Focusing on Symmetry Characterization Theorems for Homogeneous Siegel Domains

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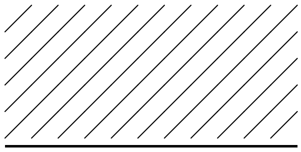
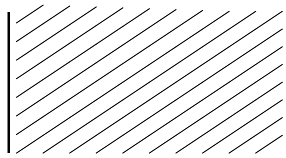
Siegel domains (Piatetski–Shapiro, 1957)

- generalization of  or 

  to higher dimensions
- holomorphically equivalent to bdd dom.

Examples.

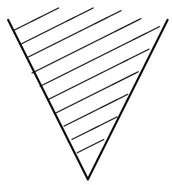
(1) $V = \mathrm{Sym}(r,\mathbb{R})$, $\Omega = \mathrm{Pos}(r,\mathbb{R})$,
$\Omega + iV$: (Siegel right half-space)

(2) $\{(u,w) \in \mathbb{C}^m \times \mathbb{C}\,;\, w + \overline{w} - \frac{1}{2}\|u\|^2 > 0\}$.

In general:

$$D = \{(u,w) \in U \times V_{\mathbb{C}}\,;\, w + \overline{w} - Q(u,u) \in \Omega\}.$$

$\Omega \subset V$: open convex cone
containing no entire line



$Q(u,v)$: $V_{\mathbb{C}}$-valued Hermitian form
Ω-positive ($Q(u,u) \in \overline{\Omega} \setminus \{0\}$ if $u \neq 0$)

- $U = \{0\}$ is allowed $\rightsquigarrow D = \Omega + iV$

Piatetski–Shapiro's motivation (1957)

Application to automorphic functions
needed a half-plane type description of
Hermitian symmetric space

Before Piatetski–Shapiro

E. Cartan : $\forall$homogeneous bdd dom in $\mathbb{C}^2$, $\mathbb{C}^3$ are
(1935) symm.

• D is symmetric
$\stackrel{\text{def}}{\Longleftrightarrow}$ $\forall z \in D$, $\exists \sigma_z \in \mathrm{Hol}(D)$ with $\sigma_z^2 = \mathrm{Id}$ s.t.
z is an isolated fixed point of σ_z.

Cartan left a question:

What happens in $\mathbb{C}^n$ for $n \geqq 4$?

1959: P.-S.'s example of non-symmetric hom. Siegel (hence bdd) domains in $\mathbb{C}^4$, $\mathbb{C}^5$.

Later: In $\mathbb{C}^n$ $(n \geqq 7)$, $\exists$mutually inequivalent non-symmetric Siegel domains with cont. param.

Symmetry characterization theorems

• Before P.-S.'s example

A. Borel (1954), L. Koszul (1955):
$\mathscr{D}$ (bdd dom.) is symm. if it is a hom. space of a semisimple Lie group
weakened to "unimodular" by J. Hano (1957)
($\exists$ left and right inv. Haar measure)

• In terms of def. data of Siegel domains

I. Satake (book, 1980)
J. Dorfmeister (Habilitationsschrift, 1978)

• Geometric conditions (curvature etc. . .)

J. D'Atri and I. Dotti (1983)
K. Azukawa (1985)

Siegel domains. — Definition —

V : a real vector space ($\dim V < \infty$)

$\cup$

Ω : a regular open convex cone

($\overset{\text{def}}{\Longleftrightarrow}$ contains *no* entire line)

$W := V_{\mathbb{C}}$ ($w \mapsto w^*$: conjugation w.r.t. V)

U : another complex vector space ($\dim U < \infty$)

$Q : U \times U \to W$, Hermitian sesquilinear Ω-positive

i.e., $\begin{cases} Q(u', u) = Q(u, u')^* \\ Q(u, u) \in \overline{\Omega} \setminus \{0\} \ (0 \neq \forall u \in U) \end{cases}$

Siegel domain (of type II)

$$D := \{(u, w) \in U \times W \ ; \ w + w^* - Q(u, u) \in \Omega\}$$

- $U = \{0\}$ is allowed. In this case $D = \Omega + iV$.

Assume that D is homogeneous

i.e., $\mathrm{Hol}(D) \curvearrowright D$ transitively.

Then Ω is also homogeneous.

D : a homogeneous Siegel domain
$\mathbf{G} := \mathrm{Hol}(D)^\circ$: identity component
Fix $\mathsf{e} \in D$, $\mathbf{K} := \mathrm{Stab}_{\mathsf{e}}\mathbf{G}$.
Then $\mathbf{K} \curvearrowright T_{\mathsf{e}}(D)$ linearly (isotropy representation)

D'Atri–Dorfmeister–Zhao's work (1985)

The following (1)∼(4) are equivalent:
(1) D is symmetric.
(2) Almost $\mathbb{C}$ structure on $T_{\mathsf{e}}(D)$ is represented by an operator of the infinitesimal isotropy representation.
(3) $\nexists$ non-trivial $\mathbf{G}$-invariant vector field.
(4) $\mathbf{D}(D)^{\mathbf{G}}$ is commutative.
(algebra of <u>$\mathbf{G}$-invariant</u> differential operators on D)

$$\begin{pmatrix} TL(g) = L(g)T \quad (\forall g \in \mathbf{G}) \\ L(g)f(x) = f(g^{-1}x) \end{pmatrix}$$

(2) is well-known for Hermitian symm. spaces.
(4) is well-known for Riemannian symm. spaces.
More is known: $\mathbf{D}(D)^{\mathbf{G}} \cong$ polynom. alg.
For Herm. symm. spaces, generators are of even degrees $\rightsquigarrow$ (3).

What is interesting here is . . .

Well-known properties for symm. spaces are already characteristic of symm. domains among homogeneous Siegel domains.

Theorem 1 (N. 2001). D : a hom. Siegel dom. Berezin transform commutes with the Laplace–Beltrami operator (w.r.t. the Bergman metric) $\Longleftrightarrow$ D is symmetric.

Remarks. • Bergman space $\rightsquigarrow$ Bergman kernel $K(z,w)$ (reproducing kernel)

$\rightsquigarrow$ Berezin kernel $\dfrac{|K(z,w)|^2}{K(z,z)K(w,w)} := A(z,w)$,

the integral kernel of the Berezin transform.

• weighted Bergman space $\rightsquigarrow$ $K(z,w)^\lambda$

$\rightsquigarrow$ (weighted) Berezin kernel $\dfrac{|K(z,w)|^{2\lambda}}{K(z,z)^\lambda K(w,w)^\lambda}$

• generalize Bergman metric to some standard Kähler metrics $\rightsquigarrow$ corresponding L.-B. operators.

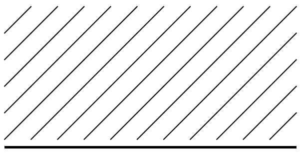
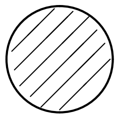
<u>Conclusion of the theorem $\rightsquigarrow$</u>

D is symm. and the metric considered is pos. number multiple of the Bergman metric.

• Berezin transform is an important operator for Berezin quantization.

• If D is symm., then Helgason's (spherical) Fourier transform. theory gives an explicit spectral decomposition of the Berezin transform.
(Berezin (1978), Unterberger–Upmeier (1994),
Arazy–Zhang (1995),. . .)

• For general D, Arazy–Upmeier (2004) made analysis by using non-unimodular Plancherel theory for simply transitive split solvable Lie group.
However, its relation to the "ordinary" spectral decomposition is not so clear
(in particular for symmetric case. . .).

In  or  the Szegö kernel S and the Poisson kernel P are related by

$$(*) \quad P(z,\zeta) = \frac{|S(z,\zeta)|^2}{S(z,z)} \quad (z \in \text{domain},\ \zeta \in \text{bdry}).$$

In a general Siegel domain D, we still have Szegö kernel. Then Hua defined Poisson kernel by $(*)$, where bdry $=$ Shilov bdry Σ:

$$\Sigma = \{(u,w)\ ;\ w + w^* - Q(u,u) = 0\}.$$

Theorem 2 (Hua–Look '59, Korányi '65, Xu '79)**.**
$\mathscr{L}$: L.-B. operator on D. Then
$\mathscr{L}P(\cdot,\zeta) = 0 \ (\forall \zeta \in \Sigma) \iff D$ is symm.
(+ metric $= c\cdot$Bergman [N., 2003]).

Remark. $\Longleftarrow$ Hua–Look (classical domains)
Korányi (general)
$\Longrightarrow$ Xu (difficult computation)

Interpretation via Cayley transform

$\exists G$: split solvable $\subset \mathrm{Aff}_{\mathbb{C}}(D)$ ($\subset \mathrm{Hol}(D)$) s.t.
$G \curvearrowright D$ simply transitively.
Fix $\mathrm{e} \in D$ Then $G \approx D$ (diffeo) by $g \mapsto g \cdot \mathrm{e}$.

Berezin transform on G.
written as right convolution operator by $a \in L^1(G)$, where $a(g) := A(g \cdot \mathrm{e}, \mathrm{e})$ and

$$Bf(x) = \int_G f(y) a(y^{-1}x)\, dy = f * a(x) \quad (f \in L^2(G)).$$

(dy : left Haar measure).

L.-B. operator $\mathscr{L}$ on G is left invariant diff. operator ($\rightsquigarrow$ expressed by an element of $U(\mathfrak{g})$, $\mathfrak{g} := \mathrm{Lie}(G)$).

Key formula.
$\mathscr{L} a(g) = (-\|\mathscr{C}(g \cdot \mathrm{e})\|^2 + c_1) a(g) \quad (\forall g \in G)$.

$c_1 > 0$: explicitly given, indep. of $g \in G$.
$\mathscr{C}$: Cayley transform $D \xrightarrow{\sim}$ bounded domain
$\|\cdot\|$ comes from the metric of D considered

Observations.

- selfadjointness of $B \rightsquigarrow a(g) = a(g^{-1})$ $(\forall g \in G)$.
- $B\mathscr{L} = \mathscr{L}B \iff \mathscr{L}a(g) = \mathscr{L}a(g^{-1})$ $(\forall g \in G)$.

Therefore

$B\mathscr{L} = \mathscr{L}B \iff \|\mathscr{C}(g\cdot \mathrm{e})\| = \|\mathscr{C}(g^{-1}\cdot \mathrm{e})\|$ $(\forall g \in G)$.

For $\mathscr{D} := \mathscr{C}(D)$, the norm equality is (note $\mathscr{C}(\mathrm{e}) = 0$):

$(**)$ $\quad \|h\cdot 0\| = \|h^{-1}\cdot 0\| \quad (\forall h \in \mathscr{C}\circ G \circ \mathscr{C}^{-1})$.

For symmetric $\mathscr{D}$, $(**)$ is just the geodesic symmetry (use Cartan decomposition of $\mathrm{Hol}(\mathscr{D})^{\circ}$).

For the unit disk in $\mathbb{C}$, we have the following picture (next sheet)

The case of unit disk $\mathbb{D} \subset \mathbb{C}$

$$SU(1,1) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \ ; \ |\alpha|^2 - |\beta|^2 = 1 \right\}$$

acts on $\mathbb{D}$ by $g \cdot z = \dfrac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \quad (z \in \mathbb{D})$.

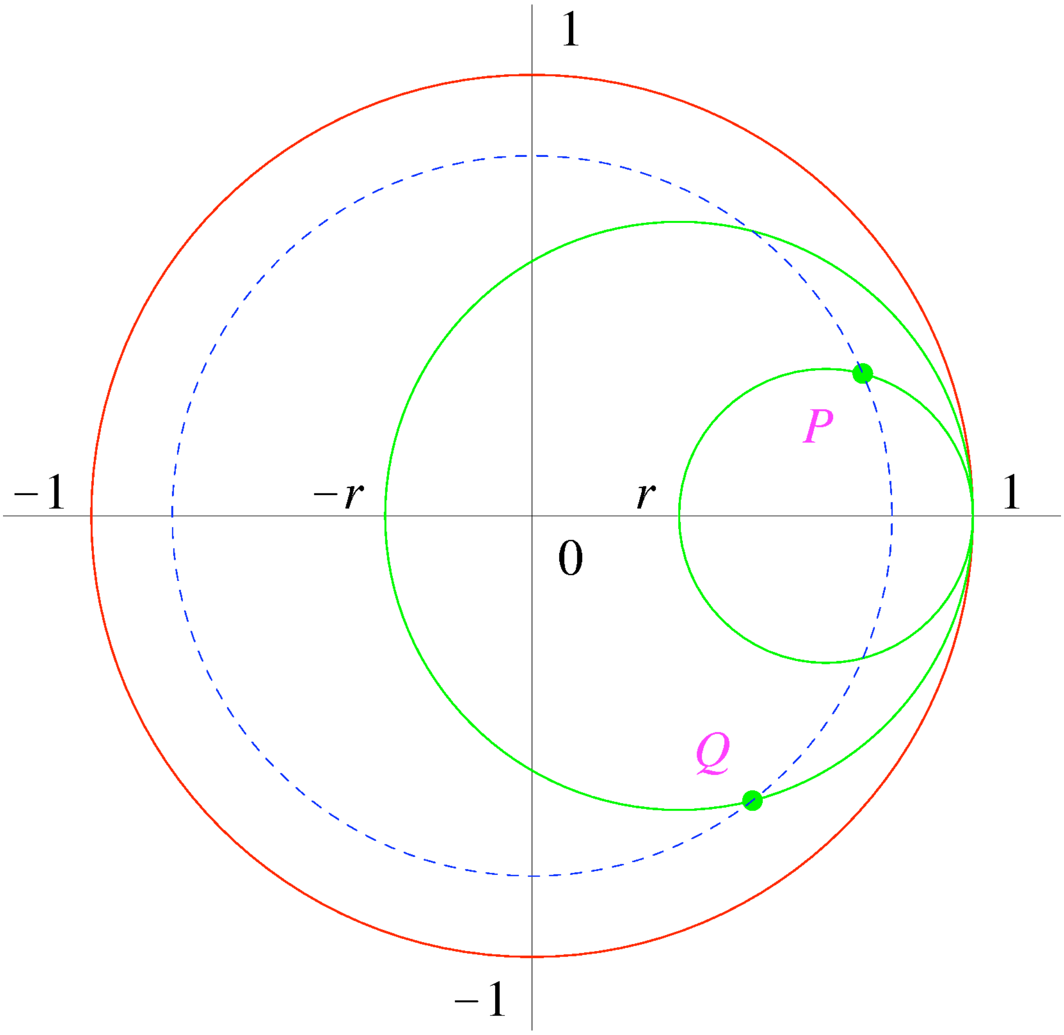
$$\begin{cases} g \cdot 0 = \dfrac{\beta}{\overline{\alpha}} \\ g^{-1} \cdot 0 = -\dfrac{\beta}{\alpha} \end{cases} \implies |g \cdot 0| = |g^{-1} \cdot 0|.$$

However, if one stays within the Iwasawa solvable subgroup, we have an interesting picture.

$$\mathsf{A} := \left\{ a_t := \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} \ ; \ t \in \mathbb{R} \right\},$$

$$\mathsf{N} := \left\{ n_\xi := \begin{pmatrix} 1 - \frac{i}{2}\xi & \frac{i}{2}\xi \\ -\frac{i}{2}\xi & 1 + \frac{i}{2}\xi \end{pmatrix} \ ; \ \xi \in \mathbb{R} \right\}.$$

Then $\mathscr{C} \circ G \circ \mathscr{C}^{-1} = \mathsf{NA}$.



$r := a_t \cdot 0 = \tanh(t/2)$

P : $n_\xi a_t \cdot 0 = n_\xi \cdot r \in \mathsf{N} \cdot r$:
horocycle emanating from $1 \in \partial\mathbb{D}$ cutting $\mathbb{R}$ at r.

Q : $(n_\xi a_t)^{-1} \cdot 0 = n_{-e^{-t}\xi} a_{-t} \cdot 0 = n_{-e^{-t}\xi} \cdot (-r)$
$\in \mathsf{N} \cdot (-r)$:
horocycle emanating from $1 \in \partial\mathbb{D}$ cutting $\mathbb{R}$ at $-r$.

Poisson kernel

$p_\zeta(g) := P(g \cdot \mathsf{e}, \zeta)$ $(g \in G, \zeta \in \Sigma)$.

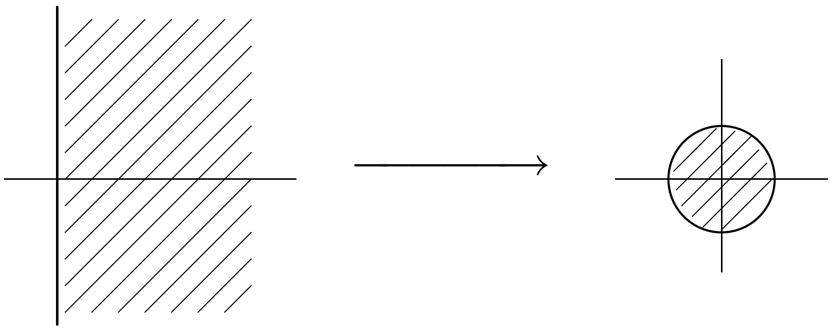
Key formula.

$$\mathscr{L} p_\zeta = \big(-\|\mathscr{C}_1(\zeta)\|^2 + c_2\big) p_\zeta.$$

$\mathscr{C}_1$: the Cayley transform (assoc. to Szegö kernel).
(previous $\mathscr{C}$ is assoc. to Bergman kernel)
c_2 : explicitly given const.(> 0) independent of ζ.

Therefore

$$\mathscr{L} p_\zeta = 0 \ (\forall \zeta \in \Sigma) \iff \|\mathscr{C}_1(\zeta)\|^2 = c_2 \ (\forall \zeta \in \Sigma).$$



Theorem 1′.

$\|\mathscr{C}(g \cdot \mathsf{e})\| = \|\mathscr{C}(g^{-1} \cdot \mathsf{e})\|$ $(\forall g \in G)$

$\iff$ D is symmetric

(and $\|\cdot\|$ is Bergman upto pos. multiple.)

Theorem 2′.

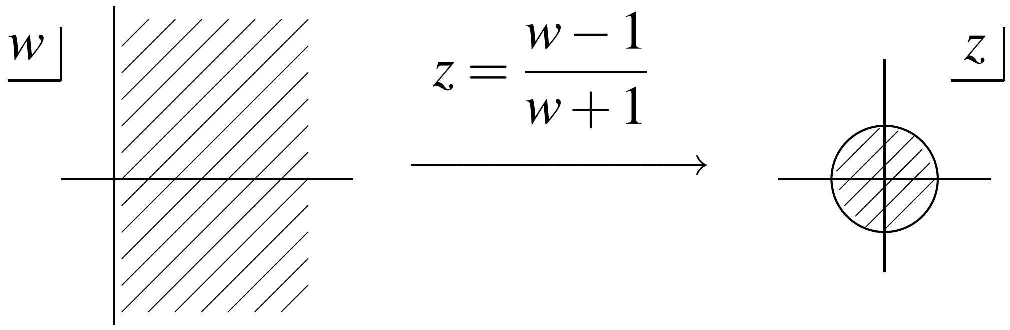
$\|\mathscr{C}_1(\zeta)\|^2 = c_2$ $(\forall \zeta \in \Sigma)$

$\iff$ D is symmetric

(and $\|\cdot\|$ is Bergman upto pos. multiple.)

Cayley transforms:

(I)



(II) $V = \mathrm{Sym}(r, \mathbb{R})$, $\Omega = \mathrm{Pos}(r, \mathbb{R})$, $\Omega + iV$

$z = (w - E)(w + E)^{-1}$ (E : unit matrix)

Siegel right half space $\ni w \mapsto z \in$ Siegel disk

(III) General symmetric tube domain

$\Omega + iV$ (Ω: a selfdual open convex cone in V)

V (hence $V_{\mathbb{C}}$) can be equipped with JA structure.

$z = (w - e)(w + e)^{-1}$ (e: the unit elemt. in V)

For $\mathrm{Sym}(r, \mathbb{C})$, JA product is:

$$\begin{cases} A \circ B = \frac{1}{2}(AB + BA), \\ \text{JA inverse} = \text{inverse matrix.} \end{cases}$$

Thus $(w - e) \circ (w + e)^{-1} = (w - e)(w + e)^{-1}$.

Symm. tube domain $\longrightarrow$ Open unit ball (w.r.t some norm)

(IV) $D := \{(u,w) \in \mathbb{C}^m \times \mathbb{C}\ ;\ w + \overline{w} - \frac{1}{2}\|u\|^2 > 0\}$.
(rank 1 (symmetric) Siegel domain)

$$\mathscr{C}(u,w) := \left(\frac{u}{w+1},\ \frac{w-1}{w+1}\right)$$

$\boxed{\mathscr{C} : D \longrightarrow \text{open unit ball in } \mathbb{C}^{m+1} = \mathbb{C}^m \times \mathbb{C}}$

(V) For general Siegel domain,
one first needs something like $(w+e)^{-1}$. Then

$$(w-e)(w+e)^{-1} = e - 2(w+e)^{-1}.$$

Recall

- $-\dfrac{d}{dt}\log\det(x+tv)^{-1}\Big|_{t=0} = \operatorname{tr}(x^{-1}v)$

$$(x \in \operatorname{Pos}(r,\mathbb{R}),\ v \in \operatorname{Sym}(r,\mathbb{R}))$$

- $\operatorname{tr}(xy)$ is the inner product to identify $\operatorname{Sym}(r,\mathbb{R})$ with the dual vector space $\operatorname{Sym}(r,\mathbb{R})^*$ so that the cone $\operatorname{Pos}(r,\mathbb{R})$ coincides with its dual cone.

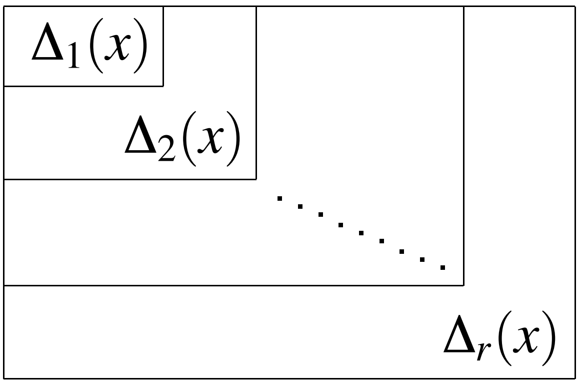
Generalization of determinant functions

(compound power functions by Gindikin)

Example. $\mathrm{Pos}(r,\mathbb{R})$: For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$

$\Delta_{\mathbf{s}}(x) := \Delta_1(x)^{s_1 - s_2} \Delta_1(x)^{s_2 - s_3} \cdots \Delta_r(x)^{s_r}$, where

$x \mapsto$



$$\Delta_1(x) := x_{11}$$

$$\Delta_2(x) := x_{11}x_{22} - x_{21}^2$$

$$\vdots$$

$$\Delta_r(x) := \det x$$

$\exists H$: split solvable $\subset G$ such that

$H \curvearrowright \Omega$ linearly and simply transitively.

Fix $E \in \Omega$. Then $H \approx \Omega$ (diffeo) by $h \mapsto hE$.

For each positive 1-dim. representation χ of H, define Δ_χ on Ω by

$$\Delta_\chi(hE) := \chi(h) \qquad (h \in H).$$

Recall $\det x \to 0$ if $\mathrm{Pos}(r,\mathbb{R}) \ni x \to x_0 \in \partial \mathrm{Pos}(r,\mathbb{R})$.

Definition. χ is <u>admissible</u>
$\overset{\text{def}}{\Longleftrightarrow}$ $\Delta_\chi(x) \to 0$ if $\Omega \ni x \to x_0 \in \partial\Omega$.

For each admissible χ, define $I_\chi(x) \in V^*$ $(x \in \Omega)$ by

$$\langle v, I_\chi(x)\rangle = -\frac{d}{dt}\log \Delta_\chi(x+tv)^{-1}\Big|_{t=0} \quad (v \in V).$$

Example. ϕ: the characteristic ftn of Ω, i.e.,

$$\phi(x) := \int_{\Omega^*} e^{-\langle x,\lambda\rangle}\, d\lambda \quad (x \in \Omega),$$

where Ω^* is the dual cone of Ω:

$$\Omega^* := \big\{\lambda \in V^*\,;\, \langle x,\lambda\rangle > 0 \;\; \forall x \in \overline{\Omega}\setminus\{0\}\big\}.$$

Then, ϕ is of the form

$$\phi(x) = C\Delta_{\chi_0}(x)^{-1} \quad (C>0 : \text{const.})$$

for some admissible χ_0. Then $I_{\chi_0}(x) = x^*$, the $*$-map defined by Vinberg. □

Facts. (1) $I_\chi : \Omega \to \Omega^*$ is a bijection.
(2) I_χ has an analytic continuation to a birational map $V_{\mathbb{C}} =: W \to W^*$.
(3) I_χ is holomorphic on $\Omega + iV$.

Theorem (Kai-N. 2005).
$I_\chi(\Omega + iV) = \Omega^* + iV^*$
$\iff$ Ω is selfdual and $\Delta_\chi(x) = \det(x)^\lambda$.
($\lambda > 0$ and det is the JA alg. determinant ftn.)

Ω is selfdual $\overset{\text{def}}{\iff}$ $\Omega = \Omega^*$ by some inner product through which we identify V^* with V.

Example.

$$\big(\mathrm{Pos}(r,\mathbb{R}) + i\,\mathrm{Sym}(r,\mathbb{R})\big)^{-1} = \mathrm{Pos}(r,\mathbb{R}) + i\,\mathrm{Sym}(r,\mathbb{R}).$$

Cayley transform for tube domains:

For admissible χ, we define

$$C_\chi(w) := I_\chi(E) - 2I_\chi(w+E) \quad (w \in \Omega + iV).$$

Cayley transform for homogeneous Siegel domains:

For admissible χ,

$$\mathscr{C}_\chi(u,w) := 2\langle Q(u,\cdot), I_\chi(w+E)\rangle \oplus C_\chi(w) \ \ ((u,w) \in D).$$

Note $U \ni u' \mapsto \langle Q(u,u'), I_\chi(w+E)\rangle$ is $\mathbb{C}$ anti-linear.

Thus $\mathscr{C}_\chi(u,w) \in U^\dagger \oplus W^*$.

($U^\dagger$: the space of anti-linear forms on U)

Theorem (N. 2003). $\mathscr{C}_\chi(D)$ is bounded.

Lemma (Gindikin 1975, Ishi 2000)**.**
For any χ, the function Δ_χ extends holomorphically to $\Omega + iV$ (Laplace transform of a distribution supported by $\overline{\Omega^*}$).

K: Bergman kernel, S: Szegö kernel
Then for $z_j = (u_j, w_j) \in D$ $(j = 1, 2)$, we have

$$K(z_1, z_2) = \Delta_{\chi_1}(w_1 + w_2^* - Q(u_1, u_2))^{-1},$$
$$S(z_1, z_2) = \Delta_{\chi_2}(w_1 + w_2^* - Q(u_1, u_2))^{-1},$$

upto positive # multiples for some admissible χ_1, χ_2.
Then we have the corresponding Cayley transforms:

$$\mathscr{C}_{\chi_1} = \mathscr{C}, \quad \mathscr{C}_{\chi_2} = \mathscr{C}_1.$$

If $\chi = \chi_0$ is s.t. $\phi = C\Delta_{\chi_0}^{-1}$, then the Cayley transform $\mathscr{C}_{\chi_0}$ is the one introduced by Penney in 1996.

Theorem (Kai, preprint 2006). $\mathscr{C}_\chi(D)$ is convex

$$\iff \begin{cases} D \text{ is symmetric,} \\ \chi \text{ is a positive power of } \chi_1. \end{cases}$$

Remark. If D is symmetric, then $C_{\chi_1}(D)$ is the Harish-Chandra model of a noncompact Hermitian symmetric space. Thus it is the open unit ball w.r.t a certain norm (the spectral norm of the underlying JTS), so that it is convex.