Realization of Homogeneous Convex Cones

through Oriented Graphs

(Joint work with Takashi Yamasaki)

Takaaki Nomura

Kyushu University (Fukuoka)

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$GL(N,\mathbb{R}) \curvearrowright \mathscr{P}(N,\mathbb{R})$ transitively by $GL(N,\mathbb{R}) \times \mathscr{P}(N,\mathbb{R}) \ni (g,x) \mapsto gx\,{}^t g$

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$H^+(N,\mathbb{R}) := \{g \in GL(N,\mathbb{R})\ ;$ lower triangular with diagonals $> 0\}$

$\Longrightarrow$ the action is simply transitive (stabilizer is trivial)

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V: a real vector space $(\dim V < \infty)$ with an inner product

$V \supset \Omega$: a regular open convex cone (containing no entire line)

$GL(\Omega) := \{g \in GL(V)\ ;\ g(\Omega) = \Omega\}$: the linear automorphism group of Ω

(a Lie group as a closed subgroup of $GL(V)$)

Ω is **homogeneous** $\overset{\text{def}}{\Longleftrightarrow}$ $GL(\Omega) \curvearrowright \Omega$ is transitive.

Vinberg (1963) introduced a non-associative matrix algebra with $*$.

This algebra is called a T-algebra.

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Another purpose:

- Stop using T-algebras (too many requirements in the definition).
- Rewrite the basics in the language of **Vinberg algebras** (renamed from clans). It should have been *klans* following Russian original, for there is an English (and French) word clan.

Vinberg's theory for homogeneous convex cones

Given a homogeneous convex cone $\Omega \subset V$

$\Longrightarrow$ $\exists H$ (unique upto conjugation) split solvable s.t. $H \curvearrowright \Omega$ simply transitively.

$\Longrightarrow$ Fix $E \in \Omega$. Then $H \ni h \mapsto hE \in \Omega$ is a diffeomorphism.

$\Longrightarrow$ Its derivative at I, *i.e.*, the map $\mathfrak{h} \ni T \mapsto TE \in V$ is a linear isomorphism.
($\mathfrak{h} := \mathrm{Lie}(H)$)

$\Longrightarrow$ $\forall x \in V, \quad \exists 1\, L(x) \in \mathfrak{h}$ s.t. $L(x)E = x$.
(note: $V \ni x \mapsto L(x) \in \mathfrak{h} \subset \mathscr{L}(V)$ is also linear)

$\Longrightarrow$ We introduce a bilinear product by $x \triangle y = L(x)y$ in V.
(we do not mind the associative law)

$\Longrightarrow$ V is a **Vinberg algebra**, and E is the unit element of V.

$\Longrightarrow$ The H-orbit HE through E is an open convex cone linearly equiv. to Ω.

Vinberg Algebras (Vinberg 1963)

Definition 1

V is a real VS with a bilinear product $x \triangle y = L(x)y$.

V is a **Vinberg algebra** $\stackrel{\text{def}}{\Longleftrightarrow}$

(1) $[L(x), L(y)] = L(x \triangle y - y \triangle x)$ $(\forall x, y \in V)$,

(2) $\exists s \in V^*$ s.t. $s(x \triangle y)$ defines an inner product of V,

(3) Each $L(x)$ has only real eigenvalues.

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- Associative law is not assumed for $\triangle$.
- In this talk we always assume that V has a unit element.
- (1) $\Longleftrightarrow$ $[x, y, z] = [y, x, z]$ $(\forall x, y, z \in V)$,
  where $[x, y, z] := x \triangle (y \triangle z) - (x \triangle y) \triangle z$: the associator.
- Algebras with (1) are called left-symmetric.
- We sometimes encounter left-symmetric algebras in mathematics and physics.

Realization of homogeneous cones

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$$\Longrightarrow \; V = \bigoplus_{1 \leqq i \leqq j \leqq r} V_{ji} \; (V_{jj} = \mathbb{R}c_j; j = 1, \ldots, r):$$

the normal decomposition w.r.t. a Vinberg frame $c_1, \ldots . c_r$.

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- In short we can regard V as

$$V = \begin{pmatrix} \mathbb{R}c_1 & V_{21} & \cdots & V_{r-1,1} & V_{r1} \\ V_{21} & \mathbb{R}c_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ V_{r-1,1} & & & \mathbb{R}c_{r-1} & V_{r,r-1} \\ V_{r1} & \cdots & \cdots & V_{r,r-1} & \mathbb{R}c_r \end{pmatrix} \quad (r\text{: the rank of } \Omega \text{ or of } V)$$

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- Fix an inner product $\langle\, x \,|\, y \,\rangle := s_0(x \triangle y)$ of V.
  $\rightsquigarrow$ ① is an orthogonal decomposition.

Example: $V = \mathrm{Sym}(r, \mathbb{R})$, $\Omega = \mathscr{P}(r, \mathbb{R})$.

- $GL(r, \mathbb{R})$-action on Ω: $GL(r, \mathbb{R}) \times \Omega \ni (g, x) \mapsto gx\,{}^t g \in \Omega$
- Product in V as a Vinberg algebra:

$$x \triangle y = \underline{x}\, y + y\, {}^t(\underline{x}),$$

where for $x = (x_{ij}) \in \mathrm{Sym}(r, \mathbb{R})$,

we put $\underline{x} := \begin{pmatrix} \frac{1}{2}x_{11} & & & 0 \\ x_{21} & \frac{1}{2}x_{22} & & \\ \vdots & \ddots & \ddots & \\ x_{r1} & \cdots & x_{r,r-1} & \frac{1}{2}x_{rr} \end{pmatrix}$.

Thus $x = \underline{x} + {}^t(\underline{x})$.

- $L(x)y = R(y)x = \underline{x}\, y + y\, {}^t(\underline{x})$.

- Let $d_{ji} := \dim V_{ji}$ $(j > i)$, and draw a weighted oriented graph by defining

$$\mathscr{V} := \{1, \ldots, r\}, \quad \mathscr{A} := \{[j \to i]\ ;\ i < j, \text{ and } d_{ji} > 0\}.$$

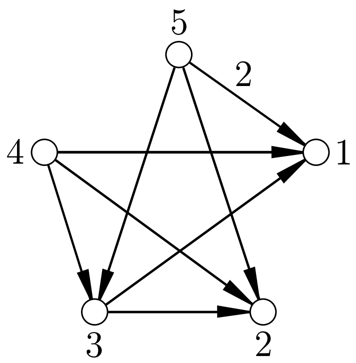
$[j \to i]$ or simply $j \to i$ deotes the arc leaving j and enters i. Thus

$$\overset{j}{\circ} \xrightarrow{d_{ji}} \overset{i}{\circ} \qquad \text{if } \dim V_{ji} > 0.$$

The graph $\Gamma = \Gamma(V) = (\mathscr{V}, \mathscr{A})$ is clearly oriented:

we do not have both $j \to i$ and $i \to j$. Moreover no $i \to i$ exists.

Example. If $d_{ji} = 1$, we do not write it in the graph for simplicity.



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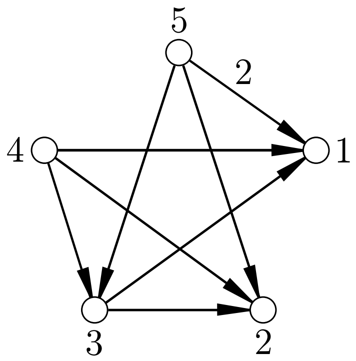

- Pick up the sources of Γ. (source $=$ vertex having no incoming arc)
  Let $\mathscr{S}$ be the source set of Γ. Note $\mathscr{S} \neq \varnothing$, since we always have $r \in \mathscr{S}$.
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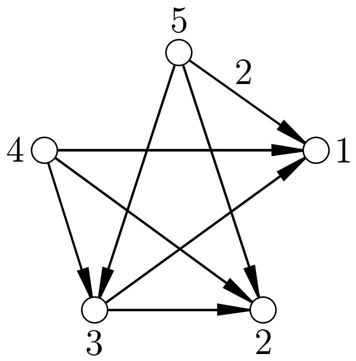

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- For each $\omega \in \mathscr{S}$ pick up its out-neighbors, i.e., the vertices k s.t. $[\omega \longrightarrow k] \in \mathscr{A}$.
  Let $N^{\mathrm{out}}(\omega) := \{$out-neighbors of $\omega\}$, and $N^{\mathrm{out}}[\omega] := N^{\mathrm{out}}(\omega) \cup \{\omega\}$.
  In the example, $N^{\mathrm{out}}[4] = \{1, 2, 3, 4\}$, $N^{\mathrm{out}}[5] = \{1, 2, 3, 5\}$.

- Form the oriented sub-graphs $\Gamma_{[\omega]}$ from $N^{\mathrm{out}}[\omega]$.

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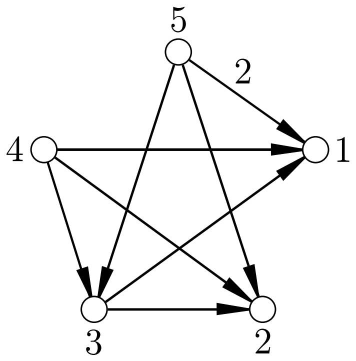

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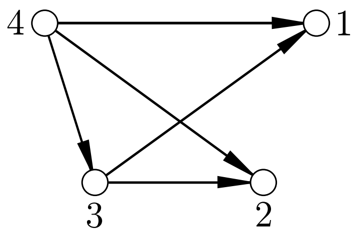
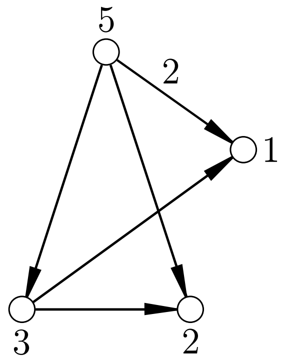


$\Gamma_{[4]}$



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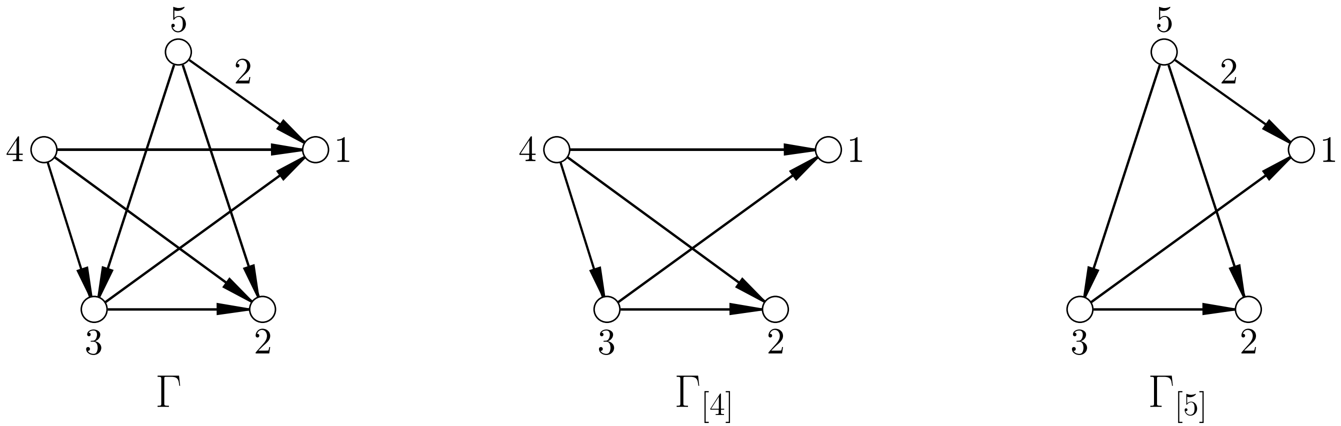


For each $\omega \in \mathscr{S}$, let

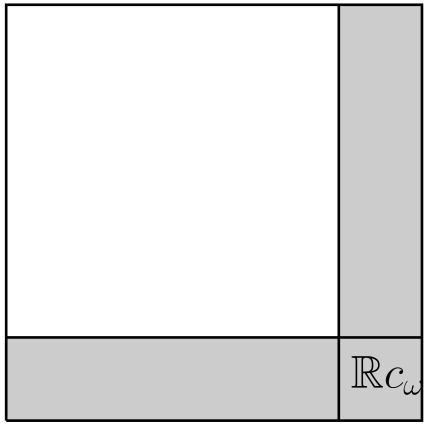
- $V_{[\omega]} := \bigoplus_{\substack{i \le j \\ i,j \in N^{\text{out}}[\omega]}} V_{ji}$. Then $V_{[\omega]}$ is a subalgebra of V

  (the source subalgebra corresponding to ω).

- $E_{[\omega]} := \bigoplus_{i \in N^{\text{out}}[\omega]} V_{\omega i}$ is a two-sided ideal of $V_{[\omega]}$.

By ignoring the unrelated entries, you just image $V_{[\omega]}$ and $E_{[\omega]}$ as

$V_{[\omega]} =$



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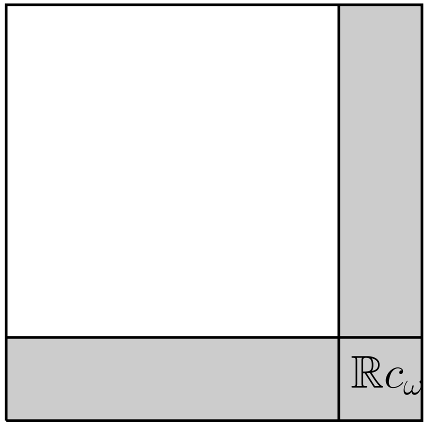

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In the above example, we have $N^{\mathrm{out}}[5] = \{1, 2, 3, 5\}$, so that

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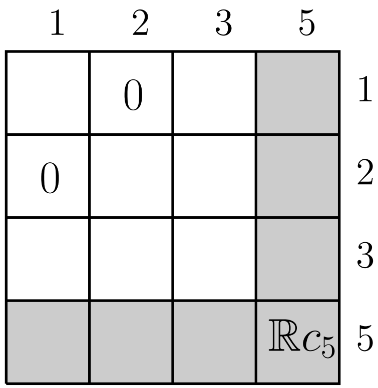

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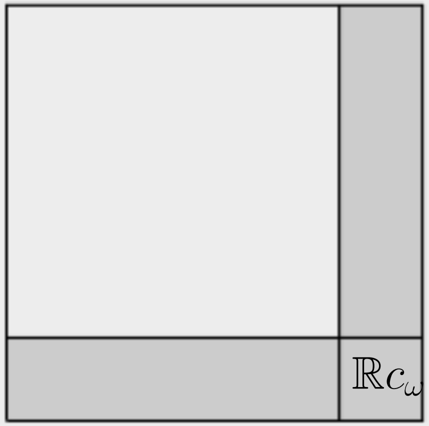

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- $\varphi_{[\omega]}(x)\eta := \eta \triangle x$ $(x \in V_{[\omega]},\ \eta \in E_{[\omega]})$.

  After a minor change of the inner product of $E_{[\omega]}$, we have $\varphi_{[\omega]}(x) \in \mathrm{Sym}(E_{[\omega]})$.

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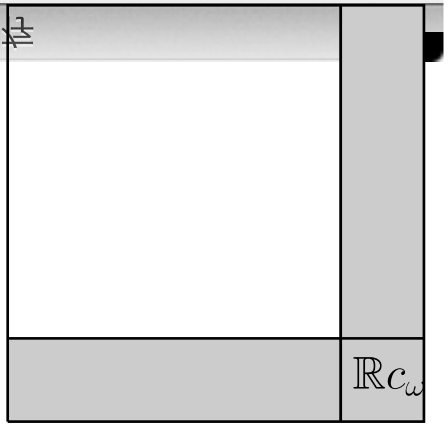

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  After a minor change of the inner product of $E_{[\omega]}$, we have $\varphi_{[\omega]}(x) \in \mathrm{Sym}(E_{[\omega]})$.
- $\Omega_{[\omega]}$: the homogeneous cone corresponding to $V_{[\omega]}$
- We name $\Omega_{[\omega]}$ the **source homogeneous cone** corresponding to the source ω.
- $V_{[\omega]} \ni x \mapsto \varphi_{[\omega]}(x)$ is faithful: $\varphi_{[\omega]}(x) = 0$ implies $x = 0$.
- $\varphi_{[\omega]}(V_{[\omega]})$ is a subalgebra of the Vinberg algebra $\mathrm{Sym}(E_{[\omega]})$.

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Theorem 2

Let $x \in V_{[\omega]}$. Then, $x \in \Omega_{[\omega]} \iff \varphi_{[\omega]}(x) \gg 0$.

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- If $\mathscr{S} = \{r\}$, we are done. We have $V = V_{[r]}$, $\Omega = \Omega_{[r]}$, and $\Omega^0_{[r]} := \varphi_{[r]}(\Omega_{[r]})$ is our realization of Ω by pos.-def. operators in $\mathrm{Sym}(E_{[r]})$.
- $\varphi_{[r]}$ intertwines the simply transitive groups
$$H \curvearrowright \Omega \quad \text{and} \quad \exp L(V^0_{[r]}) \curvearrowright \Omega^0_{[r]},$$
where $\quad V^0_{[r]} := \varphi_{[r]}(V_{[r]}) \subset \mathrm{Sym}(E_{[r]}), \quad \exp L(V^0_{[r]}) \subset GL(E_{[r]})$.
$\exp L(V^0_{[r]})$: the simply transitive Lie group with Lie algebra cosisting of the left multiplication operators of the Vinberg algebra $V^0_{[r]}$.
- $\varphi_{[r]}$ is minimal in the sense that if $\Phi : V \to \mathrm{Sym}(N, \mathbb{R})$ is an injective LSA homomorphism, then $N \geq \dim E_{[r]}$.

In general, we have

Proposition 3

Let $x \in V$. Then, with $\pi_{[\omega]} : V \to V_{[\omega]}$: orthogonal projector,

(1) $x = 0 \iff \varphi_{[\omega]}(\pi_{[\omega]}(x)) = 0$ for $\forall \omega \in \mathscr{S}$.

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Our next task is to assemble $\Omega^0_{[\omega]} := \varphi_{[\omega]}(\Omega_{[\omega]}) \; (\omega \in \mathscr{S})$.

Let $\mathscr{S} = \{\omega_1, \ldots, \omega_s\}$ $(s > 1)$.

$V^0_{[\omega_i]} := \varphi_{[\omega_i]}(V_{[\omega_i]}) \subset \mathrm{Sym}(E_{[\omega_i]})$.

$V^0 := V^0_{[\omega_1]} \oplus \cdots \oplus V^0_{[\omega_s]}$: the outer dierct sum vector space of $V^0_{[\omega_i]}$ $(i = 1, \ldots, s)$.

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Recall $\pi_{[\omega]} : V \longrightarrow V_{[\omega]}$, the orthogoal projector.

$V^0_{[\mathscr{S}]} := \left\{(X_1, \ldots, X_s) \in V^0 \;;\; \pi_{[\omega_j]} \circ \varphi^{-1}_{[\omega_i]}(X_i) = \pi_{[\omega_i]} \circ \varphi^{-1}_{[\omega_j]}(X_j) \text{ for any } i \neq j\right\}$.

We write $V^0_{[\mathscr{S}]} = \left[V^0_{[\omega_1]}, \ldots, V^0_{[\omega_s]}\right]$, which we call the **stapling** of $V^0_{[\omega_1]}, \ldots, V^0_{[\omega_s]}$.

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- For $s = 2$, you just observe $(W := V_{[\omega_1]} \cap V_{[\omega_2]})$

$$V^0_{[\omega_1]} \oplus V^0_{[\omega_2]} = \varphi_{[\omega_1]}\big(W + (V_{[\omega_1]} \cap W^\perp)\big) \oplus \varphi_{[\omega_2]}\big(W + (V_{[\omega_2]} \cap W^\perp)\big)$$

and thus

$$\left[V^0_{[\omega_1]}, V^0_{[\omega_2]}\right] = \left\{\big(\varphi_{[\omega_1]}(w + x_1),\ \varphi_{[\omega_2]}(w + x_2)\big) \ ; \ w \in W,\ x_j \in V_{[\omega_j]} \cap W^\perp\right\}.$$

Accordingly define a linear isomorphism $\varphi_{[\mathscr{S}]} : V \to V^0_{[\mathscr{S}]}$ in a natural way.

$V = \sum_{i=1}^{s} V_{[\omega_i]}$ (sum of vector subspaces; not necessarily direct) implies

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- Staple $\Omega^0_{[\omega_i]}$ following the stapling $(*)$, so that $\Omega^0_{[\mathscr{S}]} := \left[\Omega^0_{[\omega_1]}, \ldots, \Omega^0_{[\omega_s]}\right]$.
- Staple also the simple transitive matrix groups

$$\begin{array}{ccccc} H^0_{[\omega_i]} := & \exp L(V^0_{[\omega_i]}) & \curvearrowright & \Omega^0_{[\omega_i]} \\ & \cap & & \cap \\ & GL(E_{[\omega_i]}) & \curvearrowright & \mathrm{Sym}(E_{[\omega_i]}) \end{array}$$

so that

$$H^0_{[\mathscr{S}]} := \left[H^0_{[\omega_1]}, \ldots, H^0_{[\omega_s]}\right].$$

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Then, $V_{[\omega_i]} \cap V_{[\omega_j]} = \bigoplus_{\substack{k \leq l \\ k,l \in \mathscr{J}(\omega_i,\omega_j)}} V_{lk}$ is the normal decomposition of $V_{[\omega_i]} \cap V_{[\omega_j]}$.

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$\mathscr{J}(\omega_i,\omega_j) \rightsquigarrow \Gamma_{\mathscr{J}(\omega_i,\omega_j)}$: the corresponding oriented subgraph of $\Gamma = \Gamma(V)$.

$\mathscr{J}_0(\omega_i,\omega_j) = \mathscr{S}(\Gamma_{\mathscr{J}(\omega_i,\omega_j)})$: the source set for $\Gamma(\mathscr{J}(\omega_i,\omega_j))$

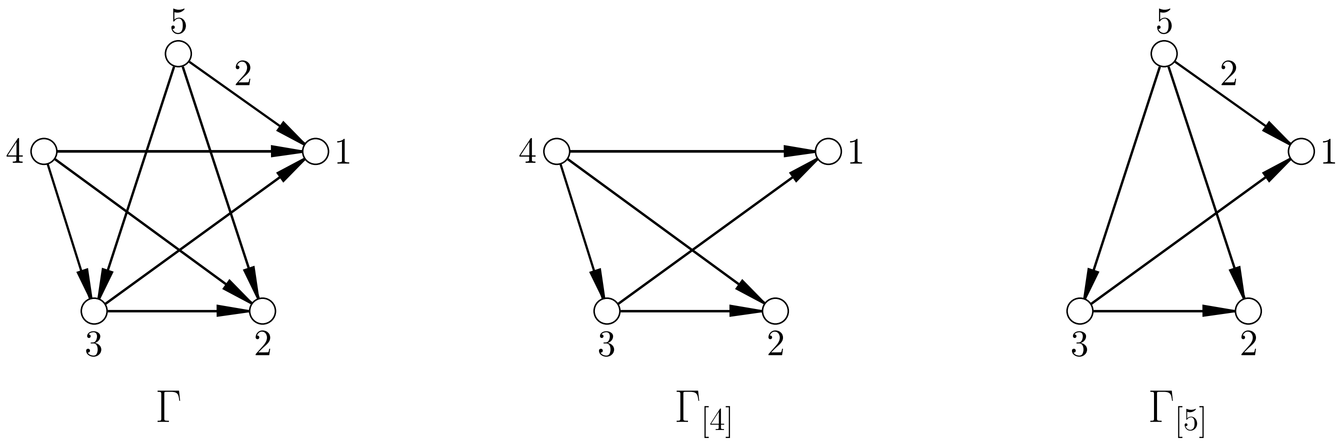
(the **reduced junction set** for ω_i,ω_j).

Let $\mathscr{J}_0(\omega_i,\omega_j) = \{j_1,\dots,j_t\}$, and put $\Omega^0_{[\mathscr{J}_0(\omega_i,\omega_j)]} := \left[\Omega^0_{[j_1]},\dots,\Omega^0_{[j_t]}\right]$.

We have $\Omega_{[\omega_i\omega_j]} \cong \Omega^0_{[\mathscr{J}_0(\omega_i,\omega_j)]}$, and we say that

$\Omega^0_{[\omega_i]}$ and $\Omega^0_{[\omega_j]}$ are stapled at $\Omega^0_{[\mathscr{J}_0(\omega_i,\omega_j)]}$.

We retuen to the example. $\mathscr{S} = \{4, 5\}$.



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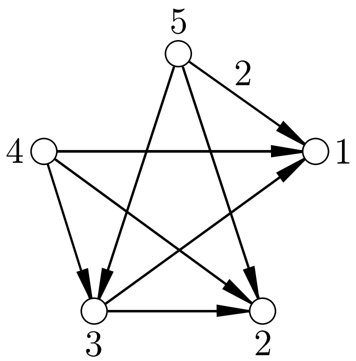
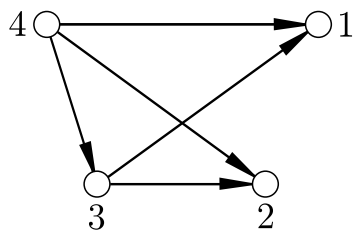
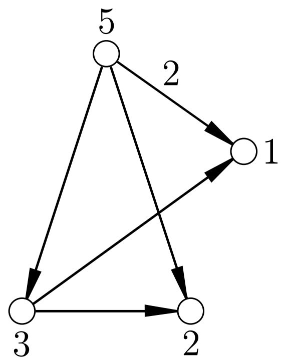


Γ



$\Gamma_{[4]}$



$\Gamma_{[5]}$

$$\Omega^0_{[4]} = \left\{ \begin{pmatrix} \lambda_1 & 0 & x_{31} & x_{41} \\ 0 & \lambda_2 & x_{32} & x_{42} \\ x_{31} & x_{32} & \lambda_3 & x_{43} \\ x_{41} & x_{42} & x_{43} & \lambda_4 \end{pmatrix} \gg 0 \right\}, \quad \Omega^0_{[5]} = \left\{ \begin{pmatrix} \lambda_1 I_2 & \mathbf{0}_2 & x_{31}\boldsymbol{e}_1 & \boldsymbol{x}_{51} \\ {}^t\mathbf{0}_2 & \lambda_2 & x_{32} & x_{52} \\ x_{31}{}^t\boldsymbol{e}_1 & x_{32} & \lambda_3 & x_{53} \\ {}^t\boldsymbol{x}_{51} & x_{52} & x_{53} & \lambda_5 \end{pmatrix} \gg 0 \right\}$$

The shaded parts are stapled. Note $\mathscr{J}(4,5) = \{1,2,3\}$, $\mathscr{J}_0(4,5) = \{3\}$.

In $\Omega^0_{[4]}$, the shaded block is the minimal realization of the dual Vinberg cone.

In $\Omega^0_{[5]}$, the shaded block is not the minimal realization of the dual Vinberg cone.

$$H^0_{[4]} := \left\{ \begin{matrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ x_{31} & x_{32} & \lambda_3 & 0 \\ x_{41} & x_{42} & x_{43} & \lambda_4 \end{matrix} \right\}_{(\lambda_j > 0)}, \quad H^0_{[5]} = \left\{ \begin{matrix} \lambda_1 I_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ {}^t\mathbf{0}_2 & \lambda_2 & 0 & 0 \\ x_{31}{}^t\boldsymbol{e}_1 & x_{32} & \lambda_3 & 0 \\ {}^t\boldsymbol{x}_{51} & x_{52} & x_{53} & \lambda_5 \end{matrix} \right\}_{(\lambda_j > 0)},$$

The shaded parts are stapled: $H^0_{[\mathscr{S}]} = [H^0_{[4]},\ H^0_{[5]}]$.

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$\Omega \rightsquigarrow V$: the corresponding Vinberg algebra

$\rightsquigarrow \Gamma = \Gamma(V)$: the corresponding oriented graph

$\rightsquigarrow \mathscr{S} = \{\omega_1, \dots, \omega_s\}$: the sources of Γ

$\rightsquigarrow \Omega_{[\omega_1]}, \dots, \Omega_{[\omega_s]}$: the source homogeneous cones

$\rightsquigarrow \Omega^0_{[\omega_1]}, \dots, \Omega^0_{[\omega_s]}$: the minimal realizations of the source cones

$\rightsquigarrow \Omega^0_{[\mathscr{S}]} := \left[\Omega^0_{[\omega_1]}, \dots, \Omega^0_{[\omega_s]}\right]$: stapling of the $\Omega^0_{[\omega_i]}$'s

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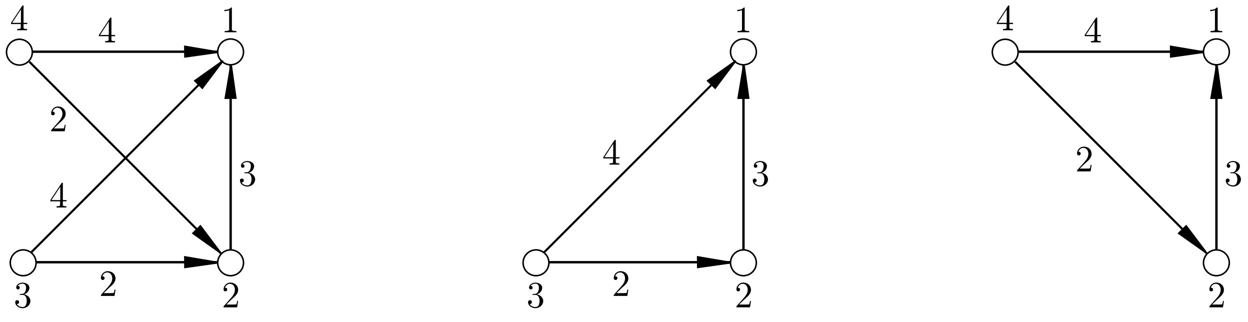

Γ $\qquad$ $\Gamma_{[3]}$ $\qquad$ $\Gamma_{[4]}$

In this case, we have two non-isomorphic Vinberg algebras V and W such that

$$\Gamma(V) = \Gamma(W) = \Gamma.$$

Then, let $\Omega^V \leftrightarrow V$ and $\Omega^W \leftrightarrow W$. We have $\Omega^V \not\cong \Omega^W$.

For $j = 3, 4$, we obatin the source cones $\Omega^V_{[j]}$ and $\Omega^W_{[j]}$ through $\Gamma_{[j]}$.

It is shown that $\Omega^V_{[3]}, \Omega^V_{[4]}, \Omega^W_{[3]}, \Omega^W_{[4]}$ are all linearly equivalent.

Note that $\dim \Omega^V = \dim \Omega^W = 19$.

Kaneyuki–Tsuji condition (1974)

Γ: a transitive oriented graph, $\mathscr{A} := \mathscr{A}(\Gamma)$: the arc set of Γ,

$$[k \to j] \in \mathscr{A} \text{ and } [j \to i] \in \mathscr{A} \implies [k \to i] \in \mathscr{A}.$$

c: a capacity (weight) function $\mathscr{A} \to \mathbb{Z}_{>0}$

We say that (Γ, c) satisfies the **Kaneyuki–Tsuji condition** $\overset{\text{def}}{\iff}$

(KT1) Suppose $i < j < k$.
If there is a path $k \to j \to i$, then one has $\max\big(c_{kj}, c_{ji}\big) \leq c_{ki}$.

(KT2) Suppose $i < j < k < l$.
If there are two paths $l \to k \to i$ and $l \to j \to i$ with $j \notin N^{\text{out}}(k)$, then

$$c_{li} \geq \max(c_{lk}, c_{ki}) + \max(c_{lj}, c_{ji}).$$

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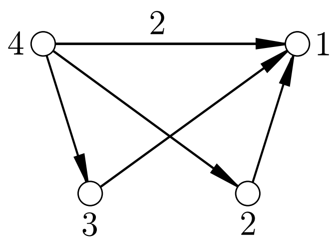
- Example of (KT2):



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(1) $\Gamma = \Gamma(V)$ for a Vinberg algebra V and $c([j \to i]) := \dim V_{ji}$ satisfies (KT1) and (KT2).

(2) $V \mapsto (\Gamma(V), (\dim V_{ji}))$ is neither surjective nor injective.

(3) However, for $\dim V \leq 10$ it is bijective, which led them to the classification of homogeneous convex cones of dimension ≤ 10.

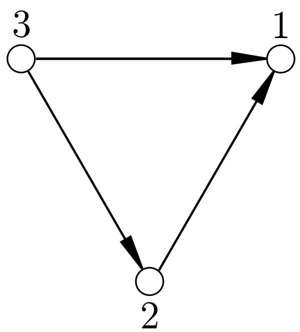
(4) For $\dim V = 11$, a family of continuously many non-isomorphic V have the same $(\Gamma(V), c)$.

(5) $\exists(\Gamma, c)$ with (KT1) and (KT2) s.t. $\Gamma = \Gamma(V)$ for no V.

For (5), the Γ below with

$$c([3 \longrightarrow 1]) = c([3 \longrightarrow 2]) = c([2 \longrightarrow 1]) = d \in \mathbb{Z}_{>0}$$

clearly satsifies (KT1) and (KT2).



But $\Gamma = \Gamma(V)$ for some $V \iff d = 1, 2, 4, 8$.

In this case the corresponding cone $\Omega^V \cong \mathscr{P}(3, \mathbb{K})$, where

$$\mathbb{K} = \mathbb{R}\ (d = 1), \quad \mathbb{K} = \mathbb{C}\ (d = 2), \quad \mathbb{K} = \mathbb{H}\ (d = 4), \quad \mathbb{K} = \mathbb{O}\ (d = 8).$$