Right Multiplication Operators in the Clan Structure of a Euclidean Jordan Algebra

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Homogeneous Open Convex Cones

V: a real vector space with an inner product
$V \supset \Omega$: a regular open convex cone (contains no entire line)

- $G(\Omega) := \{g \in GL(V)\ ;\ g(\Omega) = \Omega\}$: linear automorphism group of Ω
  This is a Lie group as a closed subgroup of $GL(V)$.
- Ω is **homogeneous** $\overset{\text{def}}{\iff}$ $G(\Omega) \curvearrowright \Omega$ is transitive

Example: $V = \mathrm{Sym}(r, \mathbb{R}) \supset \Omega := \mathrm{Sym}(r, \mathbb{R})^{++}$:
$GL(r, \mathbb{R}) \curvearrowright \Omega$ by $GL(r, \mathbb{R}) \times \Omega \ni (g, x) \mapsto gx{}^tg \in \Omega$

This is a selfdual homogeneous open convex cone (**symmetric cone**).
Ω is **selfdual** $\overset{\text{def}}{\iff}$ $\exists\langle\cdot\,|\,\cdot\rangle$ s.t. $\Omega = \big\{y \in V\ ;\ \langle\, x \,|\, y\,\rangle > 0 \quad (\forall x \in \overline{\Omega} \setminus \{0\})\big\}$
(the RHS is the dual cone taken relative to $\langle\,\cdot\,|\,\cdot\,\rangle$)

Symmetric Cones $\rightleftarrows$ Euclidean Jordan Algebras

$\Omega \rightleftarrows V$: algebraic str. in the ambient VS ($\equiv$ tangent space at a ref. pt.)

- V with a bilinear product xy is called a **Jordan algebra** if for all $x, y \in V$
  (1) $xy = yx$,
  (2) $x^2(xy) = x(x^2y)$.

- A real Jordan algebra is said to be **Euclidean** if $\exists\langle\,\cdot\,|\,\cdot\,\rangle$ s.t.

$$\langle\, xy \,|\, z \,\rangle = \langle\, x \,|\, yz \,\rangle \qquad (\forall x, y)$$

List of Irreducible Symmetric Cones:

$\Omega = \mathrm{Sym}(r, \mathbb{R})^{++} \subset V = \mathrm{Sym}(r, \mathbb{R})$, $\quad A \circ B := \frac{1}{2}(AB + BA)$

$\Omega = \mathrm{Herm}(r, \mathbb{C})^{++} \subset V = \mathrm{Herm}(r, \mathbb{C})$

$\Omega = \mathrm{Herm}(r, \mathbb{H})^{++} \subset V = \mathrm{Herm}(r, \mathbb{H})$

$\Omega = \mathrm{Herm}(3, \mathbb{O})^{++} \subset V = \mathrm{Herm}(3, \mathbb{O})$

$\Omega = \Lambda_n \subset V = \mathbb{R}^n$ (n-dimensional Lorentz cone)

By Vinberg's theory (1963)

Homogeneous Open Convex Cones $\rightleftarrows$ Clans with unit element

$\Omega \rightleftarrows V$: algebraic str. in the ambient VS ($\equiv$ tangent space at a ref. pt.)

- V with a bilinear product $x \triangle y = L(x)y = R(y)x$ is called a **Clan** if

(1) $[L(x), L(y)] = L(x \triangle y - y \triangle x)$,
(2) $\exists s \in V^*$ s.t. $\langle x \triangle y, s \rangle$ defines an inner product,
(3) Each $L(x)$ has only real eigenvalues.

- The Case of Symmetric Cones: $G(\Omega)$ is reductive.

JA str. of V: $V \equiv T_e(\Omega) \equiv \mathfrak{p}$ of the Cartan decomposition $\mathfrak{g}(\Omega) = \mathfrak{k} + \mathfrak{p}$
Indeed $\mathfrak{p} = \{M(x) \; ; \; x \in V\}$. (The Jordan product is commutative.)

- The Case of General Homogeneous Convex Cones:

simply transitive action of Iwasawa subgroup of $G(\Omega)$

Clan str. of V: $V \equiv T_e(\Omega) \equiv$ Iwasawa subalgebra $\mathfrak{s} := \mathfrak{a} + \mathfrak{n}$ of $\mathfrak{g}(\Omega)$
Indeed $\mathfrak{s} = \{L(x) \; ; \; x \in V\}$. (The clan product is non-commutative, in general.)

Ω: homogeneous open convex cone, $G(\Omega)$: linear automorphism group of Ω,
H: Iwasawa subgroup of $G(\Omega)$.
H is a split solvable Lie group, acting simply transitively on Ω.

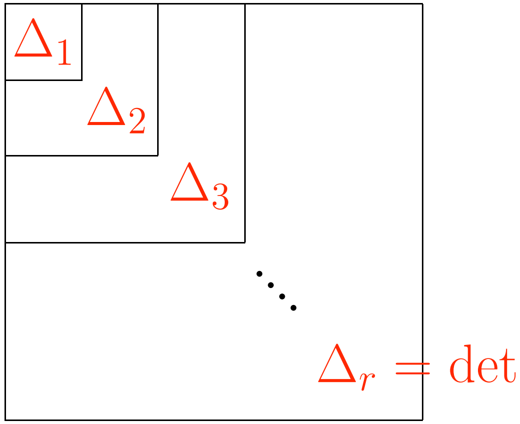
a function f on Ω, is **relatively invariant** (w.r.t. H)
$\overset{\text{def}}{\Longleftrightarrow}$ $\exists\chi$: 1-dim. rep. of H s.t. $f(gx) = \chi(g)f(x)$ (for all $g \in H$, $x \in \Omega$).

Theorem [Ishi 2001].
$\exists\Delta_1, \dots, \Delta_r$ $(r := \mathrm{rank}(\Omega))$: relat. inv. <u>irred</u>. polynomial functions on V s.t
any relat. inv. polynomial function $P(x)$ on V is written as

$$P(x) = c\,\Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r} \quad (c = \mathrm{const.},\ (m_1, \dots, m_r) \in \mathbb{Z}^r_{\geqq 0}).$$

Theorem [Ishi–N. 2008].
W: the complexification of the clan V,
$R(w)$: the right multiplication operator by w in W
$\Longrightarrow$ irreducible factors of $\det R(w)$ are just $\Delta_1(w), \dots, \Delta_r(w)$.

Example: $\Omega = \mathrm{Sym}(r,\mathbb{R})^{++} \subset V = \mathrm{Sym}(r,\mathbb{R})$



- Product in V as a clan: $x \triangle y = \underline{x}\, y + y\, {}^t(\underline{x})$, where for $x = (x_{ij}) \in \mathrm{Sym}(r,\mathbb{R})$,

we put $\underline{x} :=$

$$\begin{array}{c c} & \begin{array}{cccccc} \phantom{\tfrac{1}{2}x_{11}} & \phantom{\tfrac{1}{2}x_{22}} & i & & & \end{array} \\ \begin{array}{c} \\ \\ \\ \\ j \\ \\ \end{array} & \begin{pmatrix} \frac{1}{2}x_{11} & & & 0 & & \\ & \frac{1}{2}x_{22} & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & x_{ji} & & & \\ & & & & & \frac{1}{2}x_{nn} \end{pmatrix} \end{array}$$

$(i < j)$. Thus $x = \underline{x} + {}^t(\underline{x})$.

In this case we have $\det R(y) = \Delta_1(y) \cdots \Delta_r(y)$.

The case of general irreducible symmetric cone $\Omega \subset V$

V: a simple Euclidean Jordan algebra of rank r with unit element e,
$\Omega := \mathrm{Int}\{x^2 \,;\, x \in V\}$: the symmetric cone in V,
$G := G(\Omega)^\circ$: the connected component of $G(\Omega)$,
$\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{k} := \mathrm{Der}(V)$, $\mathfrak{p} := \{M(x) \,;\, x \in V\}$.

Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ with $\theta X = -{}^tX$,

We fix $\begin{cases} \langle\, x \,|\, y \,\rangle := \mathrm{tr}(xy)\text{: the trace inner product of } V, \\ c_1, \ldots, c_r\text{: a Jordan frame of } V\text{, so that } c_1 + \cdots + c_r = e. \end{cases}$

$V = \bigoplus_{1 \leqq j \leqq k \leqq r} V_{jk}$: the corresponding Peirce decomposition, where

$$V_{jj} := \mathbb{R}c_j \qquad (j = 1, \ldots, r),$$
$$V_{jk} := \left\{x \in V \,;\, M(c_i)x = \tfrac{1}{2}(\delta_{ij} + \delta_{ik})x \quad (i = 1, 2, \ldots, r)\right\} \quad (1 \leqq j < k \leqq r).$$

$\mathfrak{a} := \mathbb{R}M(c_1) \oplus \cdots \oplus \mathbb{R}M(c_r)$: maximal abelian in $\mathfrak{p}$,
$\alpha_1, \dots, \alpha_r$: basis of $\mathfrak{a}^*$ dual to $M(c_1), \dots, M(c_r)$.
Then the positve $\mathfrak{a}$-roots are $\frac{1}{2}(\alpha_k - \alpha_j)$ $(j < k)$,
the corresponding root spaces are described as

$$\mathfrak{n}_{kj} := \mathfrak{g}_{(\alpha_k - \alpha_j)/2} = \{ z \,\Box\, c_j \; ; \; z \in V_{jk} \} \quad (a \,\Box\, b := M(ab) + [M(a), M(b)]).$$

With $\mathfrak{n} := \sum_{j<k} \mathfrak{n}_{kj}$, we get Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.
Let $A := \exp \mathfrak{a}$, $N := \exp \mathfrak{n}$.
Then $H := N \rtimes A$ acts on Ω simply transitively, so that we have an diffeomorphism

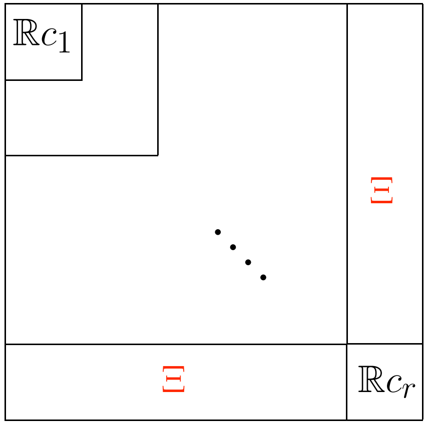
$$H \ni h \mapsto he \in \Omega.$$

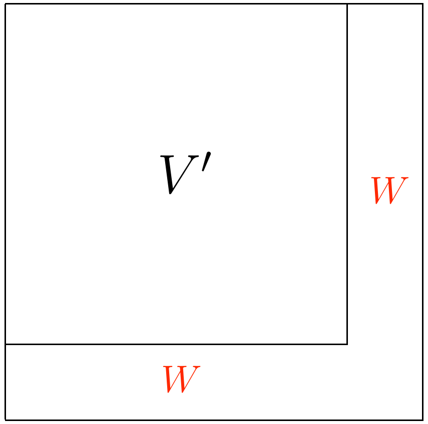
This gives rise to a linear isomorohism $\mathfrak{h} := \mathrm{Lie}(H) \ni X \mapsto Xe \in V$.
Its inverse map is denoted as $V \ni v \mapsto X_v \in \mathfrak{h}$, so that $X_v e = v$.
Now the clan $\triangle$ product in V is by definition

$$v_1 \triangle v_2 := X_{v_1} v_2 = R(v_2) v_1.$$

Lemma. (1) If $v = a_1 c_1 + \cdots + a_r c_r$ $(a_{,} \in \mathbb{R}, \dots, a_r \in \mathbb{R})$, then $X_v = M(v)$.
(2) If $v \in V_{jk}$ $(j < k)$, then $X_v = 2(v \,\Box\, c_j)$.

Let $\Xi := V_{1r} \oplus \cdots \oplus V_{r-1,r}$.





$$W := \Xi \oplus \mathbb{R}c_r, \qquad V' := \bigoplus_{1\leqq i\leqq j\leqq r-1} V_{ij}$$

Proposition. W is a two-sided ideal in the clan V:

$$X_v(W) \subset W, \qquad R_v(W) \subset W \qquad (\forall v \in V).$$

In what follows, we put $X_v^W := X_v\big|_W$, $R_v^W := R_v\big|_W$. Then

Corollary. By writing $v \in V$ as $v = v' + w$ $(v' \in V',\ w \in W)$, the operator R_v is of the form $R_v = \left(\begin{array}{c|c} R'_{v'} & O \\ \hline * & R_v^W \end{array}\right)$.

Analysis of R_v^W

Recall $V' = V_0(c_r)$, $\Xi = V_{1/2}(c_r)$, the Peirce 0- and $1/2$- spaces respectively.
The Jordan subalgebra V' has a representation ϕ on Ξ: given by $\phi(v')\xi = 2v'\xi$.
$\phi : V' \longrightarrow \mathrm{End}(\Xi)$ satisfies:

(1) $\phi(e') = \mathrm{Id}_\Xi \ \ (e' := c_1 + \cdots + c_{r-1})$,

(2) $\phi(v'_1 v'_2) = \frac{1}{2}\big(\phi(v'_1)\phi(v'_2) + \phi(v'_2)\phi(v'_1)\big)$.

On the other hand, if $v' \in V'$, we have $R_{v'}(\Xi) \subset \Xi$, so that we set $R_{v'}^{\Xi} := R_{v'}\big|_{\Xi}$.

Proposition. $R_{v'}^{\Xi} = \phi(v')$.

Proposition. By writing $v \in V$ as $v = v' + \xi + v_r c_r$ $(v' \in V', \quad \xi \in \Xi,$ $v_r \in \mathbb{R})$, the operator R_v^W is of the form

$$R_v^W = \left(\begin{array}{c|c} \phi(v') & \frac{1}{2}\langle \cdot \,|\, c_r \rangle \xi \\ \hline \langle \cdot \,|\, \xi \rangle c_r & v_r I_{\mathbb{R}c_r} \end{array} \right).$$

We renormalize the inner product in $W = \Xi + \mathbb{R}c_r$:

$$\langle \eta + y_r c_r \,|\, \eta' + y_r' c_r \rangle_W = \langle \eta \,|\, \eta' \rangle + \frac{1}{2} y_r y_r' \qquad (\eta, \eta' \in \Xi \text{ and } y_r, y_r' \in \mathbb{R}).$$

Then we have a more comfortable expression:

$$R_v^W = \left(\begin{array}{c|c} \phi(v') & \langle \cdot \,|\, c_r \rangle_W \, \xi \\ \hline \langle \cdot \,|\, \xi \rangle_W \, c_r & v_r I_{\mathbb{R}c_r} \end{array} \right) \qquad (v = v' + \xi + v_r c_r).$$

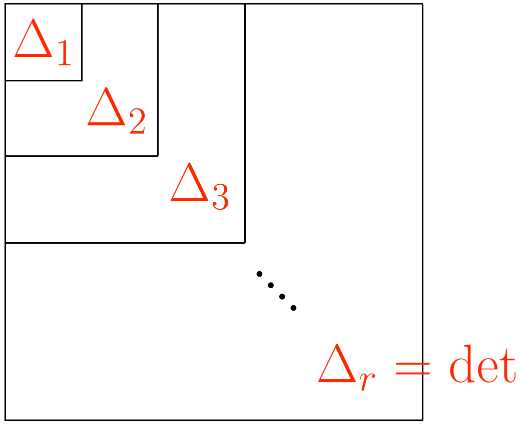
Jordan algebra principal minors

Jordan frame $c_1, \dots, c_r$ gives

$$\mathbb{R}c_1 = V^{(1)} \subset V^{(2)} \subset \cdots \subset V^{(r)} = V, \quad V^{(k)} := \bigoplus_{1 \leqq i \leqq j \leqq k} V_{ij}$$

$\det^{(k)}$: the deterninant of the Jordan algebra $V^{(k)}$,
$P^{(k)}$: the orthogonal projector $V \to V^{(k)}$,
$\Delta_k(x) := \det^{(k)}(P_k x)$: the k-th Jordan algebra prioncipal minor.

Fact. $\Delta_1, \dots, \Delta_r$ are the basic relative invariants associated to the symmetric cone Ω of V.



Theorem. $\det R(v) = \Delta_1(v)^d \cdots \Delta_{r-1}(v)^d \Delta_r(v)$, where
$d =$ common dimension of V_{ij} $(i < j)$.

$d = 1$ for $\mathrm{Sym}(r, \mathbb{R})$, $\quad d = \dim_{\mathbb{R}} \mathbb{K}$ for $\mathrm{Herm}(r, \mathbb{K})$ $(\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O})$,
$r = 2$, $d = n - 2$ for $\Omega = \Lambda_n$ $(n \geqq 3)$.

The formula is nice in view of $\dim V = r + \frac{d}{2} \cdot r(r-1)$, because

$$\deg\bigl(\Delta_1(v)^d \cdots \Delta_{r-1}(v)^d \Delta_r(v)\bigr) = d\bigl(1 + \cdots + (r-1)\bigr) + r = r + \tfrac{d}{2} \cdot r(r-1).$$

Non-selfdual Irreducible Homogeneous Convex Cones linearly isomorphic to the dual cones

- Ω is selfdual $\iff$ $\exists T$: positive definite selfadjoint operator s.t. $T(\Omega) = \Omega^*$
  $\left(\Omega^* := \left\{ y \in V \; ; \; \langle\, x \,|\, y \,\rangle > 0 \quad \text{for } \forall x \in \overline{\Omega} \setminus \{0\} \right\}\right)$

- Even though there is no positive definite selfadjoint operator T s.t. $T(\Omega) = \Omega^*$, we might find such T if we do not require the positive definiteness.

- If one accepts reducible ones, then $\Omega_0 \oplus \Omega_0^*$ just gives an example.
  Thus the irreducibility counts for much here.

- The list in [Kaneyuki–Tsuji] of irreducible homogeneous open convex cones $(\dim \leq 10)$ is described up to linear isomoprhisms. There is one non-selfdual irreducible homogeneous convex cone (7-dimensional) identified with its dual cone.

- In an exercise of Faraut–Korányi's book, a hint is given to prove that the Vinberg cone is never linerly isomorphic to its dual cone. Then the non-selfduality follows from this immediately.

We asuume $m \geqq 1$.

$$\boldsymbol{e} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{m+1}, \qquad I_{m+1}\colon\ (m+1)\times(m+1) \text{ unit matrix}$$

$$V := \left\{ x := \left(\begin{array}{c|cc} x_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}' & \boldsymbol{\xi} \\ \hline \boldsymbol{x}'\,{}^t\boldsymbol{e} & X & \boldsymbol{x}'' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}'' & x_2 \end{array}\right) \;;\; \begin{array}{lll} x_1 \in \mathbb{R}, & x_2 \in \mathbb{R}, & X \in \mathrm{Sym}(m,\mathbb{R}) \\ \boldsymbol{\xi} \in \mathbb{R}^{m+1}, & \boldsymbol{x}' \in \mathbb{R}^m, & \boldsymbol{x}'' \in \mathbb{R}^m \end{array} \right\}$$

We note $V \subset \mathrm{Sym}(2m+2,\mathbb{R})$, and take $\Omega := \{x \in V \;;\; x \gg 0\}$.

When $m = 1$, we have $x = \begin{pmatrix} x_1 & 0 & x' & \xi_1 \\ 0 & x_1 & 0 & \xi_2 \\ x' & 0 & X & x'' \\ \xi_1 & \xi_2 & x'' & x_2 \end{pmatrix}$.

Homogeneity of Ω

$$\mathbf{H} := \left\{ h := \left(\begin{array}{c|cc} h_1 I_{m+1} & 0 & 0 \\ \hline \boldsymbol{h}'\,{}^t\boldsymbol{e} & H & 0 \\ {}^t\boldsymbol{\zeta} & {}^t\boldsymbol{h}'' & h_2 \end{array} \right) \ ; \ \begin{array}{c} h_j > 0 \ (j = 1, 2),\ \boldsymbol{h}' \in \mathbb{R}^m,\ \boldsymbol{h}'' \in \mathbb{R}^m \\ \boldsymbol{\zeta} \in \mathbb{R}^{m+1}, H \in GL(m, \mathbb{R}) \end{array} \right\}$$

$\mathbf{H}$ acts on Ω by $\mathbf{H} \times \Omega \ni (h, x) \mapsto hx\,{}^th$. The action is in fact simply transitive.
We have $\mathbf{H} = \mathbf{N} \rtimes \mathbf{A}$ with

$$\mathbf{A} := \left\{ a := \left(\begin{array}{c|cc} a_1 I_{m+1} & 0 & 0 \\ \hline 0 & A & 0 \\ 0 & 0 & a_2 \end{array} \right) \ ; \ \begin{array}{c} a_j > 0 \quad (j = 1, 2), \\ A \in GL(m, \mathbb{R}) \text{ is a diagonal} \\ \text{matrix with positive diagonals} \end{array} \right\},$$

$$\mathbf{N} := \left\{ n := \left(\begin{array}{c|cc} I_{m+1} & 0 & 0 \\ \hline \boldsymbol{n}'\,{}^t\boldsymbol{e} & N & 0 \\ {}^t\boldsymbol{\nu} & {}^t\boldsymbol{n}'' & 1 \end{array} \right) \ ; \ \begin{array}{c} \boldsymbol{n}', \boldsymbol{n}'' \in \mathbb{R}^m,\ \boldsymbol{\nu} \in \mathbb{R}^{m+1} \\ N \in GL(m, \mathbb{R}) \text{ is strictly} \\ \text{lower triangular} \end{array} \right\}.$$

In solving $x = na\,{}^tn$ ($x \in \Omega$: given) we obtain basic relative invariants.

Basic relative invariants: For $x = \left(\begin{array}{c|cc} x_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}' & \boldsymbol{\xi} \\ \hline \boldsymbol{x}'\,{}^t\boldsymbol{e} & X & \boldsymbol{x}'' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}'' & x_2 \end{array}\right) \in V$,

$$\Delta_1(x) := x_1,$$

$$\Delta_j(x) := \det\left(\begin{array}{c|c} x_1 & {}^t\boldsymbol{x}'_{j-1} \\ \hline \boldsymbol{x}'_{j-1} & X_{j-1} \end{array}\right) \qquad (j = 2, \dots, m+1)$$

$$\left(X_k := \begin{pmatrix} x_{11} & \cdots & x_{k1} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{pmatrix}, \quad \boldsymbol{x}'_k := \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \end{pmatrix} \in \mathbb{R}^k \right),$$

$$\Delta_{m+2}(x) := x_1 \det\begin{pmatrix} x_1 & {}^t\boldsymbol{x}' & \xi_1 \\ \boldsymbol{x}' & X & \boldsymbol{x}'' \\ \xi_1 & {}^t\boldsymbol{x}'' & x_2 \end{pmatrix} - \left(\|\boldsymbol{\xi}\|^2 - \xi_1^2\right) \det\left(\begin{array}{c|c} x_1 & {}^t\boldsymbol{x}' \\ \hline \boldsymbol{x}' & X \end{array}\right)$$

$$\left({}^t\boldsymbol{\xi} = (\xi_1, \dots, \xi_{m+1})\right).$$

Note $\deg \Delta_{m+2} = m + 3$.

Inner product in V: For $x = \left(\begin{array}{c|cc} x_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}' & \boldsymbol{\xi} \\ \hline \boldsymbol{x}'\,{}^t\boldsymbol{e} & X & \boldsymbol{x}'' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}'' & x_2 \end{array}\right)$, $y = \left(\begin{array}{c|cc} y_1 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{y}' & \boldsymbol{\eta} \\ \hline \boldsymbol{y}'\,{}^t\boldsymbol{e} & Y & \boldsymbol{y}'' \\ {}^t\boldsymbol{\eta} & {}^t\boldsymbol{y}'' & y_2 \end{array}\right)$

$$\langle\, x \,|\, y \,\rangle := x_1 y_1 + \operatorname{tr}(XY) + x_2 y_2 + 2(\boldsymbol{x}' \cdot \boldsymbol{y}' + \boldsymbol{x}'' \cdot \boldsymbol{y}'' + \boldsymbol{\xi} \cdot \boldsymbol{\eta})$$

We consider the dual cone Ω^* taken with respect to this inner product:

$$\Omega^* := \left\{ y \in V \;;\; \langle\, x \,|\, y \,\rangle > 0 \quad \text{for } \forall x \in \overline{\Omega} \setminus \{0\} \right\}.$$

Define a linear operator T_0 on V by $T_0 x = \left(\begin{array}{c|cc} x_2 I_{m+1} & \boldsymbol{e}\,{}^t\boldsymbol{x}'' J & \boldsymbol{\xi} \\ \hline J\boldsymbol{x}''\,{}^t\boldsymbol{e} & JXJ & J\boldsymbol{x}' \\ {}^t\boldsymbol{\xi} & {}^t\boldsymbol{x}' J & x_1 \end{array}\right)$ $(x \in V)$,

where $J \in \mathrm{Sym}(m, \mathbb{R})$ is given by $J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$.

Theorem [Ishi-N. 2009]. $\Omega^* = T_0(\Omega)$.

For $h := \left(\begin{array}{c|cc} h_1 I_{m+1} & 0 & 0 \\ \hline \boldsymbol{h}'{}^t\boldsymbol{e} & H & 0 \\ {}^t\boldsymbol{\zeta} & {}^t\boldsymbol{h}'' & h_2 \end{array}\right) \in \mathbf{H}$, we set $\check{h} := \left(\begin{array}{c|cc} h_2 I_{m+1} & 0 & 0 \\ \hline J\boldsymbol{h}''{}^t\boldsymbol{e} & J{}^tHJ & 0 \\ {}^t\boldsymbol{\zeta} & {}^t\boldsymbol{h}'J & h_1 \end{array}\right)$.

Then $h \mapsto \check{h}$ is an involutive anti-automorphism of $\mathbf{H}$.

$$\rho(h)x := hx{}^th \ \ (h \in \mathbf{H},\, x \in \Omega), \qquad \sigma(h) := T_0\rho(h)T_0 \ \ (h \in \mathbf{H}).$$

Lemma. $\langle\, \rho(h)x \,|\, y \,\rangle = \langle\, x \,|\, \sigma(\check{h})y \,\rangle \quad (x, y \in V,\ h \in \mathbf{H})$.

Conjecture. Ω with $\operatorname{rank}\Omega = r$ is selfdual $\iff$ the degrees of basic relative invariants associated to Ω and Ω^* are both $1, 2, \ldots, r$.