Symmetry of Homogeneous Siegel Domains

and

Harmonicity of the Poisson–Hua Kernel

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Motivation of this work

D : a Homogeneous Siegel domain

Σ : the Shilov boundary of D

$P(z,\zeta)$ $(z \in D,\ \zeta \in \Sigma)$:

the Poisson kernel of D defined à la Hua

$\mathscr{L}$: the Laplace–Beltrami operator of D

(with respect to the Bergman metric)

> **Theorem** (Hua-Look ('59), Korányi ('65), Xu ('79))
>
> $\mathscr{L} P(\cdot,\zeta) = 0\ \forall \zeta \in \Sigma \iff D$: symm.

D : symmetric

$\underset{\text{def}}{\iff}$ $\forall z \in D,\ \exists \sigma_z \in \mathrm{Hol}(D)$ s.t.

$$\begin{cases} \sigma_z^2 = \text{identity}, \\ z \text{ is an isolated fixed point of } \sigma_z. \end{cases}$$

[$\Leftarrow$] well known

- Hua-Look : direct and case-by-case computation for 4 classical domains

- Korányi : stronger result for general *symmetric* domains

  $P(\cdot, \zeta)$ is annihilated by *any* $\mathrm{Hol}(D)^\circ$-invariant differential operator without const. term

  ($\mathrm{Hol}(D)^\circ$ is semisimple for symmetric D)

[$\Rightarrow$] less known

- Lu Ru-Qian : An example of non-symmetric Siegel domain for which $P(\cdot, \zeta)$ is *not* killed by $\mathscr{L}$
  (Chinese Math. Acta, **7** (1965))

- Xu Yichao : though the proof is hardly traceable at least for me

(1) Needs to understand his own theory of "N-Siegel domains",

(2) Some of cited papers of his are written in Chinese

not available in English.

The purpose of this talk (my contribution)

Wants to know a geometric reason that the theorem is true

Validity of some geometric norm equality $\Longleftrightarrow$ Symmetry of the domain

Norm equality involves a Cayley transform.

In general we can consider a family of Cayley transform parametrized by admissible linear forms
(N, to appear in Diff. Geom. Appl.)

- Cayley transf. assoc. to the char. ftn of the cone
(R. Penney, 1996)
- Cayley transf. assoc. to the Szegö kernel
(N, today's talk)
- Cayley transf. assoc. to the Bergman kernel
(N, JLT, 2001)

etc. . .

Siegel Domains

V : a real vector space

$\cup$

Ω : a regular open convex cone

($\underset{\text{def}}{\Longleftrightarrow}$ contains *no* entire line)

$W := V_{\mathbb{C}}$ ($w \mapsto w^*$: conjugation w.r.t. V)

U : another *complex* vector space

$Q : U \times U \longrightarrow W$, Hermitian sesquilinear Ω-positive

i.e., $\begin{cases} Q(u', u) = Q(u, u')^* \\ Q(u, u) \in \overline{\Omega} \setminus \{0\} \ (0 \neq \forall u \in U) \end{cases}$

$$\boxed{D := \{(u, w) \in U \times W \,;\, w + w^* - Q(u, u) \in \Omega\}}$$

Siegel domain (of type II)

Assume that D is homogeneous

i.e., $\mathrm{Hol}(D) \curvearrowright D$ transitively

- If $U = \{0\}$, then $D = \Omega + iV$.

  (tube domain or type I domain)

$\exists G$: split solvable $\curvearrowright D$ simply transitively

$\mathfrak{g} := \mathrm{Lie}(G)$ has a structure of
Piatetski-Shapiro algebra (normal j-algebra)

$$\begin{cases} \exists J : \text{ integrable almost complex structure on } \mathfrak{g} \\ \exists \omega : \text{ admissible linear form on } \mathfrak{g}, \textit{ i.e.,} \\ \qquad \langle x|y\rangle_\omega := \langle [Jx, y], \omega\rangle \text{ defines a } J\text{-invariant} \\ \qquad \text{inner product on } \mathfrak{g}. \end{cases}$$

Example (Koszul '55). Koszul form.

$$\langle x, \beta\rangle := \mathrm{tr}\big(\mathrm{ad}(Jx) - J\,\mathrm{ad}(x)\big) \quad (x \in \mathfrak{g}).$$

β is admissible

- In fact, $\langle x|y\rangle_\beta$ is the real part of the Hermitian inner product defined by the Bergman metric on $D \approx G$ (up to a positive scalar multiple).

Pseudoinverse assoc. to the Szegö kernel

S : the Szegö kernel of D

(= reprod. kernel of the Hardy space)

Hardy space $H^2(D)$

holomorphic functions F on D such that

$$\sup_{t\in\Omega}\int_U\int_V \left|F\left(u, t+\tfrac{1}{2}Q(u,u)+ix\right)\right|^2 dx\,dm(u) < \infty$$

$\exists\eta$: holomorphic on $\Omega + iV$ such that

$$S(z_1,z_2) = \eta\left(w_1 + w_2^* - Q(u_1,u_2)\right)$$

$$(z_j = (u_j, w_j) \in D)$$

In more detail

$\exists H \subset G$: s.t. $H \curvearrowright \Omega$ simply transitively

$E \in \Omega$ (base pt; the id. element of the clan)

Then $H \approx \Omega$ (diffeo) by $h \mapsto hE$.

For each $\chi : H \longrightarrow \mathbb{R}_+^\times$ one dim. repre.

define Δ_χ on Ω by

$$\Delta_\chi(hE) := \chi(h) \quad (h \in H)$$

- Δ_χ extends to a holomorphic function on $\Omega + iV$ as the Laplace transform of the Riesz distribution on the dual cone Ω^* (Gindikin, Ishi (J. Math. Soc. Japan, 2000)), where

$$\Omega^* := \{\xi \in V^*; \langle x, \xi \rangle > 0 \ \forall x \in \overline{\Omega} \setminus \{0\}\}.$$

- $\exists \chi$, $\exists c > 0$ s.t. $\eta = c\Delta_\chi$

For each $x \in \Omega$, define $\mathscr{I}(x) \in V^*$ by

$$\langle v, \mathscr{I}(x) \rangle := -D_v \log \eta(x)$$

$$\left(D_v f(x) := \frac{d}{dt} f(x+tv)\Big|_{t=0}\right)$$

- $\mathscr{I}(\lambda x) = \lambda^{-1} \mathscr{I}(x) \quad (\lambda > 0)$

Prop. (1) $\mathscr{I}(x) \in \Omega^*$ and $\mathscr{I} : \Omega \twoheadrightarrow \Omega^*$ is bij.

(2) $\mathscr{I}$ extends analytically to a rational map $W \longrightarrow W^*$.

(3) One also has an explicit formula for $\mathscr{I}^{-1} : \Omega^* \twoheadrightarrow \Omega$, which continues analytically to a rational map $W^* \longrightarrow W$.
Thus $\mathscr{I}$ is birational.

(4) $\mathscr{I} : \Omega + iV \twoheadrightarrow \mathscr{I}(\Omega + iV)$ is biholo.

Remark. If $\chi : H \longrightarrow \mathbb{R}_+^\times$ is defined in a natural way by an admissible linear form, then the above proposition holds for $\mathscr{I} = \mathscr{I}_\chi$ [N, to appear in Diff. Geom. Appl.].

Cayley transform

$E^* := \mathscr{I}(E) \in \Omega^*$. $\quad \left(1 - \frac{2}{w+1} = \frac{w-1}{w+1}\right)$

$C(w) := E^* - 2\,\mathscr{I}(w+E)$ for tube domains

$$\mathscr{C}(u,w) := \underbrace{2\langle Q(u,\cdot), \mathscr{I}(w+E)\rangle}_{\in U^\dagger} \oplus \underbrace{C(w)}_{\in W^*}$$

$U^\dagger$: the space of antilinear forms on U

> Prop. (1) $\mathscr{C} : D \twoheadrightarrow \mathscr{C}(D)$ is birational and biholomorphic.
> (2) $\mathscr{C}^{-1}$ can be written explicitly.

> Theorem [N]. $\mathscr{C}(D)$ is bounded (in $U^\dagger \oplus W^*$).

Remark. (1) C_χ and $\mathscr{C}_\chi$ can be defined similarly from $\mathscr{I}_\chi$. One can prove that $\mathscr{C}_\chi(D)$ is bounded [N].

(2) For general χ, $\mathscr{C}_\chi(D)$ for symmetric D is *not* the standard Harish-Chandra model of a Hermitian symmetric space.

Norm equality

$\mathrm{e} := (0, E) \in D$: base point

$\langle x | y \rangle_\omega$: J-inv. inner prod. on $\mathfrak{g}$

$\rightsquigarrow$ Upon $G \equiv D$ by $g \mapsto g \cdot \mathrm{e}$, we have
Hermitian inner prod. on $T_{\mathrm{e}}(D) \equiv U \oplus W$

$\rightsquigarrow$ Herm. inner prod. $(\cdot | \cdot)_\omega$ and norm $\| \cdot \|_\omega$
on the 'dual' vector space $U^\dagger \oplus W^*$.

Σ : the Shilov boundary of D

$\Sigma = \{ (u, w) \in U \times W \,;\, 2\,\mathrm{Re}\, w = Q(u, u) \}$

- $\Psi_\omega \in \mathfrak{g}$: $\mathrm{tr\,ad}\,(x) = \langle x | \Psi_\omega \rangle_\omega \quad (\forall x \in \mathfrak{g})$

$S(z_1, z_2) = \eta(w_1 + w_2^* - Q(u_1, u_2))$ with $\eta = c\Delta_\chi$
$\Delta_\chi(hE) = \chi(h) = e^{-\langle \log h, \alpha \rangle} \; (\alpha \in \mathfrak{h}^* \subset \mathfrak{g}^*)$.

Theorem [N].

$\|\mathscr{C}(\zeta)\|_\omega^2 = \langle \Psi_\omega, \alpha \rangle$ for $\forall \zeta \in \Sigma$
$\iff D$ is symm. and $\omega|_{[\mathfrak{g},\mathfrak{g}]} = \gamma \cdot \beta|_{[\mathfrak{g},\mathfrak{g}]} \; (\gamma > 0)$.

$\langle x, \beta \rangle = \mathrm{tr}\,(\mathrm{ad}\,(Jx) - J\,\mathrm{ad}\,(x))$: Koszul form

Laplace–Beltrami operators

$\langle x | y \rangle_\omega$ inner prod. on $\mathfrak{g}$

$\rightsquigarrow$ left invariant Riemannian metric on G

$\rightsquigarrow$ Laplace–Beltrami operator $\mathscr{L}_\omega$ on G

Upon $G \equiv D$ by $g \mapsto g \cdot \mathrm{e}$,

we have, for $\omega = \beta$, $\mathscr{L}_\beta = c' \mathscr{L}$ $(c' > 0)$

($\mathscr{L}$: Laplace–Beltrami operator $\leftrightsquigarrow$ the Bergman metric of D).

Prop (Urakawa '79). $\mathscr{L}_\omega = -\Lambda + \Psi_\omega$.

- $\Lambda := X_1^2 + \cdots + X_{\dim \mathfrak{g}}^2 \in U(\mathfrak{g})$,
- $\{X_1, \ldots, X_{\dim \mathfrak{g}}\}$ is an ONB of $\mathfrak{g}$ w.r.t. $\langle \cdot | \cdot \rangle_\omega$
  (Λ is independent of choice of ONB.)
- $\langle \cdot | \Psi_\omega \rangle_\omega = \operatorname{tr} \operatorname{ad}(\cdot)$,
- Elements of $U(\mathfrak{g})$ are regarded as left invariant differential operators on G — thus if $X \in \mathfrak{g}$,

$$Xf(x) = \frac{d}{dt} f(x \exp tX)\Big|_{t=0}.$$

Poisson kernel

$S(z_1, z_2)$: the Szegö kernel of the Siegel domain D

We know

$$S(z_1, z_2) = \eta(w_1 + w_2^* - Q(u_1, u_2)), \quad \eta = c\Delta_\chi.$$

$S(z, \zeta)$ for $z \in D$ and $\zeta \in \Sigma$ has a meaning.

$$P(z, \zeta) := \frac{|S(z, \zeta)|^2}{S(z, z)} \quad (z \in D,\ \zeta \in \Sigma) :$$

the Poisson kernel of D

$$P_\zeta^G(g) := P(g \cdot \mathrm{e},\ \zeta) \quad (g \in G).$$

Theorem [N].

$$\mathscr{L}_\omega P_\zeta^G(e) = (-\|\mathscr{C}(\zeta)\|_\omega^2 + \langle \Psi_\omega, \alpha \rangle) P_\zeta^G(e),$$

where α is related to χ by $\chi(\exp T) = e^{-\langle T, \alpha \rangle}$.

Remark. By $P(g \cdot z,\ \zeta) = \chi(g) P(z,\ g^{-1} \cdot \zeta)$ $(g \in G)$,

$$\mathscr{L}_\omega P_\zeta^G = 0 \ \forall \zeta \in \Sigma \iff \mathscr{L}_\omega P_\zeta^G(e) = 0 \ \forall \zeta \in \Sigma.$$

Theorem. $\mathscr{L}_\omega P_\zeta^G = 0$ for $\forall \zeta \in \Sigma$

$\iff$ D is symm. and $\omega|_{[\mathfrak{g}, \mathfrak{g}]} = \gamma \cdot \beta|_{[\mathfrak{g}, \mathfrak{g}]}$ $(\gamma > 0)$.

Validity of the norm equality for symmetric D $(\omega = \beta)$

D : symmetric $\Longrightarrow$ $\mathscr{D} := \mathscr{C}(D)$ is the Harish-Chandra model of a Hermitian symmetric space

In particular, $\mathscr{D}$ is circular (Note $\mathscr{C}(\mathrm{e}) = 0$).

$\mathsf{G} := \mathrm{Hol}(\mathscr{D})^\circ$: semisimple Lie gr. (with trivial center)

$\mathsf{K} := \mathrm{Stab}_{\mathsf{G}}(0)$: maximal cpt subgr. of G

Circularity of $\mathscr{D}$ ($\Longrightarrow$ K is linear)
$+$ K-inv. of the Bergman metric
$\Longrightarrow$ K $\subset$ Unitary group

$$\begin{cases} \mathscr{C} : \Sigma \ni 0 \mapsto -E^*, \\ \text{Shilov boundary } \Sigma_{\mathscr{D}} \text{ of } \mathscr{D} = \mathsf{K} \cdot (-E^*). \end{cases}$$

Since $\Sigma_{\mathscr{D}}$ is also a G-orbit $\Sigma_{\mathscr{D}} = \mathsf{G} \cdot (-E^*)$ and since Σ is an orbit of a nilpotent subgroup of $G \subset \mathrm{Hol}(D)^\circ$, we get

$$\begin{aligned} \mathscr{C}(\Sigma) \subset \mathsf{G} \cdot (-E^*) &= \Sigma_{\mathscr{D}} \\ &= \mathsf{K} \cdot (-E^*) \\ &\subset \{z\,;\, \|z\|_\beta = \|E^*\|_\beta\}. \end{aligned}$$

We see easily that $\|E^*\|_\beta^2 = \langle \Psi_\beta, \alpha \rangle$ in this case.

Norm equality $\Longrightarrow$ symmetry of D

Assumption : $\|\mathscr{C}(\zeta)\|_\omega^2 = \langle \Psi_\omega, \alpha \rangle$ for $\forall \zeta \in \Sigma$.

(1) Reduction to a quasisymmetric domain

κ : the Bergman kernel of D

$$\left(\begin{array}{l} \kappa(z_1, z_2) = \eta_0(w_1 + w_2^* - Q(u_1, u_2)), \\ \exists \chi_0 : H \to \mathbb{R}_+^\times, \ \exists c_0 > 0 \ \text{s.t.} \ \eta_0 = c_0 \Delta_{\chi_0}, \\ \Delta_{\chi_0}(hE) = \chi_0(h): \ \Delta_{\chi_0} \rightsquigarrow \text{hol. ftn on } \Omega + iV \end{array} \right.$$

$\langle x | y \rangle_\kappa := D_x D_y \log \Delta_{\chi_0}(E)$: inner prod. of V

Def. $D = D(\Omega, Q)$ is *quasisymmetric* $\underset{\text{def}}{\Longleftrightarrow}$ Ω is selfdual w.r.t. $\langle \cdot | \cdot \rangle_\kappa$.

Define a non-associative prod. xy in V by

$$\langle xy | z \rangle_\kappa = -\tfrac{1}{2} D_x D_y D_z \log \Delta_{\chi_0}(E).$$

Prop. (Dorfmeister, D'Atri, Dotti, Vinberg).
D is quasisymmetric $\Longleftrightarrow$ prod. xy is Jordan.

In this case, V is a Euclidean Jordan algebra.

$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{n}$

$\mathfrak{a}$: abelian, $\mathfrak{n}$: sum of $\mathfrak{a}$-root spaces

(positive roots only)

Possible forms of roots:

$\frac{1}{2}(\alpha_k \pm \alpha_j)\ (j < k), \quad \alpha_k, \quad \frac{1}{2}\alpha_k$

Always $\dim \mathfrak{g}_{\alpha_k} = 1\ (\forall k)$.

Prop. (D'Atri and Dotti '83; D : irred.)

D is quasisymmetric

$$\iff \begin{cases} (1) & \dim \mathfrak{g}_{(\alpha_k+\alpha_j)/2} \text{ is indep. of } j,k, \\ (2) & \dim \mathfrak{g}_{\alpha_k/2} \text{ is indep. of } k. \end{cases}$$

Extend $\langle \cdot | \cdot \rangle_\kappa$ to a $\mathbb{C}$-bilinear form on $W \times W$.

$(u_1 \,|\, u_2)_\kappa := \langle Q(u_1, u_2) \,|\, E \rangle_\kappa$

defines a Hermitian inner product on U.

For each $w \in W$, define $\varphi(w) \in \mathrm{End}_{\mathbb{C}}(U)$ by

$$(\varphi(w)u_1 \,|\, u_2)_\kappa = \langle Q(u_1, u_2) \,|\, w \rangle_\kappa.$$

Clearly $\varphi(E) =$ identity operator on U.

Prop. (Dorfmeister). D is quasisymmetric

$\Longrightarrow$ $w \mapsto \varphi(w)$ is a Jordan $*$-repre. of $W = V_{\mathbb{C}}$

$$\begin{cases} \varphi(w^*) = \varphi(w)^*, \\ \varphi(w_1 w_2) = \frac{1}{2}\big(\varphi(w_1)\varphi(w_2) + \varphi(w_2)\varphi(w_1)\big). \end{cases}$$

(2) Reduction : quasisymm $\Longrightarrow$ symm

Quasisymmetric Siegel domain

$\leftrightarrow$ $\begin{cases} \text{Euclidean Jordan algebra } V \text{ and} \\ \text{Jordan } *\text{-representation } \varphi \text{ of } W = V_{\mathbb{C}}. \end{cases}$

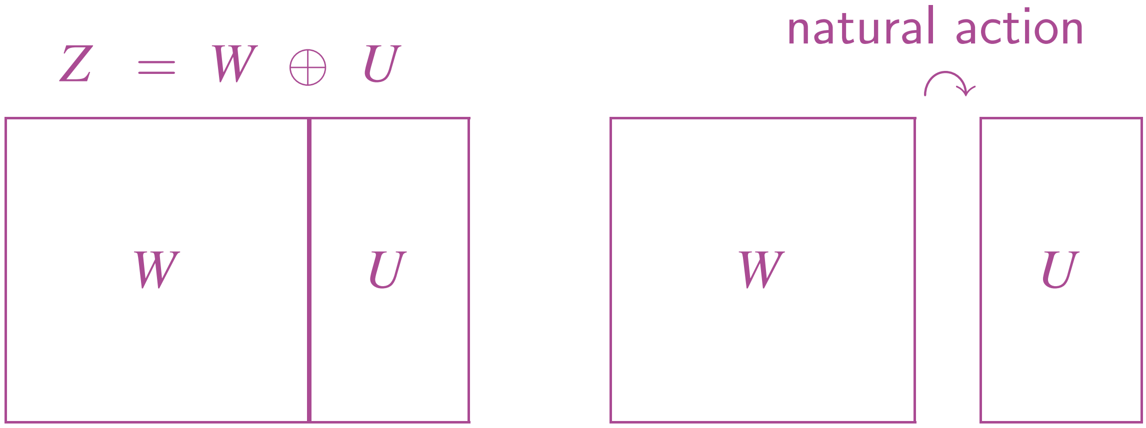
Symmetric Siegel domain

$\leftrightarrow$ Positive Hermitian JTS

The following strange formula fills the gap:

$$\varphi(w)\varphi(Q(u,u'))u = \varphi(Q(\varphi(w)u,u'))u,$$

where $u, u' \in U$ and $w \in W$.



complex semisimple Jordan algebra
$W = V_{\mathbb{C}}$
with V Euclidean JA

Jordan alg. $*$-repre. of W

Prop. (Satake). Quasisymm. D is symm.
$\iff$ V and φ come from a positive Hermitian JTS this way.

Definition of triple product

$z_j = (u_j, w_j)$ $(j = 1, 2, 3)$, $\{z_1, z_2, z_3\} := z = (u, w)$
where

$$\begin{aligned} u &:= \tfrac{1}{2}\varphi(w_3)\varphi(w_2^*)u_1 + \tfrac{1}{2}\varphi(w_1)\varphi(w_2^*)u_3 \\ &\quad + \tfrac{1}{2}\varphi(Q(u_1,u_2))u_3 + \tfrac{1}{2}\varphi(Q(u_3,u_2))u_1, \\ w &:= (w_1 w_2^*)w_3 + w_1(w_2^* w_3) - w_2^*(w_1 w_3) \\ &\quad + \tfrac{1}{2}Q(u_1, \varphi(w_3^*)u_2) + \tfrac{1}{2}Q(u_3, \varphi(w_1^*)u_2). \end{aligned}$$

Prop. (Dorfmeister).

Irreducible quasisymmetric D is symmetric

$\iff$ $\exists f_1, \ldots, f_r$: Jordan frame of V s.t.
with $U_k := \varphi(f_k)U$ we have

$$\varphi(Q(u_1,u_2))u_1 = 0$$

for $\forall u_1 \in U_1$ and $\forall u_2 \in U_2$.