lemma 1.3 to the present case with $r = 2n + 1$. Then we get $\operatorname{Re} a_k > 0$ $(k = 1, 2, 3)$. $\square$

**Lemma 1.6** Let $n \in N_{\mathbb{C}}$ and $a_1 \in \mathbb{C}, \ldots, a_r \in \mathbb{C}$.
If $w := n \cdot (a_1 c_1 + \cdots + a_r c_r) \in \Omega + iV$, then

$$\operatorname{Re} a_1 > 0, \ldots, \operatorname{Re} a_r > 0.$$

This can be generalized to nonsymmetric cases.

$\{$homogeneous regular open convex cones$\}$

$\leftrightarrow$ $\{$clans with unit element$\}$

$V$ : a real vector space with product $\triangle$ ($L_x y := x \triangle y$)
$V$ is a **clan** $\stackrel{\text{def}}{\Longleftrightarrow}$

$$\begin{cases} (1)\ [L_x, L_y] = L_{x\triangle y - y\triangle x}. \\ (2)\ \exists s \in V^* \text{ s.t. } \langle x\triangle y, s\rangle \text{ defines an inner product in } V. \\ (3)\ \text{Each } L_x \text{ has only real eigenvalues.} \end{cases}$$

Let $V$ be a clan with unit element $E$.
(1) $\rightsquigarrow$ $\mathfrak{h} := \{L_x \ ; \ x \in V\}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, and is split solvable due to (3).

$H := \exp \mathfrak{h} \subset GL(V)$, $\quad \Omega := HE$: the $H$-orbit through $E$.
$\Omega$ is a homogeneous regular open convex cone, every such arises this way.

Just as in the symmetric cases, one can find:
$E_1, \ldots, E_r$ : primitive idempotents with $E = E_1 + \cdots + E_r$

$\mathcal{A} := \mathbb{R}E_1 \oplus \cdots \oplus \mathbb{R}E_r$,
$\alpha_1, \ldots \alpha_r$ : the basis of $\mathcal{A}^*$ dual to $E_1, \ldots, E_r$.
Then, $V = \bigoplus_{k \geqq j} V_{kj}$ with $V_{ii} = \mathbb{R}E_i$ and

$$V_{kj} = \left\{ x \in V \ ; \ \begin{array}{l} L_a x = \frac{1}{2}\langle a, \alpha_k + \alpha_j \rangle x \\ R_a x = \langle a, \alpha_j \rangle x \end{array} \quad (\forall a \in \mathcal{A}) \right\}.$$

$(R_a x := x \triangle a)$

| | | | |
|---|---|---|---|
| $V_{11}$ | $V_{21}$ | $\cdots$ | $V_{r1}$ |
| $V_{21}$ | $V_{22}$ | $\cdots$ | $V_{r2}$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $V_{r1}$ | $V_{r2}$ | $\cdots$ | $V_{rr}$ |

$\mathfrak{n} := [\mathfrak{h}, \mathfrak{h}]$, $\quad N := \exp \mathfrak{n}$.
$\mathcal{A}_+ := \mathbb{R}_{>0}E_1 \oplus \cdots \oplus \mathbb{R}_{>0}E_r$.
Then $\Omega = N \cdot \mathcal{A}_+$.

$W := V_{\mathbb{C}}$,
$\mathfrak{n}_{\mathbb{C}} \subset \mathfrak{gl}(W)$ : the complexification of $\mathfrak{n}$,
$N_{\mathbb{C}} := \exp \mathfrak{n}_{\mathbb{C}} \subset GL(W)$

**Theorem 2.2.** *Let $n \in N_{\mathbb{C}}$ and $a_1 \in \mathbb{C}, \ldots, a_r \in \mathbb{C}$. If $w := n \cdot (a_1 E_1 + \cdots + a_r E_r) \in \Omega + iV$, then*

$$\operatorname{Re} a_1 > 0, \ldots, \operatorname{Re} a_r > 0.$$

Extend $x \triangle y$ to $W \times W$ by $\mathbb{C}$-bilinearity.
$R_x y := y \triangle x \; (x, y \in W)$.

**Theorem 2.3.** *The basic relative invariants*

$$\Delta_1(w), \ldots, \Delta_r(w)$$

*are the irreducible factors of the polynomial* $\det R_w$.